Graphs on Affine and Linear Spaces and Deuber Sets

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Abstract

If G is a large K_k-free graph, by Ramsey's theorem, a large set of vertices is independent. For graphs whose vertices are positive integers, much recent work has been done to identify what arithmetic structure is possible in an independent set. This paper addresses similar problems: for graphs whose vertices are affine or linear spaces over a finite field, and when the vertices of the graph are elements of an arbitrary Abelian group.

1 Introduction and Notation

For a set X and a positive integer i, the collection of all i-element subsets of X is denoted $[X]^i = \{Y \subseteq X : |X| = i\}$. For integers $a < b$, let $[a, b] = \{z \in \mathbb{Z} : a \leqslant z \leqslant b\}$. Let $[n] = [1, n] = \{1, 2, \ldots, n\}$ and let $[n]^i$ be an abbreviation for $[[n]]^i$. Let $G = (V, E)$ denote a (simple) graph with vertex-set $V = V(G)$ and edge-set $E = E(G)$, where $E \subseteq [V]^2$. The complete graph on k vertices is denoted by K_k, and a graph G is called K_k-free if G contains no copy of K_k as a subgraph. A subset $I \subseteq V(G)$ is called *independent* in G if and only if I does not contain any edges from E, *i.e.*, $[I]^2 \cap E = \emptyset$.

One form of Ramsey's theorem [15] says that for any integers $k, \ell \in \mathbb{Z}^+$, there exists an n so that any K_k-free graph on n vertices contains an independent set with at least ℓ vertices. In 1995, Erdős asked if G is triangle-free graph on $\mathbb{Z}^+$, does there exist a Schur triple $\{x, y, x+y\}$, that is independent? This was answered affirmatively by Łuczak, Rödl, and Schoen [12] by a much more general theorem for K_k-free graphs on $\mathbb{Z}^+$. Similarly, it was found by Gunderson, Leader, Prömel, and Rödl [9] that there are arbitrarily long finite arithmetic progressions that are independent in K_k-free graphs on $\mathbb{Z}^+$. Motivated by a question from Deuber, both results were generalized in [10] by the aforementioned authors to provide a characterization of those arithmetic structures that can be found in an independent set. These and more such generalizations are considered in detail in Section 5.

Here, similar questions about the richness of structure in independent sets are investigated for K_k-free graphs whose vertices are either points in a finite vector space (see Theorem 1) or linear lines in a finite vector space (see Theorem 5). In each case, entire subspaces (resp., affine or linear) are found in independent sets. We also show that there are no higher dimensional analogues of Theorems 1 and 5. As an application, in Section 5 we consider K_k-free graphs G whose vertices are elements of an Abelian group $\mathcal{G}$, and we show the existence of solutions $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in (\mathcal{G} \setminus \{0\})^\ell$ to any given partition regular system $M\mathbf{x} = \mathbf{0}$ of equations over $\mathcal{G} \setminus \{0\}$, where $\{x_1, \ldots, x_\ell\}$ is an independent set in the graph G. It turns out that for the countably infinite direct sum $G = \mathbb{Z}_p^{<\omega}$ the considerations are easier than in the case $G = \mathbb{Z}$.

2 Affine and Linear Spaces

In this section we introduce some useful notation.

When q is a power of a prime, let $\mathbb{F}_q$ denote the Galois field with q elements, and let $\mathbb{F}_q^n$ denote the n-dimensional linear vector space over $\mathbb{F}_q$. An m-dimensional linear subspace $W \subseteq \mathbb{F}_q^n$ is a set of vectors for which there exists m (basis) vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{F}_q^n$ that are linearly independent (over $\mathbb{F}_q$) and for which $W = \mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. For any vector $\mathbf{a} \in \mathbb{F}_q^n$, the set $U = \mathbf{a} + W = \{\mathbf{a} + \mathbf{w} : \mathbf{w} \in W\}$ is called an *affine subspace*, and the vector $\mathbf{a}$ is called the *translation vector* or *initial vector* of U. The dimension of $U = \mathbf{a} + W$ is the dimension of W.

A vector $\mathbf{v} \in \mathbb{F}_q^n$ is written with respect to the standard basis vectors $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, \ldots, 0, 1)$, so the notation $\mathbf{v} = (v_1, \ldots, v_n)$ means that $\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i$. Each $v_i \in \mathbb{F}_q$ is called the i-th coordinate of $\mathbf{v}$, also denoted $\mathbf{v}(i) = v_i$.

For any (linear or affine) subspace W of $\mathbb{F}_q^n$, there is a certain kind of basis for W that is convenient to work with here. By elementary linear algebra, for any m-dimensional (linear) subspace W of $\mathbb{F}_q^n$, there is a unique ordered basis $(\mathbf{w}_1, \ldots, \mathbf{w}_m)$ satisfying:

(i) the first non-zero coordinate of each $\mathbf{w}_i$ is equal to 1,

(ii) for each i, if the first non-zero coordinate of $\mathbf{w}_i$ occurs at position c_i, then $c_1 < c_2 < \cdots < c_m$, and

(iii) for each i and each $j \neq i$, $\mathbf{w}_j(c_i) = 0$.

Such an ordered basis satisfying (i)–(iii) is said to be the *Schur normal form* (SNF) basis for W. If $(\mathbf{w}_1, \ldots, \mathbf{w}_m)$ is the SNF basis for a linear subspace W, write

$$W = \langle \mathbf{w}_1, \ldots, \mathbf{w}_m \rangle.$$

For an affine space $U = \mathbf{a} + W$, the translation vector $\mathbf{a}$ is not unique, however, if W is in SNF as above, there is a unique initial vector $\mathbf{a}_0$ so that for $i = 1, \ldots, m$, the c_i-th coordinate is zero, that is, $\mathbf{a}_0(c_i) = 0$. In this case,

$$U = \langle \mathbf{a}_0; \mathbf{w}_1, \ldots, \mathbf{w}_m \rangle$$

is the (unique) SNF for U.

To illustrate the notation, consider the space $V = \mathbb{F}_5^3$. The set $A = \{(x, y, z) : y = 3\}$ of points is an affine plane with translation vector $\mathbf{a}_0 = (0, 3, 0)$, and so $A = \langle (0, 3, 0); (1, 0, 0), (0, 0, 1)\rangle$.

