On Blow-Ups and Injectivity of Quivers

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Abstract

This work connects the idea of a "blow-up" of a quiver with that of injectivity, showing that for a class of monic maps Φ, a quiver is Φ-injective if and only if all blow-ups of it are as well. This relationship is then used to characterize all quivers that are injective with respect to the natural embedding of P_n into C_n.

1   Introduction

This paper continues the study of injectivity for quivers begun in [7]. Recall that a quiver is a directed multigraph with loops, and a quiver homomorphism is a pair of maps preserving the adjacency structure. Explicitly, the definitions below will be used throughout the paper, where **Set** is the category of sets with functions.

Definition (Quiver, [5, Definition 2.1]). A *quiver* is a quadruple (V, E, σ, τ), where $V, E \in \mathrm{Ob}(\mathbf{Set})$ are sets, and $\sigma, \tau \in \mathbf{Set}(E, V)$ are functions. Elements of V are *vertices*, and V the *vertex set*. Elements of E are *edges*, and E the *edge set*. The function σ is the *source map*, and τ the *target map*. For $e \in E$, $\sigma(e)$ is the *source* of e, and $\tau(e)$ the *target* of e.

Definition (Quiver map, [5, Definition 2.4]). Given quivers G and H, a *quiver homomorphism* from G to H is a pair (ϕ_V, ϕ_E), where $\phi_V \in \mathbf{Set}(V_G, V_H)$ and $\phi_E \in \mathbf{Set}(E_G, E_H)$ satisfy $\phi_V \circ \sigma_G = \sigma_H \circ \phi_E$ and $\phi_V \circ \tau_G = \tau_H \circ \phi_E$. The function ϕ_V is the *vertex map*, and ϕ_E the *edge map*.

Let **Quiv** denote the category of quivers in which the composition of quiver homomorphisms is defined component-wise. The notation of this paper will follow that established in [7]. Please also observe that to ease notation, the functors V and E will be omitted on maps, as is convention in graph theory. In [7], the class of quivers which are injective with

respect to all monic quiver maps was characterized using the following generalization of a complete graph.

Definition ([7, Definition 3.1.1])**.** For a quiver J and $v, w \in V(J)$, let

$$\text{edges}_J(v, w) := \sigma_J^{-1}(v) \cap \tau_J^{-1}(w)$$

be the set of all edges in J with source v and target w. A quiver J is *loaded* if for every $v, w \in V(J)$, $\text{edges}_J(v, w) \neq \emptyset$.

Essentially, a loaded quiver is a complete digraph, which may have multiple edges. From [7, Proposition 3.2.1], a quiver is injective with respect to all monic quiver maps if and only if it is loaded and has a nonempty vertex set.

The notion of a "blow-up" of a quiver, analogous to the concept for simple graphs in [8], is introduced and shown to be intimately tied to quiver injectivity in Theorem 3.5. That is, if a quiver is injective, all of its blow-ups are also injective. As injectivity classes are closed on retraction, an injective quiver yields a family of injectives: its retracts and their blow-ups. Section 2 characterizes quiver sections and retractions, which are used in Section 3 to show that a blow-up retracts onto the original quiver and yield the main result in Theorem 3.5.

The remainder of the paper is dedicated to an example of using blow-ups to characterize a class of injective objects. For $n \geqslant 2$, let $P_n \xrightarrow{\phi_n} C_n$ be the natural embedding of the directed path P_n into the directed cycle C_n. Propositions 4.4 and 5.1 and Theorem 6.1 characterize a ϕ_n-injective quiver as a quiver such that the connected components are either blow-ups of C_m for $m \mid n$ or quivers with no walk of length $n - 1$. This seems to show that injectivity yields a propagation of structure, as observed in Lemmas 5.2, 5.3, and 5.4.

Studying quivers injective relative to specific maps seems to be an interesting way in which a local condition propagates to give rise to global structure. In the case of embedding P_n into C_n, this process yields the surprisingly simple structure of blow-ups of C_n. The case of embedding P_3 into a transitive triangle in the natural way gives rise to "transitive digraphs" [2, §4.3], which correspond to directed sets. It may be that embeddings not yet studied give rise to other interesting structures.

Because injectivity is a categorical notion, any structure resulting from injectivity can be compared to an algebraic counterpart. Since the notion of a retract is vital to the proof of Theorems 3.5 and 6.1, this is also one of the very few times categorical methods have been used in graph theory to prove combinatorial results.

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2 Sections and Retractions in Quiv

Recall from [1, Definition 7.19] that a *section* in a category is a morphism that is left-invertible. Also, [1, Definition 7.24] gives the dual notion of a section as a *retraction*, a

right-invertible morphism. Traditionally, the codomain of a retraction, or equivalently the domain of a section, is termed a *retract*.

For quivers, a section can be recognized by finding a certain type of partition of the codomain's vertices and edges.

Proposition 2.1 (Characterization of sections). *A homomorphism* $G \xrightarrow{\ j\ } H \in \mathbf{Quiv}$ *is a section if and only if there are partitions* $(A_v)_{v \in V(G)}$ *of* $V(H)$ *and* $(B_e)_{e \in E(G)}$ *of* $E(H)$ *satisfying*

1. *$j(v) \in A_v$,*

2. *$j(e) \in B_e$,*

3. *$\sigma_H(f) \in A_{\sigma_G(e)}$,*

4. *$\tau_H(f) \in A_{\tau_G(e)}$,*

for all $v \in V(G)$, $e \in E(G)$, and $f \in B_e$. In this case, G is a retract of H.

