On the Maximum Number of k-Hooks of Partitions of n

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Abstract

Let $\alpha_k(\lambda)$ denote the number of k-hooks in a partition λ and let $b(n,k)$ be the maximum value of $\alpha_k(\lambda)$ among partitions of n. Amdeberhan posed a conjecture on the generating function of $b(n,1)$. We give a proof of this conjecture. In general, we obtain a formula that can be used to determine $b(n,k)$. This leads to a generating function formula for $b(n,k)$. We introduce the notion of nearly k-triangular partitions. We show that for any n, there is a nearly k-triangular partition which can be transformed into a partition of n that attains the maximum number of k-hooks. The operations for the transformation enable us to compute the number $b(n,k)$.

Keywords: partition; hook length; nearly k-triangular partition

1 Introduction

The objective of this paper is to derive a generating function formula for the maximum number of k-hooks in the Young diagrams of partitions of n. For $k = 1$, the problem was posed by Amdeberhan [1]. Let $\alpha_1(\lambda)$ be the number of 1-hooks in the partition λ, or equivalently, the number of distinct parts in λ. Let

$$b_n = \max\{\alpha_1(\lambda)\colon \lambda \in P(n)\},$$

where $P(n)$ denotes the set of partitions of n.

Amdeberhan [1] posed the following conjecture.

Conjecture 1.1. *We have*

$$\sum_{n\geqslant 0} b_n q^n = \frac{1}{1-q}\left(\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} - 1\right), \tag{1.1}$$

where $(q,q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots$.

Following the notation $\alpha_1(\lambda)$ of Amdeberhan, we use $\alpha_k(\lambda)$ to denote the number of k-hooks in λ, and let

$$b(n,k) = \max\{\alpha_k(\lambda)\colon \lambda \in P(n)\}.$$

The main result of this paper is the following generating function formula for $b(n,k)$.

Theorem 1.2. *For $k \geqslant 1$, we have*

$$\sum_{n\geqslant 0} b(n,k) q^n = \frac{1}{1-q}\left(\sum_{t\geqslant 1} q^{\binom{t}{2}k^2}\frac{1-q^{tk^2}}{1-q^{tk}} - 1\right). \tag{1.2}$$

Clearly, Theorem 1.2 reduces to Theorem 1.1 when $k=1$. Pak [6] gave a generating function formula for the statistic $\alpha_1(\lambda)$, where he used $\gamma(\lambda)$ to denote $\alpha_1(\lambda)$:

$$\sum_{n\geqslant 0}\sum_{\lambda\in P(n)} \gamma(\lambda) q^n = \frac{q}{(1-q)(q;q)_\infty}. \tag{1.3}$$

In general, the statistic $\alpha_k(\lambda)$ has been studied by Han [5, Eq. 1.5]. More precisely, he showed that

$$\sum_{n\geqslant 0}\sum_{\lambda\in P(n)} x^{\alpha_k(\lambda)} q^{|\lambda|} = \prod_{j\geqslant 1}\frac{(1+(x-1)q^{kj})^k}{1-q^j}. \tag{1.4}$$

Taking logarithms of both sides of (1.4) and differentiating with respect to x, we obtain the following relation by setting $x=1$:

$$\sum_{n\geqslant 0}\sum_{\lambda\in P(n)} \alpha_k(\lambda) q^n = \frac{kq^k}{(1-q^k)(q;q)_\infty}. \tag{1.5}$$

It can be seen that relation (1.5) becomes (1.3) when $k=1$.

Let us recall some basic notation and terminology on partitions as used in [2]. A partition λ of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_r = n$. The entries λ_i are called parts of λ. The number of parts of λ is called the length of λ, denoted by $l(\lambda)$. The weight of λ is the sum of parts, denoted $|\lambda|$.

A partition can be represented by a Young diagram. For each cell u in the Young diagram of λ, we define the hook length $h_u(\lambda)$ of u by the number of cells v in the Young

Figure 1.1: A 3-hook and the three cells of hook length 3.

diagram of λ such that $v = u$, or v appears below u in the same column, or v lies to the right of u in the same row. A hook of length k is called a k-hook, see Figure 1.1.

To prove Theorem 1.2, we introduce a class of partitions, called nearly k-triangular partitions.

Definition 1.3. *For fixed $m \geqslant 0$ and $k \geqslant 1$, let $m = sk+r$, where $s \geqslant 0$ and $0 \leqslant r \leqslant k-1$. Let $T_m^{(k)}$ denote the nearly k-triangular partition with m parts as given by*

$$T_m^{(k)} = (\underbrace{(s+1)k, \ldots, (s+1)k}_{r}, \underbrace{sk, \ldots, sk}_{k}, \ldots, \underbrace{2k, \ldots, 2k}_{k}, \underbrace{k, \ldots, k}_{k}).$$

It can be checked that in each row of the Young diagram of $T_m^{(k)}$, there is exactly one cell of hook length k. We use the symbol $*$ to mark cells in $T_m^{(k)}$ of hook length k. Figure 1.2 gives a nearly 3-triangular partition with eight parts.