The 1-dimensional affine subspace $B = \{(x, y, z) : x + 3z = 1, y = 3\}$ of A has SNF $B = \langle (0, 3, 2); (1, 0, 3)\rangle$. If the above affine line B is written as $\langle \mathbf{a}_0; \mathbf{a}_1\rangle$ in SNF, the t'th point $\mathbf{a}_0 + t \cdot \mathbf{a}_1$ on the line is uniquely determined.

3 Affine Points

In this section we consider graphs with vertices being points in a vector space over some finite field. Here is the first main result in this paper.

Theorem 1. *Fix a prime power q and let k, m with $k \geqslant 3$ be positive integers. Then, there exists a positive integer $n_1 = n_1(q, k, m)$ such that for every integer $n \geqslant n_1$, every K_k-free graph G on the vertex-set $\mathbb{F}_q^n$ contains an m-dimensional affine subspace whose set of points forms an independent set in the graph G.*

To prove Theorem 1, we use a higher-dimensional partition result. Such a strategy turned out to be fruitful in Canonizing Ramsey Theory. Here we use the following result of Graham, Leeb, and Rothschild [7].

Theorem 2. *Let q be a prime power, and let ℓ, m, r be nonnegative integers with $\ell \leqslant m$. Then, there exists a positive integer $n_2 = n_2(q, \ell, m, r)$ such that for every integer $n \geqslant n_2$ and every r-coloring of the ℓ-dimensional affine subspaces of $\mathbb{F}_q^n$ there exists an m-dimensional affine subspace $S \subseteq \mathbb{F}_q^n$ such that all ℓ-dimensional affine subspaces in S are colored the same.*

Proof. (of Theorem 1) Assume, without loss of generality, that $k \leqslant m$, for then the case for smaller m follows directly.

From Theorem 2, we set $n_2 = n_2(q, 1, m, 2^{q(q-1)/2})$. We claim that $n_1 = n_1(q, k, m) \leqslant n_2$ for each $k \leqslant m$. Let $n \geqslant n_2$, and let $G = (V, E)$ be a K_k-free graph on the vertex-set $V = \mathbb{F}_q^n$. It remains to find an m-dimensional affine subspace whose vertices form an independent set in G.

Color the 1-dimensional affine subspaces L of $\mathbb{F}_q^n$ (affine lines) according to the pattern of the edges in L, *i.e.*, color the line $\langle \mathbf{a}_0; \mathbf{a}_1\rangle$ in SNF by the set of all unordered pairs $\{g, h\}$, $g, h \in \mathbb{F}_q$, for which $\{\mathbf{a}_0 + g \cdot \mathbf{a}_1,\ \mathbf{a}_0 + h \cdot \mathbf{a}_1\} \in E$. Hence, at most $2^{q(q-1)/2}$ colors are used.

By Theorem 2 there exists an m-dimensional affine subspace S of $\mathbb{F}_q^n$ such that all its 1-dimensional affine subspaces are colored the same, hence they all have the same pattern with respect to the occurring edges. Let the m-dimensional affine subspace $S = \langle \mathbf{a}_0; \mathbf{a}_1, \ldots, \mathbf{a}_m\rangle$ be in SNF, where $\mathbf{a}_0$ is the initial vector.

We show that the points in S form an independent set in G. For a contradiction first assume that in the 1-dimensional affine subspace given by $\langle \mathbf{a}_0; \mathbf{a}_1\rangle$ in SNF, for some $i \neq j$,

$$\{\mathbf{a}_0 + i \cdot \mathbf{a}_1, \mathbf{a}_0 + j \cdot \mathbf{a}_1\} \in E. \tag{1}$$

We claim that the following k points in S yield a complete graph K_k in the graph G:

$$\begin{aligned}
&\mathbf{a}_0 + i \cdot \mathbf{a}_1, \\
&\mathbf{a}_0 + j \cdot \mathbf{a}_1 + i \cdot \mathbf{a}_2, \\
&\mathbf{a}_0 + j \cdot \mathbf{a}_1 + j \cdot \mathbf{a}_2 + i \cdot \mathbf{a}_3, \\
&\vdots \\
&\mathbf{a}_0 + j \cdot \mathbf{a}_1 + \cdots + j \cdot \mathbf{a}_{k-1} + i \cdot \mathbf{a}_k.
\end{aligned}$$

To see this, for $1 \leqslant r < s \leqslant k$ consider the 1-dimensional affine subspaces given by $\langle \mathbf{x}_{r,s}; \mathbf{y}_{r,s} \rangle$ of S in SNF with initial vector $\mathbf{x}_{r,s}$, where

$$\mathbf{x}_{r,s} = \mathbf{a}_0 + j \cdot \sum_{g=1}^{r-1} \mathbf{a}_g - i \cdot j \cdot (j-i)^{-1} \cdot \sum_{h=r+1}^{s-1} \mathbf{a}_h - i^2 \cdot (j-i)^{-1} \cdot \mathbf{a}_s, \tag{2}$$

$$\mathbf{y}_{r,s} = \mathbf{a}_r + j \cdot (j-i)^{-1} \cdot \sum_{h=r+1}^{s-1} \mathbf{a}_h + i \cdot (j-i)^{-1} \cdot \mathbf{a}_s. \tag{3}$$

Note that $\sum_{h=r+1}^{s-1} a_h = \mathbf{0}$ for $s = r+1$. As $\langle \mathbf{a}_0; \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle$ is in SNF, so are the affine lines $\langle \mathbf{x}_{r,s}; \mathbf{y}_{r,s} \rangle$. Namely, the first nonzero entry of $\mathbf{y}_{r,s}$ arises from the first nonzero entry of the vector $\mathbf{a}_r$, hence is equal to 1, and the corresponding coordinate of the vector $\mathbf{x}_{r,s}$ is equal to 0.