Proof. ($\Rightarrow$) Let $H \xrightarrow{\ q\ } G \in \mathbf{Quiv}$ satisfy that $q \circ j = id_G$. For $v \in V(G)$ and $e \in E(H)$, define $A_v := q^{-1}(v)$ and $B_e := q^{-1}(e)$, which partition $V(H)$ and $E(H)$. Notice that

$$v = (q \circ j)(v) = q\,(j(v)) \quad \text{and} \quad e = (q \circ j)(e) = q\,(j(e))$$

for all $v \in V(G)$ and $e \in E(G)$. Thus, $j(v) \in A_v$ and $j(e) \in B_e$. Also,

$$q\,(\sigma_H(f)) = \sigma_G\,(q(f)) = \sigma_G(e)$$

and

$$q\,(\tau_H(f)) = \tau_G\,(q(f)) = \tau_G(e)$$

for all $e \in V(G)$ and $f \in B_e$. Thus, $\sigma_H(f) \in A_{\sigma_G(e)}$ and $\tau_H(f) \in A_{\tau_G(e)}$

($\Leftarrow$) Define $V(H) \xrightarrow{\ q_V\ } V(G) \in \mathbf{Set}$ and $E(H) \xrightarrow{\ q_E\ } E(G) \in \mathbf{Set}$ by

$$q_V(w) := v \quad \text{and} \quad q_E(f) := e,$$

where $w \in A_v$ and $f \in B_e$. Both are well-defined, as $(A_v)_{v \in V(G)}$ and $(B_e)_{e \in E(G)}$ are partitions. For $f \in E(H)$, there is $e \in E(G)$ such that $f \in B_e$. Thus,

$$\sigma_G\,(q_E(f)) = \sigma_G(e) = q_V\,(\sigma_H(f)),$$

as $\sigma_H(f) \in A_{\sigma_G(e)}$. Likewise, $\tau_G\,(q_E(f)) = q_V\,(\tau_H(f))$ by a similar argument using $\tau_H(f) \in A_{\tau_G(e)}$. Thus, $q := (q_V, q_E)$ is a quiver homomorphism. Moreover,

$$(q \circ j)(v) = q_V\,(j(v)) = v \quad \text{and} \quad (q \circ j)(e) = q_E\,(j(e)) = e$$

as $j(v) \in A_v$ and $j(e) \in B_e$ for all $v \in V(G)$ and $e \in E(G)$. Thus, $q \circ j = id_G$. $\qquad \square$

Dually, a retraction can be recognized by finding pre-images of the vertices and edges that align with the structure maps.

Proposition 2.2 (Characterization of retractions). *A homomorphism* $H \xrightarrow{q} G \in$ **Quiv** *is a retraction if and only if there are* $(w_v)_{v \in V(G)} \subseteq V(H)$ *and* $(f_e)_{e \in E(G)} \subseteq E(H)$ *satisfying*

1. $q(w_v) = v$,

2. $q(f_e) = e$,

3. $\sigma_H(f_e) = w_{\sigma_G(e)}$,

4. $\tau_H(f_e) = w_{\tau_G(e)}$,

for all $v \in V(G)$ *and* $e \in E(G)$. *In this case, G is a retract of H.*

Proof. ($\Rightarrow$) Let $G \xrightarrow{j} H \in$ **Quiv** satisfy that $q \circ j = id_G$. Define $w_v := j(v)$ and $f_e := j(e)$ for $v \in V(G)$ and $e \in E(G)$. Observe that

$$v = (q \circ j)(v) = q(j(v)) = q(w_v)$$

and

$$e = (q \circ j)(e) = q(j(e)) = q(f_e)$$

for all $v \in V(G)$ and $e \in E(G)$. Also,

$$\sigma_H(f_e) = (\sigma_H \circ j)(e) = (j \circ \sigma_G)(e) = w_{\sigma_G(e)}$$

and

$$\tau_H(f_e) = (\tau_H \circ j)(e) = (j \circ \tau_G)(e) = w_{\tau_G(e)}$$

for all $e \in E(G)$.

($\Leftarrow$) Define $V(G) \xrightarrow{j_V} V(H) \in$ **Set** and $E(G) \xrightarrow{j_E} E(H) \in$ **Set** by

$$j_V(v) := w_v \quad \text{and} \quad j_E(e) := f_e.$$

For $e \in E(G)$,
$$(\sigma_H \circ j_E)(e) = \sigma_H(f_e) = w_{\sigma_G(e)} = (j_V \circ \sigma_G)(e)$$
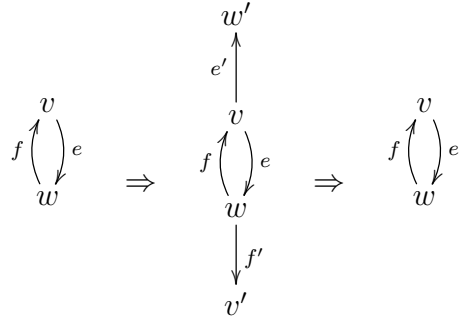
and likewise $(\tau_H \circ j_E)(e) = (j_V \circ \tau_G)(e)$, using $\tau_H(f_e) = w_{\tau_G(e)}$. Thus, $j := (j_V, j_E)$ is a quiver homomorphism. Moreover,

$$(q \circ j)(v) = q(w_v) = v \quad \text{and} \quad (q \circ j)(e) = q(f_e) = e$$

for all $v \in V(G)$ and $e \in E(G)$. Thus, $q \circ j = id_G$. $\qquad\square$

For any category, sections are monic and retractions epic. Due to the characterization of quiver monomorphisms, a copy of a retract can always be found in the codomain quiver of a section, or dually the domain quiver of a retraction, as illustrated below.

