Coloring the cube with rainbow cycles

Dhruv Mubayi[*]

Department of Mathematics, Statistics and Computer Science
University of Illinois
Chicago, IL 60607 U.S.A.

mubayi@uic.edu

Randall Stading

Department of Mathematics, Statistics and Computer Science
University of Illinois
Chicago, IL 60607, U.S.A.

rstadi1@uic.edu

Abstract

For every even positive integer $k \geqslant 4$ let $f(n,k)$ denote the minimim number of colors required to color the edges of the n-dimensional cube Q_n, so that the edges of every copy of the k-cycle C_k receive k distinct colors. Faudree, Gyárfás, Lesniak and Schelp proved that $f(n,4) = n$ for $n = 4$ or $n > 5$. We consider larger k and prove that if $k \equiv 0 \pmod 4$, then there are positive constants c_1, c_2 depending only on k such that

$$c_1 n^{k/4} < f(n,k) < c_2 n^{k/4}.$$

Our upper bound uses an old construction of Bose and Chowla of generalized Sidon sets. For $k \equiv 2 \pmod 4$, the situation seems more complicated. For the smallest case $k = 6$ we show that

$$3n - 2 \leqslant f(n,6) < n^{1+o(1)}$$

with the lower bound holding for $n \geqslant 3$. The upper bound is obtained from Behrend's construction of a subset of integers with no three term arithmetic progression.

1   Introduction

Given graphs G and H, and an integer $q \leqslant |E(H)|$, a (G, H, q)-coloring is an edge-coloring of G such that the edges of every copy of H in G receive at least q colors. Let $f(G, H, q)$ be the minimum number of colors in a (G, H, q)-coloring. This general problem is hopeless in most cases, for example, when G and H are cliques, and $q = 2$, determining it is equivalent to determining the multicolor Ramsey number $R_k(p)$ which is a longstanding open problem. There has been more success in determining $f(G, H, q)$ when G and H are not cliques or when $q > 2$ (or both). Many Ramsey problems have received considerable attention when studied on the n-dimensional cube. The papers [3, 4] are examples where anti-Ramsey problems for subcubes in cubes and problems about monochromatic cycles in cubes are investigated. In [13], Offner found the exact value for the maximum number of colors for which it is possible to edge color the hypercube so that all subcubes of dimension d contain all colors. Related Turán type problems for subcubes in cubes have been studied in [2].

Rainbow cycles have also been well studied as subgraphs of K_n. Erdős, Simonovits and Sós [9] introduced $AR(n, H)$, the maximum number of colors in an edge coloring of K_n such that it contains no rainbow copy of H, and provided a conjecture when H is a cycle and showed that their conjecture was true when $H = C_3$. Alon [1] proved their conjecture for cycles of length four and Montellano-Ballesteros and Neumann-Lara [12] proved the conjecture for all cycles. More recently, Choi [7] gave a shorter proof of the conjecture.

We continue this theme in the current note and let $G = Q_n$, the n-dimensional cube, and $H = C_k$, the cycle of length k. Our focus is on $q = |E(H)|$, in which case we will call a (G, H, q)-coloring an H-rainbow coloring, assuming that G is obvious from context (in this paper $G = Q_n$ always).

Definition 1. Fix $4 \leqslant k \leqslant 2^n$. Let $f(n, k)$ ($g(n, k)$) be the minimum number of colors in a C_k-rainbow (Q_k-rainbow) coloring of Q_n.

The smallest case $f(n, 4)$ was studied by Faudree, Gyárfás, Lesniak and Schelp [10] who proved that the trivial lower bound of n is tight by providing, for all $n \geqslant 6$, a C_4-rainbow coloring with n colors. We consider larger k. Our first result determines the order of magnitude of $f(n, k)$ for $k \equiv 0 \pmod 4$.

Theorem 2. *Fix a positive $k \equiv 0$ (mod 4). There are constants $c_1, c_2 > 0$ depending only on k such that*
$$c_1 n^{k/4} < f(n, k) < c_2 n^{k/4}.$$

The case $k \equiv 2 \pmod 4$ seems more complicated. Our results imply that for such fixed k there are positive constants c_1', c_2' with

$$c_1' n^{\lfloor k/4 \rfloor} < f(n, k) < c_2' n^{\lceil k/4 \rceil}.$$

We believe that the lower bound is closer to the truth. As evidence for this, we tackle the smallest case in this range, $k = 6$. As we will observe later, the lower bound $f(n, 6) \geqslant 3n - 2$ is easy to see for $n \geqslant 3$, and we obtain the following upper bound.