All affine lines in S are colored the same. With (2), and (3) we infer

$$\begin{aligned}
\mathbf{x}_{r,s} + i \cdot \mathbf{y}_{r,s} &= \mathbf{a}_0 + j \cdot \sum_{g=1}^{r-1} \mathbf{a}_g + i \cdot \mathbf{a}_r, \\
\mathbf{x}_{r,s} + j \cdot \mathbf{y}_{r,s} &= \mathbf{a}_0 + j \cdot \sum_{g=1}^{s-1} \mathbf{a}_g + i \cdot \mathbf{a}_s,
\end{aligned}$$

and by (1) for $1 \leqslant r < s \leqslant k$ we obtain

$$\{\mathbf{a}_0 + j \cdot \sum_{g=1}^{r-1} \mathbf{a}_g + i \cdot \mathbf{a}_r,\ \mathbf{a}_0 + j \cdot \sum_{g=1}^{s-1} \mathbf{a}_g + i \cdot \mathbf{a}_s\} \in E.$$

Hence the points $\mathbf{a}_0 + j \cdot \sum_{g=1}^{r-1} \mathbf{a}_g + i \cdot \mathbf{a}_r$, $r = 1, \ldots, k$, yield a complete subgraph K_k in G, which is a contradiction. As all 1-dimensional affine subspaces in S are colored the same, no affine line in S contains an edge of G. Since two distinct points determine uniquely an affine line, it follows that the set of points in S is an independent set in G. $\square$

3.1 A Different Proof

Using methods from ergodic theory, the following density version of Theorem 2 for affine points was given by Fürstenberg and Katznelson [6]. For another proof by purely combinatorial methods, following the original approach of Szemerédi [17], which he used for proving the density result for arithmetic progressions, compare Eggenwirth [5].

Theorem 3. *For every prime power q, $\varepsilon > 0$ and $\ell \in \mathbb{Z}^+$, there exists a least positive integer $n_3 = n_3(q, \varepsilon, \ell)$ such that for every integer $n \geqslant n_3$ every subset $X \subseteq \mathbb{F}_q^n$ with $|X| \geqslant \varepsilon \cdot q^n$ contains an ℓ-dimensional affine subspace.*

Theorem 3 can be used to give another proof of Theorem 1, a proof similar to one given in [10], where the corresponding problem for arithmetic progressions in $\mathbb{Z}^+$ was investigated.

Proof. (second proof of Theorem 1) Let $S(k, m, q)$ denote the following statement: There exists $n_0(k, m, q)$ such that for every integer $n \geqslant n_0(k, m, q)$, every K_k-free graph with vertex-set being the set of all q^n affine points in an n-dimensional affine vector space over $\mathbb{F}_q$ contains an m-dimensional affine subspace whose affine points yield an independent set in G.

Fix m and q; the proof of $S(k, m, q)$ is by induction on k. Certainly $S(2, m, q)$ holds, hence assume that for some $k \geqslant 3$, statement $S(k-1, m, q)$ holds. Let $n_0(k, m, q) = n_3(q, 2/q^{2m}, n_0(k-1, m, q))$ and let $n \geqslant n_0(k, m, q)$. Let $G = (V, E)$ be a K_k-free graph with vertex-set being the set of all points in an n-dimensional affine vector space over $\mathbb{F}_q$.

The number of m-dimensional linear subspaces in $\mathbb{F}_q^n$ is equal to

$$\binom{n}{m}_q = \frac{(q^n - 1) \cdot (q^{n-1} - 1) \cdots (q^{n-m+1} - 1)}{(q^m - 1) \cdot (q^{m-1} - 1) \cdots (q - 1)},$$

and the number of m-dimensional affine subspaces in $\mathbb{F}_q^n$ is $q^{n-m} \cdot \binom{n}{m}_q$. Moreover, each affine line can be extended in $\binom{n-1}{m-1}_q$ ways to an m-dimensional affine subspace in $\mathbb{F}_q^n$. Thus, if every m-dimensional affine subspace in $\mathbb{F}_q^n$ contains an edge, then G contains at least

$$\frac{q^{n-m} \cdot \binom{n}{m}_q}{\binom{n-1}{m-1}_q} = q^{n-m} \cdot \frac{q^n - 1}{q^m - 1} \geqslant q^{2n-2m}$$

edges. Hence there exists a vertex v in G such that the set $N(v)$ of its neighbors has cardinality at least $2 \cdot q^{n-2m}$. The subgraph of G induced on the vertex-set $N(v)$ is by assumption K_{k-1}-free. By Theorem 3 with $\varepsilon = 2/q^{2m}$ and by choice of $n_0(k, m, q)$ the set $N(v)$ contains an $n_0(k-1, m, q)$-dimensional affine subspace S. By the induction assumption this affine space S contains an m-dimensional affine subspace and the set of all its affine points is an independent set in G, completing the induction step. □

Remark: It would be nice to have a proof of Theorem 1 that does not use these deep Theorems 2 or 3.

3.2 Higher Dimensional Version

In Theorem 1 vertices are points (0-dimensional subspaces) and one might expect by using methods as in [18], for example, that an analogue to Theorem 1 might also hold, where vertices are higher dimensional affine subspaces. However, the following observation yields a counterexample, and shows that Theorem 1 is optimal in this sense.

Proposition 4. *Fix a prime power q and let ℓ, n be positive integers with $n \geqslant \ell + 2$. There exists a triangle-free graph $G = (V, E)$ with vertex-set V being the set of all ℓ-dimensional affine subspaces of $\mathbb{F}_q^n$, such that each $(\ell+1)$-dimensional affine subspace contains two ℓ-dimensional affine subspaces R and S of $\mathbb{F}_q^n$, which are joined by an edge, i.e., $\{R, S\} \in E$.*

Proof. Fix a total ordering $<^*$ of the ℓ-dimensional affine subspaces of $\mathbb{F}_q^n$ with the following property: for two ℓ-dimensional affine spaces $R = \langle \mathbf{a}_0; \mathbf{a}_1, \ldots, \mathbf{a}_\ell \rangle$ and $S = \langle \mathbf{b}_0; \mathbf{b}_1, \ldots, \mathbf{b}_\ell \rangle$ in SNF, if the position of the first occurrence of 1 in $\mathbf{a}_1$ is less than the position of the first occurrence of 1 in $\mathbf{b}_1$, then $R <^* S$.

Next we define the edge-set E of the graph $G = (V, E)$. For two ℓ-dimensional affine subspaces $R = \langle \mathbf{a}_0; \mathbf{a}_1, \ldots, \mathbf{a}_\ell \rangle$ and $S = \langle \mathbf{b}_0; \mathbf{b}_1, \ldots, \mathbf{b}_\ell \rangle$ in SNF of $\mathbb{F}_q^n$ with $R <^* S$, let $\{R, S\} \in E$ if and only if $\mathbf{b}_0 = \mathbf{a}_0 + \mathbf{a}_1$ and $\mathbf{b}_i = \mathbf{a}_{i+1}$, $i = 1, \ldots, \ell - 1$.