Example 2.3. The following maps are a section and retraction pair, where the primes are associated to their counterparts.

$$
\begin{array}{ccc}
& & w' \\
& & \uparrow e' \\
\quad v & & v \\
f \Big(\ \Big) e & \Rightarrow & f \Big(\ \Big) e & \Rightarrow & f \Big(\ \Big) e \\
\quad w & & w & & w \\
& & \downarrow f' \\
& & v'
\end{array}
$$

3 Blow-up of a Quiver

The notion of a "blow-up" of a simple graph G is already defined in [8, p. 1] as a new graph constructed by replacing $v \in V(G)$ with a set of vertices A_v and defining $x \in A_v$ and $y \in A_w$ adjacent if and only if v and w were adjacent in G. The introduction of multiple edges and direction obscures this intuitive idea, but the blow-up of a quiver follows the same concept.

For a quiver G, each vertex $v \in V(G)$ is replaced with a set of vertices A_v and each edge e with a set of edges B_e, satisfying appropriate adjacency conditions. If $v \xrightarrow{e} w$ is an edge in G and $f \in B_e$, the source and target of f must be in A_v and A_w, respectively. Conversely, if $x \in A_v$ and $y \in B_w$, there must be an edge from x to y in the blow-up associated to e.

Definition. Given a quiver G, a *blow-up* of G is a quiver H equipped with partitions $(A_v)_{v \in V(G)}$ of $V(H)$ and $(B_e)_{e \in E(G)}$ of $E(H)$ satisfying

1. $\sigma_H(f) \in A_{\sigma_G(e)}$,

2. $\tau_H(f) \in A_{\tau_G(e)}$,

3. $\mathrm{edges}_H(x, y) \cap B_e \neq \emptyset$,
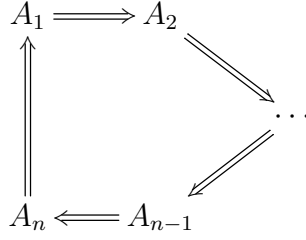
for all $e \in E(G)$, $f \in B_e$, $x \in A_{\sigma_G(e)}$, $y \in A_{\tau_G(e)}$.

Example 3.1 (Reflexivity). Any quiver G is a blow-up of itself by using $A_v := \{v\}$ and $B_e := \{e\}$ for $v \in V(G)$ and $e \in E(G)$.

Example 3.2 (Loaded). The blow-ups of the single loop bouquet C_1 are precisely all nonempty loaded quivers.

Example 3.3 (Cycles). For $n \geqslant 2$, the blow-ups of the directed cycle C_n take the form below, where the bold arrows indicate the existence of at least one arrow between any

vertex of the source set and any vertex of the target set. Note that since C_n has no loops, the sets A_j are independent.

$$A_1 \Longrightarrow A_2$$
$$\cdots$$
$$A_n \Longleftarrow A_{n-1}$$

Consequently, the middle quiver in Example 2.3 is not a blow-up of C_2.

From this definition, any blow-up of a quiver can be retracted back onto the original. This can be done by constructing a section using Proposition 2.1, or by constructing a retraction using Proposition 2.2.

Proposition 3.4. *Given a quiver G and a blow-up H of G, G is a retract of H.*

Proof. Let $(A_v)_{v \in V(G)}$ and $(B_e)_{e \in E(G)}$ be the partitions of $V(H)$ and $E(H)$, respectively. Define $V(H) \xrightarrow{q_V} V(G) \in \mathbf{Set}$ by $q_V(w) := v$, where $w \in A_v$. Likewise, define $E(H) \xrightarrow{q_E} E(G) \in \mathbf{Set}$ by $q_E(f) := e$, where $f \in B_e$. Both are well-defined, as $(A_v)_{v \in V(G)}$ and $(B_e)_{e \in E(G)}$ are partitions. For $f \in E(H)$, there is $e \in E(G)$ such that $f \in B_e$. Thus,

$$(\sigma_G \circ q_E)(f) = \sigma_G(e) = (q_V \circ \sigma_H)(f)$$

and

$$(\tau_G \circ q_E)(f) = \tau_G(e) = (q_V \circ \tau_H)(f)$$

as $\sigma_H(f) \in A_{\sigma_G(e)}$ and $\tau_H(f) \in A_{\tau_G(e)}$. Hence, $q := (q_V, q_E)$ is a quiver homomorphism. For $v \in V(G)$, choose $w_v \in A_v$. For $e \in E(G)$, choose $f_e \in \mathrm{edges}_H\left(w_{\sigma_G(e)}, w_{\tau_G(e)}\right) \cap B_e$. Then,

$$q(w_v) = v, \quad q(f_e) = e, \quad \sigma_H(f_e) = w_{\sigma_G(e)},$$

and $\tau_H(f_e) = w_{\tau_G(e)}$. By Proposition 2.2, q is a retraction, and G is a retract of H. $\square$

In a general category, retracts have a pleasant relationship with injectivity from [1, §9]. Given a category $\mathscr{C}$ and a class of morphisms Φ, an object A of $\mathscr{C}$ is Φ-injective if and only if all retracts of A are. In the case of $\mathscr{C} = \mathbf{Quiv}$, if a blow-up of a quiver is Φ-injective, then by Proposition 3.4, the original quiver must also be Φ-injective. Moreover, the converse of this statement about blow-ups is also true if Φ is a class of monomorphisms.

Theorem 3.5. *Let G be a quiver and Φ a class of monic quiver maps. The following are equivalent:*

1. *G is Φ-injective,*

2. *all blow-ups of G are Φ-injective,*

3. *a blow-up of G is Φ-injective.*

Proof. $(2 \Rightarrow 3)$ If all blow-ups of G are Φ-injective, then a single blow-up is.