Theorem 3. *For every $\epsilon > 0$ there exists n_0 such that for $n > n_0$*

$$3n - 2 \leqslant f(n, 6) < n^{1+\epsilon}$$

with the lower bound holding for all $n \geqslant 3$.

It is rather easy to see that $g(n, 3) = f(n, 6)$. Indeed, if Q_n is edge-colored so that every Q_3 is rainbow, then every C_6 is rainbow since each one is contained in a rainbow Q_3 and so $g(n, 3) \geqslant f(n, 6)$. On the other hand, it is easy to see that any two edges of a Q_3 lie in some C_6 and therefore if Q_n is edge-colored so that every C_6 is rainbow then every Q_3 must also be rainbow and so $g(n, 3) \leqslant f(n, 6)$.
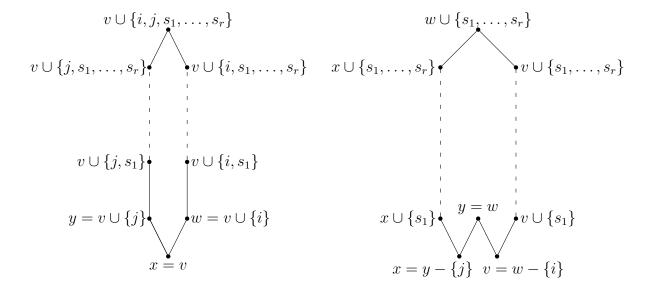
Since $C_4 = Q_2$, the following corollary can also be considered an analogue of the result [10] to subcubes.

Corollary 4. *As $n \to \infty$, we have $g(n, 3) = n^{1+o(1)}$.*
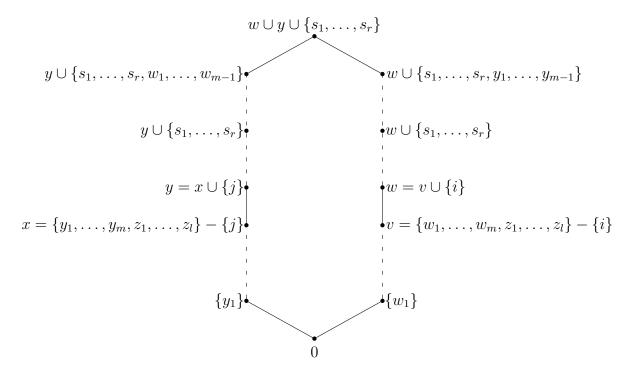
We will consider the vertices of Q_n as binary vectors of length n or as subsets of $[n] = \{1, \ldots, n\}$, depending on the context (with the natural bijection $\vec{v} \leftrightarrow v$ where $\vec{v}$ is the incidence vector for $v \subset [n]$, i.e. $\vec{v}_i = 1$ iff $i \in v$). In particular, whenever we write $v - w$ we mean set theoretic difference, $v \cup w$ or $v \cap w$ we mean set union/intersection and when we write $\vec{v} \pm \vec{w}$ we mean vector addition/subtraction modulo 2. We write e_i for the standard basis vector, so e_i is one in the ith coordinate and zero in all other coordinates. Given an edge $f = uv$ of Q_n where $\vec{v} = \vec{u} + e_s$ for some s, we say that v is the top vertex of f and u is the bottom vertex. We will say the an edge is on level i of Q_n if its bottom vertex corresponds to a vector with $i - 1$ ones and the top vertex to a vector with i ones.

2 Proof of Theorem 2

The lower bound in Theorem 2 follows from the easy observation that in a C_k-rainbow coloring all edges at level $k/4$ must receive distinct colors. Indeed, given any two such edges $f_1 = vw$ and $f_2 = xy$, where $\vec{w} = \vec{v} + e_i$ and $\vec{y} = \vec{x} + e_j$, it suffices to find a copy of C_k containing f_1 and f_2. If f_1 and f_2 are incident then it is clear that we can find a C_k containing them as long as $n > k$ which we may clearly assume. The two cases are illustrated below where $r = k/2 - 2$ and $s_i \notin w \cup y$ for all $i \in \{1, \ldots, r\}$.