We claim that G is triangle-free. Namely, assume that the vertices R, S, U with $R <^* S <^* U$ form a triangle, i.e., $R = \langle \mathbf{a}_0; \mathbf{a}_1, \ldots, \mathbf{a}_\ell \rangle$, and $S = \langle \mathbf{a}_0 + \mathbf{a}_1; \mathbf{a}_2, \ldots, \mathbf{a}_\ell, \mathbf{a}_{\ell+1} \rangle$, and $U = \langle \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2; \mathbf{a}_3, \ldots, \mathbf{a}_\ell, \mathbf{a}_{\ell+1}, \mathbf{a}_{\ell+2} \rangle$ in SNF, as $\{R, S\}, \{S, U\} \in E$. The assumption that $\{R, U\} \in E$ implies $\mathbf{a}_0 + \mathbf{a}_1 = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2$, i.e., $\mathbf{a}_2$ is the all-zeros vector, which is a contradiction.

Now let $W = \langle \mathbf{c}_0; \mathbf{c}_1, \ldots, \mathbf{c}_{\ell+1} \rangle$ in SNF be an $(\ell+1)$-dimensional affine subspace of $\mathbb{F}_q^n$. Then, by construction, for the two ℓ-dimensional affine subspaces $R = \langle \mathbf{c}_0; \mathbf{c}_1, \ldots, \mathbf{c}_\ell \rangle$ and $S = \langle \mathbf{c}_0 + \mathbf{c}_1; \mathbf{c}_2, \ldots, \mathbf{c}_{\ell+1} \rangle$ in SNF we have $R <^* S$, and R and S are joined by an edge in G. $\square$

4 Linear Lines

In this section we consider the corresponding problem to Theorem 1 for graphs with the vertex-set consisting of all linear lines of a linear vector space over $\mathbb{F}_q$. We prove the following analog of Theorem 1 for linear spaces.

Theorem 5. *Let q be a prime power, and let k, m be positive integers with $k \geqslant 3$ and $m \geqslant 2$. Then, there exists a positive integer $n_4 = n_4(q, k, m)$ such that for every $n \geqslant n_4$, every K_k-free graph G with vertex-set being the set of all 1-dimensional linear subspaces of $\mathbb{F}_q^n$ contains an m-dimensional linear subspace S such that the set of all 1-dimensional linear subspaces in S is an independent set in the graph G.*

In our arguments we use the following version of Theorem 2 for linear spaces due to Graham, Leeb and Rothschild [7].

Theorem 6. *Let q be a prime power and let ℓ, m, r be positive integers with $\ell \leqslant m$ and $r \geqslant 1$. Then, there exists a positive integer $n_5 = n_5(q, \ell, m, r)$ such that for every integer $n \geqslant n_5$ and for every r-coloring of the ℓ-dimensional linear subspaces of $\mathbb{F}_q^n$, there exists an m-dimensional linear subspace of $\mathbb{F}_q^n$ such that all its ℓ-dimensional linear subspaces are colored the same.*

As in the affine case, an m-dimensional linear space $S = \langle \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle$ is given in SNF.

Proof. (of Theorem 5) Since we may replace m by $\max\{m, k+1\}$, we may assume that $m \geqslant k+1$. Fix $n_4 = n_4(q,k,m) = n_5(q,2,m,2^{q+1})$ according to Theorem 6. Let $n \geqslant n_4$ and let $G = (V,E)$ be a K_k-free graph with vertex-set V, where V is the set of all 1-dimensional linear subspaces of $\mathbb{F}_q^n$.

Color each 2-dimensional linear subspace P of $\mathbb{F}_q^n$ according to the pattern of the edges occurring in P, *i.e.*, color the 2-dimensional linear subspace given by $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ in SNF by the set of all unordered pairs $\{(i_1,i_2),(j_1,j_2)\}$ such that the two 1-dimensional linear subspaces given by $\langle i_1 \cdot \mathbf{a}_1 + i_2 \cdot \mathbf{a}_2 \rangle$ and $\langle j_1 \cdot \mathbf{a}_1 + j_2 \cdot \mathbf{a}_2 \rangle$ are in SNF and form an edge in the graph G. Note that all 1-dimensional linear subspaces of a 2-dimensional linear subspace $P = \langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ in SNF are given by $\langle \mathbf{a}_1 + i \cdot \mathbf{a}_2 \rangle$ for some $i \in \mathbb{F}_q$ and by $\langle \mathbf{a}_2 \rangle$, so there are exactly $q+1$ such spaces. Therefore the color of P contains only the unordered pairs of the form $\{(1,i),(0,1)\}$, and $\{(1,i),(1,j)\}$ for some $i,j \in \mathbb{F}_q$, $i \neq j$. For this coloring at most 2^{q+1} colors are used.

By Theorem 6, there exists an m-dimensional linear subspace S such that all 2-dimensional linear subspaces of S are colored the same, hence they all have the same pattern with respect to the occurring edges. Let the linear subspace S be given by $\langle \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle$ in SNF.

We claim that the set of all linear lines in S is independent in G. Suppose, for a contradiction, that the color of some 2-dimensional linear subspace in S is not the empty set, hence contains some unordered pair $p = \{(p_1,p_2),(p_3,p_4)\}$. Then for the 2-dimensional linear space given by $\langle \mathbf{a}_1, \mathbf{a}_2 \rangle$ in SNF,

$$\{p_1 \cdot \mathbf{a}_1 + p_2 \cdot \mathbf{a}_2, p_3 \cdot \mathbf{a}_1 + p_4 \cdot \mathbf{a}_2\} \in E. \tag{4}$$

All 2-dimensional linear subspaces of S are colored the same. According to the type of this pair p we distinguish two cases.

Case (i): $p = \{(1,i),(0,1)\}$.

We claim that the 1-dimensional linear subspaces $\langle \mathbf{a}_j + i \cdot \sum_{\ell=j+1}^{m} \mathbf{a}_\ell \rangle$ of S, $j = 1, \ldots, k$, form a complete subgraph K_k in G. To see this, consider all 2-dimensional linear subspaces in S given by $\langle \mathbf{x}_{r,s}, \mathbf{y}_{r,s} \rangle$, $1 \leqslant r < s \leqslant k$, where

$$\mathbf{x}_{r,s} = \mathbf{a}_r + i \cdot \sum_{\ell=r+1}^{s-1} \mathbf{a}_\ell + (i - i^2) \cdot \sum_{\ell=s+1}^{m} \mathbf{a}_\ell \tag{5}$$

$$\mathbf{y}_{r,s} = \mathbf{a}_s + i \cdot \sum_{\ell=s+1}^{m} \mathbf{a}_\ell, \tag{6}$$

where $\sum_{\ell=r+1}^{s-1} \mathbf{a}_\ell = \mathbf{0}$ for $s = r+1$.