$(3 \Rightarrow 1)$ By Proposition 3.4, G is a retract of any blow-up, so G becomes Φ-injective by the general theory.

$(1 \Rightarrow 2)$ Let H be a blow-up of G, equipped with partitions $(A_v)_{v \in V(G)}$ and $(B_e)_{e \in E(G)}$ of $V(H)$ and $E(H)$, respectively. Given $D \xrightarrow{\phi} C \in \Phi$ and $D \xrightarrow{\psi} H \in \mathbf{Quiv}$, the goal is to construct a map $C \xrightarrow{\tilde{\psi}} H \in \mathbf{Quiv}$ such that $\psi = \tilde{\psi} \circ \phi$. Let $H \xrightarrow{q} G \in \mathbf{Quiv}$ be the retraction constructed in Proposition 3.4.

$$
\begin{array}{ccc}
H & \xrightarrow{\;q\;} & G \\
{\scriptstyle\psi}\Big\uparrow & & \\
D & \xrightarrowtail[\phi]{} & C
\end{array}
$$

As G is Φ-injective, there is $C \xrightarrow{\hat{\psi}} G \in \mathbf{Quiv}$ such that $q \circ \psi = \hat{\psi} \circ \phi$.

$$
\begin{array}{ccc}
H & \xrightarrow{\;q\;} & G \\
{\scriptstyle\psi}\Big\uparrow & & \Big\uparrow{\scriptstyle \exists\hat{\psi}} \\
D & \xrightarrowtail[\phi]{} & C
\end{array}
$$

To construct the quiver map from C to H, vertices and edges will be chosen much like in Proposition 3.4. First, the vertices of C will be handled. For $v \in \operatorname{ran}(\phi)$, there is a unique $a_v \in V(D)$ such that $v = \phi(a_v)$, since ϕ is monic. Let $w_v := \psi(a_v)$ and observe that

$$q(w_v) = (q \circ \psi)(a_v) = \left(\hat{\psi} \circ \phi\right)(a_v) = \hat{\psi}(v).$$

Thus, $w_v \in A_{\hat{\psi}(v)}$. For $v \notin \operatorname{ran}(\phi)$, choose $w_v \in A_{\hat{\psi}(v)}$ arbitrarily.

Next, consider the edges of C. For $e \in \operatorname{ran}(\phi)$, there is a unique $b_e \in E(D)$ such that $e = \phi(b_e)$, since ϕ is monic. Let $f_e := \psi(b_e)$ and observe that

$$q(f_e) = (q \circ \psi)(b_e) = \left(\hat{\psi} \circ \phi\right)(b_e) = \hat{\psi}(e),$$

$$\sigma_H(f_e) = (\sigma_H \circ \psi)(b_e) = (\psi \circ \sigma_D)(b_e) = \psi\left(a_{\sigma_C(e)}\right) = w_{\sigma_C(e)},$$

and $\tau_H(f_e) = w_{\tau_C(e)}$ by a similar calculation since ϕ is monic. Thus, $f_e \in \operatorname{edges}_H\left(w_{\sigma_C(e)}, w_{\tau_C(e)}\right) \cap B_{\hat{\psi}(e)}$. For $e \notin \operatorname{ran}(\phi)$, choose $f_e \in \operatorname{edges}_H\left(w_{\sigma_C(e)}, w_{\tau_C(e)}\right) \cap B_{\hat{\psi}(e)}$ arbitrarily.

Define $V(C) \xrightarrow{\alpha} V(H) \in \mathbf{Set}$ by $\alpha(v) := w_v$ and $E(C) \xrightarrow{\beta} E(H) \in \mathbf{Set}$ by $\beta(e) := f_e$. For $e \in E(C)$,

$$(\sigma_H \circ \beta)(e) = \sigma_H(f_e) = w_{\sigma_C(e)} = (\alpha \circ \sigma_C)(e)$$

and

$$\left(\tau_H \circ \beta\right)(e) = \tau_H\left(f_e\right) = w_{\tau_C(e)} = \left(\alpha \circ \tau_C\right)(e)$$

Thus, $\tilde{\psi} := (\alpha, \beta)$ is a quiver homomorphism.
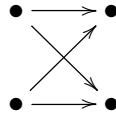
Notice that $\tilde{\psi} \circ \phi = \psi$ by design. Since ϕ was arbitrary, H is Φ-injective. $\square$

Example 3.6. Notice that the bouquet C_1 is a terminal object in **Quiv**. Consequently, it is injective with respect to any class of quiver morphisms, in particular any class Φ of monomorphisms. By Example 3.2 and Theorem 3.5, any nonempty loaded quiver is Φ-injective. The content of [7, Proposition 3.2.1] is that when Φ is the class of all monic maps, the injectivity class contains no other members.

Similarly, abstract projectivity classes are closed under retractions, but the dual statement of Theorem 3.5 is not true.
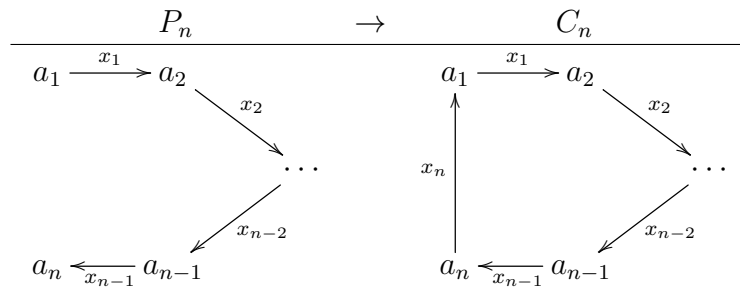
Example 3.7. By [7, Proposition 4.1.1], the directed path of order-2, P_2, is projective with respect to all quiver epimorphisms. However, the following quiver is a blow-up of P_2, but not epi-projective.