$$v \cup \{i, j, s_1, \ldots, s_r\}$$

$$v \cup \{j, s_1, \ldots, s_r\} \qquad v \cup \{i, s_1, \ldots, s_r\}$$

$$v \cup \{j, s_1\} \qquad v \cup \{i, s_1\}$$

$$y = v \cup \{j\} \qquad w = v \cup \{i\}$$

$$x = v$$

$$w \cup \{s_1, \ldots, s_r\}$$

$$x \cup \{s_1, \ldots, s_r\} \qquad v \cup \{s_1, \ldots, s_r\}$$

$$x \cup \{s_1\} \qquad y = w \qquad v \cup \{s_1\}$$

$$x = y - \{j\} \quad v = w - \{i\}$$

Now, suppose f_1 and f_2 are not incident. We know that $|x \triangle v| \leqslant k/2 - 2$ since x and v are each sets of size $k/4 - 1$. By successively deleting elements of v and x in the appropriate order, we can obtain a v, x-path of length $k/2 - 2$. Then, since w and y are sets of size $k/4$, we may find a w, y-path of length $k/2$ between them by successively adding the elements of y to w and vice versa along with extra elements as needed. The two paths along with the edges vw and xy form a cycle of length k. This is shown in the following diagram. Let $y - w = \{y_1, \ldots, y_m\}$, $w - y = \{w_1, \ldots, w_m\}$ and $w \cap y = \{z_1, \ldots, z_l\}$ where $m + l = k/4$. Let $\{s_1, \ldots, s_r\}$ again be a set such that $s_i \notin y \cup w$ with $r = k/4 - m$.

$$w \cup y \cup \{s_1, \ldots, s_r\}$$

$$y \cup \{s_1, \ldots, s_r, w_1, \ldots, w_{m-1}\} \qquad w \cup \{s_1, \ldots, s_r, y_1, \ldots, y_{m-1}\}$$

$$y \cup \{s_1, \ldots, s_r\} \qquad w \cup \{s_1, \ldots, s_r\}$$

$$y = x \cup \{j\} \qquad w = v \cup \{i\}$$

$$x = \{y_1, \ldots, y_m, z_1, \ldots, z_l\} - \{j\} \qquad v = \{w_1, \ldots, w_m, z_1, \ldots, z_l\} - \{i\}$$

$$\{y_1\} \qquad \{w_1\}$$

$$0$$

For the upper bound we need a classical construction of generalized Sidon sets by Bose and Chowla. A B_t-set $S = \{s_1, \ldots, s_n\}$ is a set of integers such that if $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t \leqslant n$ and $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_t \leqslant n$, then

$$s_{i_1} + \cdots + s_{i_t} \neq s_{j_1} + \cdots + s_{j_t}$$

unless $(i_1, \ldots, i_t) = (j_1, \ldots, j_t)$. A consequence of this is that if P, Q are nonempty disjoint subsets of $[n]$ with $|P| = |Q| \leqslant t$, then

$$\sum_{i \in P} s_i \neq \sum_{j \in Q} s_j. \tag{1}$$

The result below is phrased in a form that is suitable for our use later.

Theorem 5. (Bose-Chowla [6]) *For each fixed $t \geqslant 2$, there is a constant $A > 1$ such that for all n, there is a B_t-set $S = \{s_1, \ldots, s_n\} \subset \{1, 2, \ldots, \lfloor An^t \rfloor\}$.*

Now we provide the upper bound construction for Theorem 2.

Construction 1. Let $t = k/4 - 1$ and $S = \{s_1, \ldots, s_n\} \subset \{1, 2, \ldots, \lfloor An^t \rfloor\}$ be a B_t-set as above. For each $v \in V(Q_n)$, let

$$a(v) = \sum_{i=1}^{n} \vec{v}_i s_i = \sum_{i:\vec{v}_i=1} s_i.$$

Given $vw \in E(Q_n)$ with $\vec{w} = \vec{v} + e_j$, let $M = \lceil kAn^t \rceil$, and let

$$d(vw) = a(v) + Mj.$$

Suppose further that vw is at level p and p' is the congruence class of p modulo $k/2$. Then the color of the edge vw is
$$\chi(vw) = (d(vw), p').$$

Let us now argue that this construction yields the upper bound in Theorem 2.