With $\langle \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle$ in SNF, the 2-dimensional subspaces $\langle \mathbf{x}_{r,s}, \mathbf{y}_{r,s} \rangle$, $1 \leqslant r < s \leqslant k$, are also in SNF. Namely, the first nonzero entry of $\mathbf{x}_{r,s}$ arises from the vector $\mathbf{a}_r$ and is equal to 1, and the first nonzero entry of $\mathbf{y}_{r,s}$ arises from the vector $\mathbf{a}_s$ and is equal to 1 and the corresponding coordinate of the vector $\mathbf{x}_{r,s}$ is equal to 0. By (4) with (5) and (6),

$$\begin{aligned}\mathbf{x}_{r,s} + i \cdot \mathbf{y}_{r,s} &= \mathbf{a}_r + i \cdot \sum_{\ell=r+1}^{m} \mathbf{a}_\ell, \\ 0 \cdot \mathbf{x}_{r,s} + 1 \cdot \mathbf{y}_{r,s} &= \mathbf{a}_s + i \cdot \sum_{\ell=s+1}^{m} \mathbf{a}_\ell,\end{aligned}$$

hence, for $1 \leqslant r < s \leqslant k$

$$\left\{ \mathbf{a}_r + i \cdot \sum_{\ell=r+1}^{m} \mathbf{a}_\ell,\ \mathbf{a}_s + i \cdot \sum_{\ell=s+1}^{m} \mathbf{a}_\ell \right\} \in E,$$

and we have found a complete subgraph K_k in G, which is a contradiction. Thus every 2-dimensional linear subspace of S is colored by the empty set. As any two 1-dimensional linear subspaces of S are contained in a 2-dimensional linear space, the set of all 1-dimensional linear subspaces of S is an independent set in G.

Case (ii): $p = \{(1, i), (1, j)\}$, where $i \neq j$.

We claim that the 1-dimensional linear subspaces $\langle \mathbf{a}_1 + j \cdot \sum_{g=2}^{r} \mathbf{a}_g + i \cdot \mathbf{a}_{r+1} \rangle$ of S, $r = 1, \ldots, k-1$, and $\langle \mathbf{a}_1 + j \cdot \sum_{g=2}^{m} \mathbf{a}_g \rangle$ form a complete subgraph K_k in G. To see this, consider first the 2-dimensional linear subspaces of S given by $\langle \mathbf{x}_{r,s}, \mathbf{y}_{r,s} \rangle$, $1 \leqslant r < s \leqslant k-1$, with

$$\mathbf{x}_{r,s} = \mathbf{a}_1 + j \cdot \sum_{g=2}^{r} \mathbf{a}_g - i \cdot j \cdot (j-i)^{-1} \cdot \sum_{h=r+2}^{s} \mathbf{a}_h - i^2 \cdot (j-i)^{-1} \cdot \mathbf{a}_{s+1}, \tag{7}$$

$$\mathbf{y}_{r,s} = \mathbf{a}_{r+1} + j \cdot (j-i)^{-1} \cdot \sum_{h=r+2}^{s} \mathbf{a}_h + i \cdot (j-i)^{-1} \cdot \mathbf{a}_{s+1}, \tag{8}$$

where $\sum_{\ell=r+1}^{s-1} \mathbf{a}_\ell = \mathbf{0}$ for $s = r+1$.

As before, with $\langle \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle$ in SNF each 2-dimensional linear subspace given by $\langle \mathbf{x}_{r,s}, \mathbf{y}_{r,s} \rangle$ is in SNF, too, $1 \leqslant r < s \leqslant k-1$.

By (4) with (7) and (8),

$$\begin{aligned}\mathbf{x}_{r,s} + i \cdot \mathbf{y}_{r,s} &= \mathbf{a}_1 + j \cdot \sum_{g=2}^{r} \mathbf{a}_g + i \cdot \mathbf{a}_{r+1}, \\ \mathbf{x}_{r,s} + j \cdot \mathbf{y}_{r,s} &= \mathbf{a}_1 + j \cdot \sum_{g=2}^{s} \mathbf{a}_g + i \cdot \mathbf{a}_{s+1},\end{aligned}$$

hence for $1 \leqslant i < j \leqslant k-1$ we infer

$$\left\{ \mathbf{a}_1 + j \cdot \sum_{g=2}^{r} \mathbf{a}_g + i \cdot \mathbf{a}_{r+1},\ \mathbf{a}_1 + j \cdot \sum_{g=2}^{s} \mathbf{a}_g + i \cdot \mathbf{a}_{s+1} \right\} \in E, \tag{9}$$

giving so far a copy of K_{k-1} in G. This we will extend to a copy of K_k in G.

Next consider the 2-dimensional linear subspaces of S given by $\langle \mathbf{x}_r, \mathbf{y}_r \rangle$, $r = 1, \ldots, k-1$, with

$$\mathbf{x}_r = \mathbf{a}_1 + j \cdot \sum_{g=2}^{r} \mathbf{a}_g - i \cdot j \cdot (j-i)^{-1} \cdot \sum_{h=r+2}^{m} \mathbf{a}_h, \tag{10}$$

$$\mathbf{y}_r = \mathbf{a}_{r+1} + j \cdot (j-i)^{-1} \cdot \sum_{h=r+2}^{m} \mathbf{a}_h. \tag{11}$$

Again, with $\langle \mathbf{a}_1, \ldots, \mathbf{a}_m \rangle$ in SNF each linear subspace given by $\langle \mathbf{x}_r, \mathbf{y}_r \rangle$ is in SNF, too, $r = 1, \ldots, k-1$.

By (4) with (10) and (11),

$$\begin{aligned} \mathbf{x}_r + i \cdot \mathbf{y}_r &= \mathbf{a}_1 + j \cdot \sum_{g=2}^{r} \mathbf{a}_g + i \cdot \mathbf{a}_{r+1}, \\ \mathbf{x}_r + j \cdot \mathbf{y}_r &= \mathbf{a}_1 + j \cdot \sum_{g=2}^{m} \mathbf{a}_g, \end{aligned}$$

hence for $r = 1, \ldots, k-1$ we infer

$$\left\{ \mathbf{a}_1 + j \cdot \sum_{g=2}^{r} \mathbf{a}_g + i \cdot \mathbf{a}_{r+1},\ \mathbf{a}_1 + j \cdot \sum_{g=2}^{m} \mathbf{a}_g \right\} \in E.$$

With (9) we obtain a copy of K_k in G, which is a contradiction. Thus each 2-dimensional linear subspace of S does not contain any edge of G. As two 1-dimensional linear spaces are contained in a 2-dimensional linear space, the set of all 1-dimensional linear subspaces in S is an independent set in G. □

With arguments similar to those used in the proof of Proposition 4, the next result follows.