4   Definition and Abstract Relationships

The remainder of this paper will consider some particular cases of quiver injectivity, where the concept of a blow-up will become quite telling and useful.

For an integer $n \geqslant 2$, let P_n be the directed path of order n and C_n the directed cycle of order n. In this section, P_n will be embedded into C_n by connecting the two endpoints of the path in the obvious way. This embedding will be denoted as the quiver monomorphism $\phi_n : P_n \to C_n$. This situation is drawn below.

The goal will be to describe all quivers which are ϕ_n-injective. Much like [7, Proposition 3.2.1], the property of being ϕ_n-injective can be recovered with only a condition on the quiver itself.

Proposition 4.1 (Characterization of ϕ_n-injectivity). *A quiver J is ϕ_n-injective if and only if for any walk of length $n-1$, there is an edge from the terminal vertex to the initial vertex.*

Proof. ($\Rightarrow$) Let $(v_j)_{j=1}^n \subseteq V(J)$ with $e_j \in \text{edges}_J(v_j, v_{j+1})$ for $1 \leqslant j \leqslant n-1$. Define $\psi_V : V(P_n) \to V(J)$ and $\psi_E : E(P_n) \to E(J)$ by

$$a_j \mapsto v_j \quad \text{and} \quad x_k \mapsto e_k,$$

for all $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n-1$. A routine check shows that $\psi := (\psi_V, \psi_E)$ is a quiver map from P_n to J. As J is ϕ_n-injective, there is a quiver map $\hat{\psi} : C_n \to J$ such that $\hat{\psi} \circ \phi_n = \psi$. Let $e_n := \hat{\psi}(x_n)$. A calculation shows that $\sigma_J(e_n) = v_n$ and $\tau_J(e_n) = v_1$.
($\Leftarrow$) Given $\psi : P_n \to J$, let

$$v_j := \psi(a_j) \quad \text{and} \quad e_k := \psi(x_k),$$

for all $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n-1$. A calculation shows that $e_j \in \text{edges}_J(v_j, v_{j+1})$ for $1 \leqslant j \leqslant n-1$. By assumption, there is $e_n \in \text{edges}_J(v_n, v_1)$. Define $\hat{\psi}_V : V(C_n) \to V(J)$ and $\hat{\psi}_E : E(C_n) \to E(J)$ by

$$a_j \mapsto v_j \quad \text{and} \quad x_j \mapsto e_j,$$
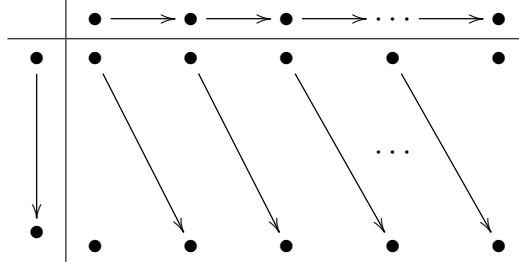
for all $1 \leqslant j \leqslant n$. A routine check shows that $\hat{\psi} := \left(\hat{\psi}_V, \hat{\psi}_E\right)$ is a quiver map from C_n to J and that $\hat{\psi} \circ \phi = \psi$. $\qquad\square$

Example 4.2 (ϕ_n-injectivity of some quivers). Using this criterion, some quivers can immediately be shown to be ϕ_n-injective or not.
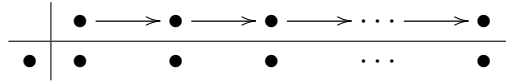
1. Any nonempty loaded quiver is mono-injective by [7, Proposition 3.2.1] and, therefore, trivially ϕ_n-injective for all $n \geqslant 2$.

2. For $n, m \geqslant 2$, P_m is ϕ_n-injective if and only if $n > m$.

3. For any $n, m \geqslant 2$, C_m is ϕ_n-injective if and only if $m \mid n$.

By [1, Proposition 10.40], abstract injectivity classes are closed on the categorical product, which is no less true here. In most algebraic settings, a product object is injective if and only if all of its factors are. However, this is not the case for ϕ_n-injectivity.

Example 4.3 (Failure of products for ϕ_n-injectivity). Observe that $P_2 \prod P_m$ is ϕ_n-injective for $n \geqslant 3$ and $m \geqslant 2$.

Let I_m denote the independent set of vertices of order-m. Then, $I_1 \prod P_m \cong I_m$ is ϕ_2-injective.

On the other hand, these particular injectivity classes have a very tight relationship with the coproduct, the disjoint union, which can be proven quickly with Proposition 4.1. This result is due to the weak connectivity of P_n and C_n.

Proposition 4.4 (Disjoint unions and ϕ_n-injectivity)**.** *Let Λ be an index set and J_λ a quiver for each $\lambda \in \Lambda$. Then, $\coprod\limits_{\lambda \in \Lambda} J_\lambda$ is ϕ_n-injective if and only if J_λ is ϕ_n-injective for each $\lambda \in \Lambda$.*

Proof. Let $J := \coprod\limits_{\lambda \in \Lambda} J_\lambda$, and regard each J_λ as a subquiver of J.