Proof. First, the number of colors is at most

$$\max_{vw} d(vw) \times \frac{k}{2} \leqslant (n \cdot \max s_i + Mn)\frac{k}{2} \leqslant \frac{nk}{2}An^t + \frac{nk}{2}M < k^2An^{t+1} = k^2An^{k/4}$$

as desired. Now we show that this is a C_k-rainbow coloring. Suppose for contradiction that H is a copy of C_k in Q_n and $f_1 = vw, f_2 = xy$ are distinct edges of H with $\chi(f_1) = \chi(f_2)$. Since H spans at most $k/2$ levels, f_1 and f_2 cannot lie in levels that differ by more than $k/2$, so $\chi(f_1) \neq \chi(f_2)$ unless f_1 and f_2 are in the same level which we may henceforth assume. Let v, x be the bottom vertices of f_1, f_2, and $\vec{w} = \vec{v} + e_i, \vec{y} = \vec{x} + e_j$. Assume without loss of generality that $i \leqslant j$. If $v = x$, then

$$a(v) + Mi = d(vw) = d(xy) = a(x) + Mj = a(v) + Mj.$$

This implies that $i = j$ and contradicts the fact that $f_1 \neq f_2$. We may therefore assume that $v \neq x$. Similarly, if $w = y$, then $i < j$ and

$$a(w) - s_i + Mi = a(v) + Mi = d(vw) = d(xy) = a(x) + Mj = a(y) - s_j + Mj = a(w) - s_j + Mj.$$

This implies the contradiction $s_j - s_i = M(j - i) \geqslant M > An^t > s_j - s_i$. Consequently, we may assume that vw and xy share no vertex. If $|v \triangle x| > k/2$, then any v, x-path in Q_n has length more than $k/2$ so there can be no cycle of length k containing both v and x, contradiction. So we may assume that $|v \triangle x| \leqslant k/2$. Now $\chi(vw) = \chi(xy)$ implies that

$$a(v) + Mi = d(vw) = d(xy) = a(x) + Mj$$

and this yields

$$M(j - i) = Mj - Mi = a(v) - a(x) = a(v - x) - a(x - v) \leqslant \frac{|v \triangle x|}{2} An^t \leqslant \frac{k}{4} An^t < M.$$

Consequently, we may assume that $i = j$, $a(v) = a(x)$, $a(v - x) = a(x - v)$ and $|v \triangle x| = |w \triangle y|$. If $|v - x| = |x - v| \leqslant k/4 - 1$, then

$$a(v - x) = \sum_{i \in v - x} s_i \neq \sum_{j \in x - v} s_j = a(x - v)$$

due to (1), the definition of S and $t = k/4 - 1$. So we may assume that $|v - x| = |x - v| = k/4$ and $|w \triangle y| = |v \triangle x| = k/2$. This implies that $\text{dist}_{Q_n}(w, y) = \text{dist}_{Q_n}(v, x) = k/2$. Together with edges f_1, f_2, we conclude that C must have at least $k + 2$ edges, contradiction. $\qquad \square$

3 Proof of Theorem 3

We will first show the lower bound $f(n, 6) \geqslant 3n - 2$ for $n \geqslant 3$ by providing a set of edges F of size $3n - 2$ which must all receive distinct colors in a C_6-rainbow coloring of Q_n. Fix an edge f on the first level of Q_n. For each copy C of C_4 containing f, let F_C be the set of four edges in C. Let $F = \cup_C F_C$, where the union is taken over all copies C of a C_4 containing f. There are $n - 1$ such C's and any two distinct C's share at most one edge, so $|F| = 3(n - 1) + 1 = 3n - 2$.

Now, suppose $f_1, f_2 \in F$ receive the same color. If f_1 and f_2 lie on the same C_4, then let h be one of the two remaining edges of that C_4 and take a different C_4 which contains h. Delete h and observe that the remaining six edges of the two C_4's form a C_6. If f_1 and f_2 do not lie on the same C_4, then their C_4's share the edge f. Now delete f and observe that the remaining six edges form a C_6. We conclude that all the edges in F must receive distinct colors.

To obtain the upper bound, we will give an explicit coloring that makes use of a classical construction of Behrend on sets of integers with no arithmetic progression of size three. Let $r_3(N)$ denote the maximum size of a subset of $\{1, \ldots, N\}$ that contains no 3-term arithmetic progression.

Theorem 6. (Behrend [5]) *There is a $c > 0$ such that if N is sufficiently large, then*

$$r_3(N) > N^{1 - \frac{c}{\sqrt{\log N}}}.$$

Theorem 6 clearly implies that for $\epsilon > 0$ and sufficiently large N we have $r_3(N) > N^{1-\epsilon}$. The error term ϵ was improved recently by Elkin [8] (see [11] for a simpler proof) and using Elkin's result would give corresponding improvements in our result.