Proposition 7. *Fix a prime power q and let ℓ, n be positive integers with $\ell \geqslant 2$ and $n \geqslant \ell + 2$. There exists a triangle-free graph $G = (V, E)$ with vertex-set V being the set of all ℓ-dimensional linear subspaces of $\mathbb{F}_q^n$, such that each $(\ell+1)$-dimensional linear subspace contains two ℓ-dimensional linear subspaces R and S of $\mathbb{F}_q^n$, which are joined by an edge, i.e., $\{R, S\} \in E$.*

Proof. Fix an arbitrary total ordering $<^*$ of the ℓ-dimensional linear subspaces of $\mathbb{F}_q^n$ with the property that for two ℓ-dimensional linear subspaces $R = \langle \mathbf{a}_1, \ldots, \mathbf{a}_\ell \rangle$ and $S = \langle \mathbf{b}_1, \ldots, \mathbf{b}_\ell \rangle$ in SNF of $\mathbb{F}_q^n$: if the position of the first occurrence of 1 in $\mathbf{a}_1$ is smaller than the position of the first occurrence of 1 in $\mathbf{b}_1$, then $R <^* S$.

Let the edge-set E of the graph $G = (V, E)$ be defined as follows. For two ℓ-dimensional linear subspaces $R = \langle \mathbf{a}_1, \ldots, \mathbf{a}_\ell \rangle$ and $S = \langle \mathbf{b}_1, \ldots, \mathbf{b}_\ell \rangle$ in SNF of $\mathbb{F}_q^n$ with $R <^* S$, let $\{R, S\} \in E$ if and only if $\mathbf{b}_i = \mathbf{a}_{i+1}$, $i = 1, \ldots, \ell - 1$.

It is easy to see that the graph G is triangle-free. Namely, for contradiction assume that the vertices R, S, U with $R <^* S <^* U$ form a triangle. Then we have $R = \langle \mathbf{a}_1, \ldots, \mathbf{a}_\ell \rangle$, and $S = \langle \mathbf{a}_2, \ldots, \mathbf{a}_\ell, \mathbf{a}_{\ell+1} \rangle$, and $U = \langle \mathbf{a}_3, \ldots, \mathbf{a}_\ell, \mathbf{a}_{\ell+1}, \mathbf{a}_{\ell+2} \rangle$ in SNF, as $\{R, S\}, \{S, U\} \in E$. By assumption, $\{R, U\} \in E$, which implies that the vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ are identical, hence with $\ell \geqslant 2$ this contradicts the fact that the ℓ-dimensional linear space S is in SNF.

Now let $W = \langle \mathbf{c}_1, \ldots, \mathbf{c}_{\ell+1} \rangle$ be an $(\ell + 1)$-dimensional linear subspace of $\mathbb{F}_q^n$ in SNF. Then, by construction, for the two ℓ-dimensional linear subspaces $R = \langle \mathbf{c}_1, \ldots, \mathbf{c}_\ell \rangle$ and $S = \langle \mathbf{c}_2, \ldots, \mathbf{c}_{\ell+1} \rangle$ in SNF we have $R <^* S$, and R and S are joined by an edge in G. $\square$

5 Partition Regular Systems of Equations

In this section we give an application of Theorem 5 to solutions of systems of linear equations.

For an integer-valued $k \times \ell$-matrix M, the system $M\mathbf{x} = \mathbf{0}$ is called *partition regular* if and only if for every coloring of $\mathbb{Z}^+$ with finitely many colors, there exists a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in (\mathbb{Z}^+)^\ell$ to $M\mathbf{x} = \mathbf{0}$ such that $\{x_1, \ldots, x_\ell\}$ is monochromatic. Define M to be partition regular if and only if the system $M\mathbf{x} = \mathbf{0}$ is partition regular. For example, the matrix $[1, 1, -1]$ is partition regular, which is just Schur's theorem [16], i.e., under any finite coloring of $\mathbb{Z}^+$, there exists x, y, z all with the same color satisfying $x + y = z$. (See, *e.g.*, [8] for more examples of partition regular matrices.)

The above definition of "partition regular" has been extended. A $k \times \ell$ matrix M with integer entries is *partition regular over a set* X if and only if for any finite coloring of X, there exists a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in X^\ell$ to $M\mathbf{x} = \mathbf{0}$ so that $\{x_1, \ldots, x_\ell\}$ is monochromatic. So a partition regular matrix defined above is partition regular over $\mathbb{Z}^+$. Rado [13] observed that partition regularity over $\mathbb{Z}^+$ is equivalent to partition regularity over $\mathbb{Z} \setminus \{0\}$ or to partition regularity over $\mathbb{Q} \setminus \{0\}$.

The similar definition can be made over a group: Let $\mathcal{G}$ be an (additive) Abelian group. An integer-valued $k \times \ell$-matrix M is *partition regular* over $\mathcal{G} \setminus \{0\}$ if and only if for every coloring of $\mathcal{G} \setminus \{0\}$ with a finite number of colors there exists a monochromatic solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in (\mathcal{G} \setminus \{0\})^\ell$ to the system $M\mathbf{x} = \mathbf{0}$.

In 1933, a characterization of the set of all partition regular systems was given by Rado [13] in terms of a property of the columns of the matrix. An integer-valued matrix M has the *columns property* over $\mathbb{Z}$ if and only if there exists a partition $I_0, I_1, \ldots, I_m$ of the set of column indices of M such that (i) the sum of all columns with indices from I_0 is the all zero-vector, and (ii) for each $j = 1, \ldots, m$, the sum of all column vectors with indices from I_j is a (rational) linear combination of the columns with indices from $I_0 \cup \cdots \cup I_{j-1}$ (equivalently, there is $c_j \in \mathbb{Z}$ so that c_j times the sum of the columns with indices from I_j is a linear combination with coefficients in $\mathbb{Z}$ of the columns with indices from $I_0 \cup \cdots \cup I_{j-1}$). Rado [13] proved that if M is a matrix with integer entries, then $M\mathbf{x} = \mathbf{0}$ is partition regular over $\mathbb{Z} \setminus \{0\}$ if and only if M has the columns property over $\mathbb{Z}$. Rado [14, Thm VII] generalized his theorem for partition regularity and columns condition over any subring of complex numbers. Bergelson, Deuber, and Hindman [1] showed that for any finite field $\mathbb{F}_q$ and any $k \times \ell$ matrix M over $\mathbb{F}_q$, M has the columns property over $\mathbb{F}_q$ if and only if for every $r \in \mathbb{Z}^+$, there exists n_0 so that for every $n \geqslant n_0$, if $\mathbb{F}_q^n \setminus \{\mathbf{0}\}$ is r-colored, then there exists a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T$ to $M\mathbf{x} = \mathbf{0}$ so that $\{x_1, \ldots, x_\ell\}$ is monochromatic.