$(\Rightarrow)$ Fix $\lambda \in \Lambda$. Let $(v_j)_{j=1}^n \subseteq V(J_\lambda)$ with $e_j \in \text{edges}_{J_\lambda}(v_j, v_{j+1})$ for $1 \leqslant j \leqslant n-1$. Since J is ϕ_n-injective and J_λ is a subquiver of J, there must be $e_n \in \text{edges}_J(v_n, v_1)$ by Proposition 4.1. As a disjoint union, the only edges in J between v_n and v_1 arise from J_λ, forcing $e_n \in \text{edges}_{J_\lambda}(v_n, v_1)$. Proposition 4.1 states that J_λ must be ϕ_n-injective.

$(\Leftarrow)$ Let $(v_j)_{j=1}^n \subseteq V(J)$ with $e_j \in \text{edges}_J(v_j, v_{j+1})$ for $1 \leqslant j \leqslant n-1$. As a disjoint union, e_1 arises from some J_λ, forcing that v_1 and v_2 must also be from the same J_λ. By induction, e_k and v_j must be from J_λ also for all $1 \leqslant j \leqslant n$ and $1 \leqslant k \leqslant n-1$. Since J_λ is ϕ_n-injective, there is $e_n \in \text{edges}_{J_\lambda}(v_n, v_1)$, which also exists in J. $\square$

By this result, one need only consider the weakly connected components of a quiver to determine ϕ_n-injectivity.

5 Trivial Cases

Looking at the examples thus far, most quivers have been ϕ_n-injective for trivial reasons, either being loaded or having no walk of length $n-1$. The latter property can be used to completely describe the structure of the quiver.

Proposition 5.1 (No walk of length n). *If J is a quiver with no walk of length n, then J consists of $n-1$ sets of vertices $B_1, \ldots, B_{n-1}$, where every edge of J goes from a vertex in B_i to a vertex in B_j for some $i < j$.*

Proof. Let B_1 be the set of all sources of J. If J has no walk of length n, then J has no cycles, and hence B_1 is non-empty if J is non-empty. For every vertex $v \notin B_1$, place $v \in B_i$ if i is the maximum length of a path ending with v.

Hence all that needs to be shown is that every edge goes from a vertex B_i to a vertex B_j for $i < j$. Suppose there is an edge from $u \in B_i$ to $v \in B_j$. By definition, there is a path of length i ending on u. By taking the edge $u \to v$, this means there is a path of length $i+1$ ending on v. Since j is the length of the longest path ending on v, it must be that $j \geqslant i+1 > i$, as desired. $\qquad\square$

For $n \geqslant 3$, the loaded case can be identified readily by one characteristic feature, having a loop at some vertex. This fact will be proven incrementally over three lemmas. First, an edge terminating in a loop spawns a copy of the full quiver on two vertices.

Lemma 5.2 (Propagating loops). *For $n \geqslant 3$, suppose J is a ϕ_n-injective quiver. If J has a walk of length 2, which has distinct endpoints and either starts or ends with a loop, then there is an edge in reverse and a loop on the opposite vertex.*

Proof. Say the following is a subquiver of J.

$$ u \xrightarrow{\;g\;} w \,\circlearrowright f $$

Then, consider the walk given by

$$
\begin{array}{ccc}
v_1 & := & u, \\
v_j & := & w,
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
e_1 & := & g, \\
e_k & := & f,
\end{array}
$$

for $2 \leqslant j \leqslant n$ and $2 \leqslant k \leqslant n-1$. By Proposition 4.1, there is $h \in \text{edges}_J(w, u)$.

$$ u \underset{h}{\overset{g}{\rightleftarrows}} w \,\circlearrowright f $$

Now, consider the walk given by

$$
\begin{array}{ccc}
v_1 := v_n & := & u, \\
v_j & := & w,
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
e_1 & := & g, \\
e_k & := & f, \\
e_{n-1} & := & h,
\end{array}
$$

for $2 \leqslant j \leqslant n-1$ and $2 \leqslant k \leqslant n-2$. By Proposition 4.1, there is $d \in \text{edges}_J(u, u)$.

$$ d \,\circlearrowright u \underset{h}{\overset{g}{\rightleftarrows}} w \,\circlearrowright f $$

If g was reversed, the subquiver appears between u and w by dualizing the proof. $\qquad\square$
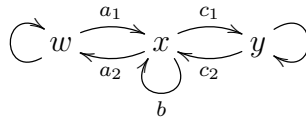
Next, two consecutive edges give rise to a copy of the full quiver on three vertices, allowing the quiver to be triangulated.

Lemma 5.3 (Triangulation). *For $n \geqslant 3$, suppose J is a ϕ_n-injective quiver. If J has two consecutive edges ending with a loop, then an edge exists between each pair of vertices involved.*

Proof. Say the following subquiver exists in J,

$$w \text{———} x \text{———} y \circlearrowright$$
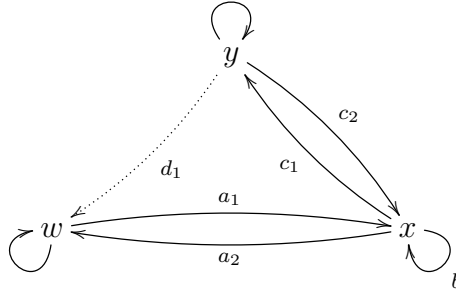
where the two edges can be oriented in either direction. By Lemma 5.2, each edge becomes a 2-cycle with a loop at each vertex.