Construction 2. Let $\epsilon > 0$ and n be sufficiently large. Put $N = \lceil n^{1+\epsilon} \rceil$ and let $S = \{s_1, \ldots, s_n\} \subset \{1, \ldots, N\}$ contain no 3-term arithmetic progression. Such a set exists by Theorem 6 since

$$n > n^{1-\epsilon^2} = n^{(1-\epsilon)(1+\epsilon)} > N^{1-2\epsilon}.$$

Let

$$a(v) = \sum_{i=1}^{n} \vec{v}_i \, s_i.$$
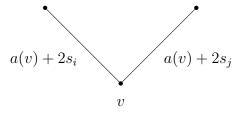
Consider the edge vw, where $\vec{w} = \vec{v} + e_k$. Let

$$d(vw) = a(v) + 2s_k \in Z_{2N}.$$

We emphasize here that we are computing $d(vw)$ modulo $2N$. Suppose further that vw is at level p and p' is the congruence class of p modulo 3. Then the color of the edge vw is
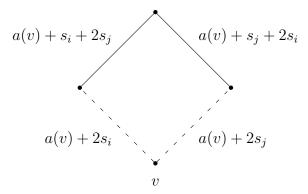
$$\chi(vw) = (d(vw), p').$$

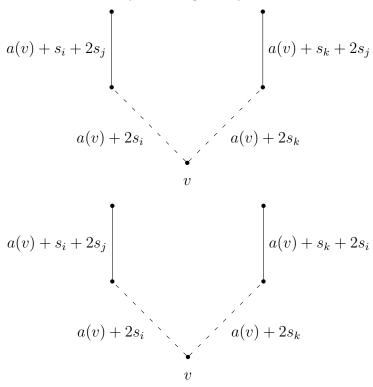The number of colors used is at most $6N < n^{1+2\epsilon}$ as required.

Proof. We will now show that this is a C_6-rainbow coloring. Due to the second coordinate, it suffices to show that any two edges f_1, f_2 of a C_6 which are on the same of level of Q_n receive different colors. If f_1 and f_2 are incident, then they meet either at their top vertices or bottom vertices. If incident at their bottom vertices, the edges are colored as follows and thus are distinctly colored:
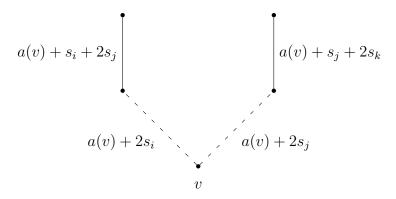

If incident at their top vertices, the edges lie on a C_4 and are therefore distinctly colored

$$a(v) + s_i + 2s_j \qquad a(v) + s_j + 2s_i$$

$$a(v) + 2s_i \qquad a(v) + 2s_j$$

$$v$$

If f_1 and f_2 are not incident, then there must be a path of length two between their bottom vertices. For if not, then they could not lie on a C_6 as the shortest path between their top vertices has length at least two. Moreover, the top vertices of f_1 and f_2 have symmetric difference precisely two since there is a path of length two between them. With these conditions, there are three ways the edges may be colored.

$$a(v) + s_i + 2s_j \qquad a(v) + s_k + 2s_j$$

$$a(v) + 2s_i \qquad a(v) + 2s_k$$

$$v$$

$$a(v) + s_i + 2s_j \qquad a(v) + s_k + 2s_i$$

$$a(v) + 2s_i \qquad a(v) + 2s_k$$

$$v$$

In the first coloring, $s_i + 2s_j \neq s_k + 2s_j$ holds due to i and k being distinct. In the second and third colorings, $s_i + 2s_j \neq s_k + 2s_i$ and $s_i + 2s_j \neq s_j + 2s_k$ hold due to our set S being free of three term arithmetic progressions. $\qquad\square$

4   Concluding Remark

Our results imply a tight connection between C_k-rainbow colorings in the cube and constructions of large generalized Sidon sets. When $k \equiv 0 \pmod 4$ Construction 1 gives the correct order of magnitude, however for $k \equiv 2 \pmod 4$ the same method does not work. In this case an approach similar to Construction 2 would work provided we can construct large sets that do not contains solutions to certain equations.

Conjecture 7. Fix $4 \leqslant k \equiv 2 \pmod 4$. Then $f(n,k) = n^{\lfloor k/4 \rfloor + o(1)}$.