Another characterization of partition regular systems over $\mathbb{Z}\setminus\{0\}$ was given by Deuber [2] in terms of so-called (m, p, c)-sets; in the following definition, the notation $[-p, p] = \{z \in \mathbb{Z} : -p \leqslant z \leqslant p\}$ is used.

Definition 8. Let m, p, c be nonnegative integers. A set $S \subset \mathbb{Z}^+$ is called an (m, p, c)-set if and only if there exist $m + 1$ positive integers $a_0, \ldots, a_m$ such that

$$\begin{aligned} S = \{ca_0 + \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m : &\quad \lambda_1, \lambda_2, \ldots, \lambda_m \in [-p, p]\} \\ \cup\{ca_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m : &\quad \lambda_2, \ldots, \lambda_m \in [-p, p]\} \\ \vdots \\ \cup\{ca_{m-1} + \lambda_m a_m : &\quad \lambda_m \in [-p, p]\} \\ \cup\{ca_m\}. \end{aligned}$$

Today, (m, p, c)-sets are sometimes called *Deuber sets.*

The next result of Deuber [2] shows that solutions of partition regular systems can be found in (m, p, c)-sets.

Theorem 9. *Let M be an integer-valued $k \times \ell$-matrix so that $M\mathbf{x} = \mathbf{0}$ is partition regular over $\mathbb{Z} \setminus \{0\}$. Then there exists a triple (m, p, c) of positive integers such that every (m, p, c)-set in $\mathbb{Z}^+$ contains a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in (\mathbb{Z}^+)^\ell$ to the system $M\mathbf{x} = \mathbf{0}$.*

The following result is by Gunderson, Leader, Prömel and Rödl [10]:

Theorem 10. *Let k, m, p, c be positive integers. Then there exists a triple (n, q, d) of positive integers such that for every (n, q, d)-set V and every K_k-free graph $G = (V, E)$, there exists an (m, p, c)-set in V which is an independent set in G.*

As a consequence of Theorem 10, if G is a K_k-free graph on the set of positive integers, any finite system $M\mathbf{x} = \mathbf{0}$ of partition regular equations can be satisfied in an independent set, and hence Theorem 10 generalizes most Ramsey-type theorems for finite arithmetic structures. Deuber, Gunderson, Hindman, and Strauss [4] gave an example of a K_3-free graph on $\mathbb{Z}^+$ that does not contain an independent Hindman set (an infinite sequence of positive integers, together with all finite sums from the sequence), so in general Theorem 10 cannot be extended to include infinite families of partition regular equations. However, there are infinite systems of partition regular equations, where a similar statement like in Theorem 10 can be shown [11].

If a system $M\mathbf{x} = \mathbf{0}$ is not partition regular over $\mathbb{Z}^+$, then for some positive integer r there is an r-coloring Δ of $\mathbb{Z}^+$ that prevents monochromatic solutions. Let $S_1, \ldots, S_r$ be the color classes. Then, consider the complete r-partite graph G with vertex-set $\mathbb{Z}^+$ and classes $S_1, \ldots, S_r$. By choice of the coloring Δ, each solution to $M\mathbf{x} = \mathbf{0}$ must intersect at least two classes, hence G is K_{r+1}-free, but any solution $\mathbf{x}$ contains an edge of G.

A q-analog of (m, p, c)-sets, which are called m-sets in [3], is now defined in the straightforward way:

Definition 11. A subset S of points in a vector space V over $\mathbb{F}_q$ is called an $[m,q,1]$-set if and only if there exist $(m+1)$ vectors $\mathbf{a}_0,\ldots,\mathbf{a}_m \in V$, which are linearly independent over $\mathbb{F}_q$, such that

$$\begin{aligned} S = \{\mathbf{a}_0 + \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_m\mathbf{a}_m &: \lambda_1,\lambda_2,\ldots,\lambda_m \in \mathbb{F}_q\} \\ \cup\{\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_m\mathbf{a}_m &: \lambda_2,\ldots,\lambda_m \in \mathbb{F}_q\} \\ &\vdots \\ \cup\{\mathbf{a}_{m-1} + \lambda_m\mathbf{a}_m &: \lambda_m \in \mathbb{F}_q\} \\ &\cup\{\mathbf{a}_m\}. \end{aligned}$$

Using Theorem 5 we obtain the following:

Theorem 12. *Let $\mathbb{F}_q$ be a finite field with q elements. For each $k,m \in \mathbb{Z}^+$ there exists $n \in \mathbb{Z}^+$ such that every K_k-free graph G with vertex-set being the set of all points of an $[n,q,1]$-set in a linear vector space over $\mathbb{F}_q$ contains an $[m,q,1]$-set, which is an independent set in G.*

Proof. Let $N = n_4(q,k,m+1)$ and let $U = \langle \mathbf{y}_0,\ldots,\mathbf{y}_{N-1}\rangle$ be an N-dimensional linear vector space over $\mathbb{F}_q$ in SNF. Let $G=(V,E)$ be a K_k-free graph with vertex-set being the set of all $(q^N-1)/(q-1)$ points in U of the form $\sum_{i=0}^{N-1}\mu_i\mathbf{y}_i$, $\mu_i \in \mathbb{F}_q$, $i=0,1,\ldots,N-1$, where the first nonzero entry in the sequence $(\mu_0,\mu_1,\ldots,\mu_{N-1})$ is equal to 1. Hence V is an $[N-1,q,1]$-set.

Construct a new graph $G'=(V',E')$ with vertex-set being the set of all 1-dimensional linear subspaces of U; hence $|V'| = (q^N-1)/(q-1)$, where edges are defined as follows: for 1-dimensional subspaces $\langle\mathbf{x}\rangle$ and $\langle\mathbf{y}\rangle$ of U in SNF, let $\{\langle\mathbf{x}\rangle,\langle\mathbf{y}\rangle\} \in E'$ if and only if $\{\mathbf{x},\mathbf{y}\} \in E$.