Consider the walk given by

$$
\begin{array}{rcl rcl}
v_1 & := & w, & e_1 & := & a_1, \\
v_j & := & x, \quad \text{and} \quad & e_k & := & b, \\
v_n & := & y, & e_{n-1} & := & c_1,
\end{array}
$$

for $2 \leqslant j \leqslant n-1$ and $2 \leqslant k \leqslant n-2$. By Proposition 4.1, there is $d_1 \in \mathrm{edges}_J(y, w)$.
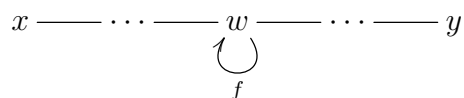

By a symmetric argument, there is $d_2 \in \mathrm{edges}_J(w, y)$. $\qquad \square$

At last, these two lemmas together yield the following characterization of when a ϕ_n-injective quiver is loaded.
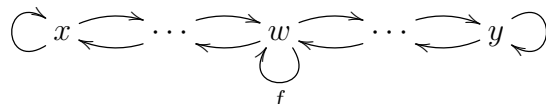
Lemma 5.4 (ϕ_n-injective components, loaded). *For $n \geqslant 3$, suppose J is a ϕ_n-injective quiver with $V(J) \neq \emptyset$. Then, J is loaded if and only if it is both weakly connected and has at least one vertex with a loop.*

Proof. ($\Rightarrow$) This follows from [7, Definition 3.1.1] and [7, Proposition 3.2.1].
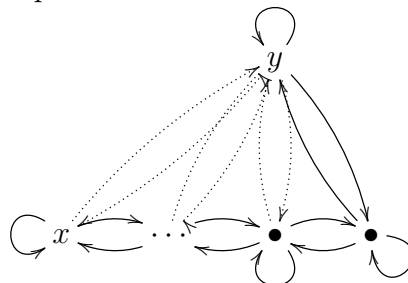
($\Leftarrow$) Let $w \in V(J)$ such that there is $f \in \text{edges}_J(w, w)$. Consider arbitrary $x, y \in V(J)$. Since J is weakly connected, there is some finite sequence of edges from x to w and some finite sequence of edges from w to y.

By induction on the length of the edge sequence from x to w to y, Lemma 5.2 yields the following.

Induction on the length of the path with Lemma 5.3 then gives the following.

Since x and y were arbitrary, J is loaded. □

This is a very useful criterion since this reduces avoiding loaded quivers to avoiding loops. However, this criterion does not hold for $n = 2$ as shown in the example below.

Example 5.5 (A ϕ_2-injective quiver). The following quiver is ϕ_2-injective, but not ϕ_n-injective for $n \geqslant 3$.

However, ϕ_2-injective quivers are readily understood from Proposition 4.1.

Corollary 5.6 (Structure of ϕ_2-injective quivers). *A quiver J is ϕ_2-injective if and only if given any non-loop edge in J, there is an edge in reverse.*

From this, the bi-directing of any undirected multigraph is naturally ϕ_2-injective, but notice that Example 5.5 is not the bi-directing of an undirected multigraph.

6 Blow-ups of Cycles

For $n \geqslant 3$, what remains to be described is the structure of quivers that are ϕ_n-injective and weakly connected with a walk of length $n-1$ and no loops. The archetype of this case is C_m, where $m \mid n$. By Theorem 3.5, all blow-ups of C_m are ϕ_n-injective for $m \mid n$. Moreover, these blow-ups exhaust this final case of ϕ_n-injectivity.

Theorem 6.1 (Structural characterization of ϕ_n-injective quivers). *For $n \geqslant 3$, consider a ϕ_n-injective and weakly connected quiver J. If J is has a walk of length $n-1$, then J is a blow-up of C_ℓ for some $\ell \mid n$.*

Proof. By hypothesis, there is a walk $(w_j)_{j=1}^{n}$ in J. By Proposition 4.1, there is an edge from w_n to w_1, meaning that J has a closed walk. Let

$$\ell := \min\{m \in \mathbb{N} : J \text{ has a closed walk of length } m\}.$$

If $\ell = 1$, Lemma 5.4 states that J is loaded since J is weakly connected. By Example 3.2, J is a blow-up of C_1.

If $\ell \neq 1$, let $(v_j)_{j=0}^{\ell-1}$ and $e_j \in \text{edges}_J\left(v_j, v_{j+1 \ (\text{mod} \ \ell)}\right)$ for $0 \leqslant j \leqslant \ell - 1$ be a closed walk of minimal length. Consequently, the vertices are distinct as a repeated vertex yields a shorter closed walk. Since $\ell \leqslant n$, $n = q\ell + r$ for some $q \in \mathbb{N}$ and $0 \leqslant r < \ell$. If $r \neq 0$, the walk $w_k := v_{k \ (\text{mod} \ \ell)}$ for $0 \leqslant k \leqslant n-1$ necessitates an edge from $v_{r-1 \ (\text{mod} \ \ell)}$ to v_0. This forms a closed walk of length r, contradicting the minimality of ℓ. Thus, $n = q\ell$.

Define $B_{j+1 \ (\text{mod} \ \ell)} := \tau_J\left(\sigma_J^{-1}(v_j)\right)$ for $0 \leqslant j \leqslant \ell - 1$, the proposed partite sets for the blow-up of C_ℓ. For $x \in B_0$, there is $f \in \text{edges}_J(v_{\ell-1}, x)$. The walk

$$w_k := \begin{cases} v_{k \ (\text{mod} \ \ell)}, & 1 \leqslant k \leqslant n-1, \\ x, & k = n, \end{cases} \quad \text{and} \quad f_k := \begin{cases} e_{k \ (\text{mod} \ \ell)}, & 1 \leqslant k \leqslant n-2, \\ f, & k = n-1, \end{cases}$$

forces an edge from x to v_1. Translation of this argument gives an edge from every element of B_j to $v_{j+1 \ (\text{mod} \ \ell)}$ for all $0 \leqslant j \leqslant \ell - 1$.