For the first open case $k = 10$, we can show that $f(n, 10) = n^{2 + o(1)}$ provided one can construct a set $S \subset [N]$ with $|S| > N^{1/2 - o(1)}$ that contains no nontrivial solution to any of the following equations:

$$x_1 + x_2 = x_3 + x_4$$
$$x_1 + x_2 + x_3 = x_4 + 2x_5$$
$$x_1 + 2x_2 = x_3 + 2x_4.$$

Ruzsa [14, 15] defined the genus $g(E)$ of an equation

$$E: \quad a_1 x_1 + \cdots + a_k x_k = 0$$

as the largest m such that there is a partition $S_1 \cup \ldots \cup S_m$ of $[k]$ where the S_i are disjoint, non-empty and for all j,

$$\sum_{i \in S_j} a_i = 0. \qquad (2)$$

A solution $(x_1, \ldots, x_k)$ of E is trivial if there are l distinct numbers among $\{x_1, \ldots, x_k\}$ and (2) holds for a partition $S_1 \cup \ldots \cup S_l$ of $[k]$ into disjoint, non-empty parts such that $x_i = x_j$ if and only if $i, j \in S_v$ for some v. Ruzsa showed that if $S \subset [n]$ has no

nontrivial solutions to E then $|S| \leqslant O(n^{1/g(E)})$. The question of whether there exists S with $|S| = n^{1/g(E)-o(1)}$ remains open for most equations E. The set of equations above has genus two so it is plausible that one can prove Conjecture 7 for $k = 10$ using this approach. For the general case, we can provide a rainbow coloring if our set S contains no nontrivial solutions to any of the three equations below with $m = \lfloor k/4 \rfloor$.

$$x_1 + \cdots + x_m = x_{m+1} + \cdots + x_{2m}$$
$$x_1 + \cdots + x_m + x_{m+1} = x_{m+2} + \cdots + x_{2m} + 2x_{2m+1}$$
$$x_1 + \cdots + x_{m-1} + 2x_m = x_{m+1} + \cdots + x_{2m-1} + 2x_{2m}.$$

The set of equations above has genus $m = \lfloor k/4 \rfloor$, so if Ruzsa's question has a positive answer, then we would be able to construct a set of the desired size.

Acknowledgements

References

[1] N. Alon, On a conjecture of Erdős, Simonovits, and Sós concerning anti-Ramsey theorems. J. Graph Theory 7 (1983) no. 1, 91–94.

[2] N. Alon, A. Krech, T. Szabó, Turán's theorem in the hypercube. SIAM J. Discrete Math. 21 (2007), no. 1, 66–72.

[3] N. Alon, R. Radoičić, B. Sudakov and J. Vondrák, A Ramsey-type result for the hypercube. J. Graph Theory 53 (2006), no. 3, 196—208.

[4] M. Axenovich, H. Harborth, M. Möller and I. Schiermeyer, Rainbows in the hypercube. (English summary) Graphs Combin. 23 (2007), no. 2, 123—133.

[5] F. A. Behrend, On sets of integers which contain no three terms in arithmetical progression. Proc. Nat. Acad. Sci. U.S.A. 32, (1946). 331—332.

[6] R. C. Bose, S. Chowla, Theorems in the additive theory of numbers. Comment. Math. Helv. 37 1962/1963 141-–147.

[7] J. Choi, A short proof of anti-Ramsey number for cycles. preprint.

[8] M. Elkin, An improved construction of progression-free sets. Israel J. Math. 184 (2011), 93—128.

[9] P. Erdős, M. Simonovits, and V. T. Sós, Anti-Ramsey theorems. Infinite and finite sets. Colloq. Math. Soc. Janos Bolyai 10 (1975) 633–643.

[10] R. J. Faudree, A. Gyárfás, L. Lesniak and R.H. Schlep, Rainbow coloring the cube. J. Graph Theory 17 (1993), no. 5, 607—612.

[11] B. Green, J. Wolf, A note on Elkin's improvement of Behrend's construction, in Additive Number Theory (Festschrift in honour of Mel Nathanson), Springer 2010, 141–144.

[12] J. J. Montellano-Ballesteros, V. Neumann-Lara, An anti-Ramsey theorem on cycles. Graphs Combin. 22 (2005) no. 3, 343–354.

[13] D. Offner, Polychromatic colorings of subcubes of the hypercube. SIAM J. Discrete Math 22 (2008) no. 2, 450-–454.

[14] I. Z. Ruzsa, Solving a linear equation in a set of integers. I. Acta Arith. 65 (1993), no. 3, 259—282.

[15] I. Z. Ruzsa, Solving a linear equation in a set of integers. II. Acta Arith. 72 (1995), no. 4, 385—397.