We claim that G' is K_k-free. Otherwise, there exist pairwise distinct 1-dimensional linear subspaces $\langle\mathbf{x}_1\rangle,\ldots,\langle\mathbf{x}_k\rangle$ in SNF such that $\{\langle\mathbf{x}_i\rangle,\langle\mathbf{x}_j\rangle\} \in E'$, $1 \leqslant i < j \leqslant k$. But then $\{\mathbf{x}_i,\mathbf{x}_j\} \in E$, $1 \leqslant i < j \leqslant k$, which contradicts G being K_k-free.

By Theorem 5, in G' there exists an $(m+1)$-dimensional linear subspace $\langle\mathbf{a}_0,\ldots,\mathbf{a}_m\rangle$ in SNF, such that the set of all its 1-dimensional linear subspaces is an independent set in G'. Then the $[m,q,1]$-set

$$S = \bigcup_{j=0}^{m}\{\mathbf{a}_j + \sum_{i=j+1}^{m}\lambda_i\mathbf{a}_i : \quad \lambda_{j+1},\ldots,\lambda_m \in \mathbb{F}_q\}$$

is an independent set in G. □

For a prime p, an integer-valued matrix M has the *p-columns property,* if M has the columns property over $\mathbb{Z}$ where all calculations are done modulo p.

Let $\mathcal{G}$ be an (additive) Abelian group viewed as a module over $\mathbb{Z}$. The *order* of an element $g \in \mathcal{G}$ is the least positive integer n such that $n \cdot g = g + \cdots + g = 0$. If no such integer n exists, then the order of the element g is ∞.

Deuber [3] characterized all partition regular matrices with respect to Abelian groups:

Theorem 13. *Let $\mathcal{G}$ be an (additive) Abelian group and let M be an integer-valued matrix. Then M is partition regular if and only if at least one of the following conditions hold:*

- *(i) There exists $x \in \mathcal{G} \setminus \{0\}$ such that $M(x, x, \ldots, x)^T = \mathbf{0}$.*
- *(ii) M has the columns property over $\mathbb{Z}$ and $\mathcal{G}$ contains elements of arbitrary large finite or infinite order.*
- *(iii) For some prime p the matrix M has the p-columns property and $\mathcal{G}$ contains the countably infinite direct sum $\mathbb{Z}_p^{<\omega}$ of $\mathbb{Z}_p$ (so only finitely many entries are non-zero), where $\mathbb{Z}_p$ is the cyclic group of order p.*

Similarly as in Theorem 9, $[m, p, 1]$-sets are universal for the solutions of integer-valued matrices with the p-columns property, as the following result of Deuber shows, see Lemma 15 in [3]:

Theorem 14. *Let p be prime. Let M be an integer-valued $k \times \ell$-matrix, which has the p-columns property. Then there exists a positive integer m such that every $[m, p, 1]$-set $S \subset \mathbb{Z}_p^{<\omega}$ contains a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in S^\ell$ to the system $M\mathbf{x} = \mathbf{0}$ (mod p).*

We now have all tools together to prove the following consequence.

Theorem 15. *Let $\mathcal{G}$ be an (additive) Abelian group and let M be an integer-valued $k \times \ell$-matrix, which is partition regular over $\mathcal{G} \setminus \{0\}$. Let $G = (V, E)$ be a K_k-free graph with vertex-set $\mathcal{G} \setminus \{0\}$. Then there exists a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in (\mathcal{G} \setminus \{0\})^\ell$ to $M\mathbf{x} = \mathbf{0}$, such that $\{x_1, \ldots, x_\ell\}$ is an independent set in the graph G.*

Proof. Since the matrix M is partition regular over $\mathcal{G} \setminus \{0\}$, at least one of the conditions (i), (ii), or (iii) from Theorem 13 hold.

If (i) holds, there is nothing to show.

Assume that (ii) holds, i.e., the matrix M has the columns property over $\mathbb{Z}$. Moreover, the Abelian group $\mathcal{G}$ contains elements of arbitrary large finite or infinite order. By Theorem 9 there exist positive integers m, p, c such that every (m, p, c)-set in $\mathbb{Z}^+$ contains a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T$ to the system $M\mathbf{x} = \mathbf{0}$.

By Theorem 10 fix positive integers n, q, d such that for every (n, q, d)-set $V \subset \mathbb{Z}^+$ and every K_k-free graph $G = (V, E)$ there exists an (m, p, c)-set $S \subseteq V$, which is an independent set in G. Fix any (n, q, d)-set T in $\mathbb{Z}^+$. Let $M = \max T$. Let g be an element in $\mathcal{G}$ with $\mathrm{ord}(g) > M$. Consider the subset $T \cdot g = \{t \cdot g \mid t \in T\}$ which is of size $|T|$, since $\mathrm{ord}(g) > M$, of the vertex-set $\mathcal{G}$. By Theorem 10 there exists an (m, p, c)-set $S \subseteq T$, such that the set $S \cdot g$ is an independent set in the graph G. Then there exist $x_1, \ldots, x_\ell \in S$, such that $\mathbf{x} = (x_1 \cdot g, \ldots, x_\ell \cdot g)^T$ is a solution to the system $M\mathbf{x} = \mathbf{0}$, and $\{x_1, \ldots, x_\ell\}$ is an independent set in $\mathcal{G}$.

If (iii) holds, then for some prime p the matrix M has the p-columns property. Moreover, the Abelian group $\mathcal{G}$ contains $\mathbb{Z}_p^{<\omega}$. By Theorem 14 there exists a positive integer

m such that every $[m, p, 1]$-set $S \subset \mathbb{Z}_p^{<\omega}$ contains a solution $\mathbf{x} = (x_1, \ldots, x_\ell)^T \in S^\ell$ to the system $M\mathbf{x} = \mathbf{0}$.

By Theorem 12 there exists $n \in \mathbb{Z}^+$ such that every K_k-free graph G with vertex-set being the set of all points of an $[n, p, 1]$-set in a vector space over $\mathbb{F}_p$ contains an $[m, p, 1]$-set which is an independent set in G. Pick any $[n, p, 1]$-set $T \subset \mathbb{Z}_p^{<\omega}$ and pick such an $[m, p, 1]$-set contained in T. This $[m, p, 1]$-set contains a solution to the system $M\mathbf{x} = \mathbf{0}$. □

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