For $x \in B_0$ and $y \in B_1$, there is $f \in \text{edges}_J(v_{\ell-1}, x)$ and $g \in \text{edges}_J(y, v_2)$. The walk

$$w_k := \begin{cases} y, & k = 1, \\ v_{k \ (\text{mod} \ \ell)}, & 2 \leqslant k \leqslant n-1, \\ x, & k = n, \end{cases} \quad \text{and} \quad f_k := \begin{cases} g, & k = 1, \\ e_{k \ (\text{mod} \ \ell)}, & 2 \leqslant k \leqslant n-2, \\ f, & k = n-1, \end{cases}$$

forces an edge from x to y. Translation of this argument gives an edge from every element of B_j to every element of $B_{j+1 \ (\text{mod} \ \ell)}$ for all $0 \leqslant j \leqslant \ell - 1$, as desired.

For $j \neq 1$, assume that $x \in B_0$, $y \in B_j$, and $h \in \text{edges}_J(x, y)$. There are $f \in \text{edges}_J(y, v_{j+1})$, $g \in \text{edges}_J(v_{\ell-1}, x)$, and $b \in \text{edges}_J(x, v_1)$. If $j > 1$, the walk

$$w_k := \begin{cases} y, & k = 1, \\ v_{j+k-1}, & 2 \leqslant k \leqslant \ell - j, \\ x, & k = \ell - j + 1, \end{cases} \quad \text{and} \quad f_k := \begin{cases} f, & k = 1, \\ e_{j+k-1}, & 2 \leqslant k \leqslant \ell - j - 1, \\ g, & k = \ell - j, \\ h, & k = \ell - j + 1, \end{cases}$$

becomes a closed walk of length $\ell - j + 1$, contradicting the minimality of ℓ. If $j = 0$, the walk

$$
w_k := \begin{cases} x, & k = 1, \\ y, & k = 2, \\ v_{k-2 \pmod{\ell}}, & 3 \leqslant k \leqslant n, \end{cases} \quad \text{and} \quad f_k := \begin{cases} h, & k = 1, \\ f, & k = 2, \\ e_{k-2 \pmod{\ell}}, & 3 \leqslant k \leqslant n - 1, \end{cases}
$$

yields $a \in \text{edges}_J(v_{\ell-2}, x)$, making the closed walk of length $\ell - 1$ below.

$$
w_k := \begin{cases} v_k, & 1 \leqslant k \leqslant \ell - 2, \\ x, & k = \ell - 1, \end{cases} \quad \text{and} \quad f_k := \begin{cases} e_k, & 1 \leqslant k \leqslant \ell - 3, \\ a, & k = \ell - 2, \\ b, & k = \ell - 1. \end{cases}
$$

This again contradicts the minimality of ℓ. Therefore, edges from B_0 terminate in B_j only when $j = 1$. Translation of this argument shows that edges from B_k terminate in B_j only when $j - k \equiv 1 \pmod{\ell}$ as desired.

Assume that $\bigcup_{j=0}^{\ell-1} B_j \neq V(J)$. For $z \notin \bigcup_{j=0}^{\ell-1} B_j$, there is some finite sequence of edges from z to v_0, since J is weakly connected. Let f be the first edge in this sequence such that one endpoint is in $\bigcup_{j=0}^{\ell-1} B_j$ and the other is not. Let $x := \sigma_J(f)$ and $y := \tau_J(f)$. If $x \notin \bigcup_{j=0}^{\ell-1} B_j$ and $y \in B_j$ for some $0 \leqslant j \leqslant \ell - 1$, there is $g \in \text{edges}_J\left(y, v_{j+1 \pmod{\ell}}\right)$. Then, the walk

$$
w_k := \begin{cases} x, & k = 1, \\ y, & k = 2, \\ v_{j+k-2 \pmod{\ell}}, & 3 \leqslant k \leqslant n, \end{cases} \quad \text{and} \quad f_k := \begin{cases} f, & k = 1, \\ g, & k = 2, \\ e_{j+k-2}, & 1 \leqslant k \leqslant n - 1, \end{cases}
$$

forces an edge from $v_{j-2 \pmod{\ell}}$ to x, meaning that $x \in B_{j-2 \pmod{\ell}}$. A similar contradiction occurs if the roles of x and y are reversed. Therefore, $V(J) = \bigcup_{j=0}^{\ell-1} B_j$, meaning J is a blow-up of C_ℓ. $\qquad\square$

7   Concluding Remarks

As seen with the example of ϕ_n, injectivity relative to a single embedding led to a propagation of structure, characterizing an entire class of quivers. A logical next step would be to consider different embeddings and to find new classes of examples that might be characterized in this way. Dually, the same question could be asked with regard to projectivity relative to a covering map, though this seems more mysterious due to Example 3.7.

$$A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B$$
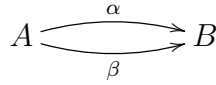
Figure 7.1: The Category $\mathscr{K}$

Further, different graph-like structures could be considered. Recall that the category of quivers can be considered as a either a functor category [1, Definition I.6.15] from the category $\mathscr{K}$ drawn in Figure 7.1 to **Set**, or as a comma category [1, Exercise I.3K] between the identity functor and diagonal functor on **Set**. In either case, replacing **Set** with the category of topological spaces creates the category of topological, or continuous, graphs considered in [3, Definition 1.1] and [9, Definition 2.1]. Similarly, replacing **Set** with a category of weighted, or normed, sets in [6, §2] creates a respective category of weighted graphs. Moreover, in the comma category setting, replacing the diagonal functor with the covariant power set functor gives rise to the category of hypergraphs considered in [4, §2]. Thus, the methods and results of this paper could be adapted to each of these settings.

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