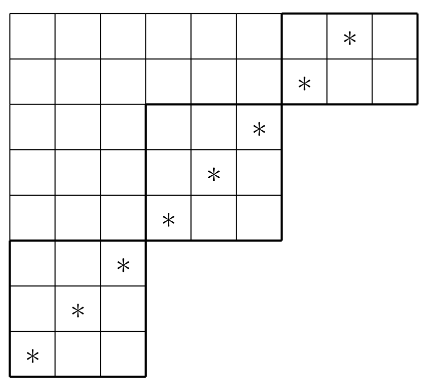

Figure 1.2: A nearly 3-triangular partition $T_8^{(3)}$.

This paper is organized as follows. In Section 2, we give the range of n such that $b_n = m$, which can be used to determine b_n of Conjecture 1.1. We then derive the generating function of b_n. In Section 3, we define two operations D_i and P_i on Young diagrams. Using these operations one can transform a partition λ with m k-hooks into a nearly k-triangular partition $T_s^{(k)}$, where $s \geqslant m$. This leads to a proof of Theorem 3.1. In Section 4, we define an operation Q_j on Young diagrams. We obtain a formula for $b(n,k)$ as well as a formula for the generating function.

2 Proof of Conjecture 1.1

In this section, we give a proof of Conjecture 1.1.

Theorem 2.1. *Assume that m is a nonnegative integer and n is an integer such that $\binom{m+1}{2} \leqslant n \leqslant \binom{m+2}{2} - 1$. Then we have $b_n = m$.*

Proof. Recall that $\alpha_1(\lambda)$ is the number of distinct parts of λ and b_n is the maximum value of $\alpha_1(\lambda)$ when λ ranges over partitions of n. We claim that $b_n < m+1$ if $1 \leqslant n < \binom{m+2}{2}$. Assume that λ is a partition with $m+1$ distinct parts. It is clear that

$$|\lambda| \geqslant 1 + 2 + \cdots + (m+1) = \binom{m+2}{2}.$$

In other words, if $1 \leqslant n < \binom{m+2}{2}$ then we have $b_n < m+1$. So the claim is verified.

Next we show that $b_n \geqslant m$ if $n \geqslant \binom{m+1}{2}$. Let

$$\lambda = (m, m-1, \cdots, 2, 1^{n-\binom{m+1}{2}+1}).$$

Clearly, λ has m distinct parts and $|\lambda| = n$. Thus $b_n \geqslant m$ if $n \geqslant \binom{m+1}{2}$. So we reach the conclusion that $b_n = m$ for $\binom{m+1}{2} \leqslant n \leqslant \binom{m+2}{2} - 1$. This completes the proof. $\square$

We are ready to prove Conjecture 1.1 with the aid of Theorem 2.1.

Proof of Conjecture 1.1. First, we may express the generating function of b_n in terms of the generating function of $b_{n+1} - b_n$. More precisely,

$$(1-q)\sum_{n\geqslant 0} b_n q^n = \sum_{n\geqslant 0}(b_{n+1} - b_n)q^{n+1}. \tag{2.1}$$

To compute $b_{n+1} - b_n$, we denote the interval $[\binom{m+1}{2}, \binom{m+2}{2} - 1]$ by I_m. By Theorem 2.1, we see that b_n is determined by the interval which n lies in. There are two cases:

Case 1: n and $n+1$ lie in the same interval I_m. Then we have $b_{n+1} = b_n = m$. It follows $b_{n+1} - b_n = 0$.

Case 2: n and $n+1$ lie in two consecutive intervals I_m and I_{m+1}. So we have $n = \binom{m+2}{2} - 1$. It follows that $b_n = m$ and $b_{n+1} = m+1$. Hence $b_{n+1} - b_n = 1$.

Combining the above two cases, we find that

$$\sum_{n\geqslant 0}(b_{n+1} - b_n)q^{n+1} = \sum_{m\geqslant 0} q^{\binom{m+2}{2}} = \sum_{m\geqslant 0} q^{\binom{m+1}{2}} - 1.$$

By Gauss' identity [3, Eq. 1.4.10]

$$\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.2}$$

we obtain that

$$(1-q)\sum_{n\geqslant 0} b_n q^n = \sum_{m\geqslant 0} q^{\binom{m+1}{2}} - 1 = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} - 1. \tag{2.3}$$

Thus, identity (1.1) can be deduced from (2.3) by dividing both sides by $(1-q)$. This completes the proof. □

3 Nearly k-triangular partitions

In this section, we introduce the structure of nearly k-triangular partitions, and we show that such a partition has minimum weight given the number of k-hooks. This property will be used in the next section to determine $b(n,k)$.

For $m \geqslant 0$ and $k \geqslant 1$, the weight of the nearly k-triangular partition $T_m^{(k)}$ is given by

$$\Delta(m,k) = m\left(\left\lfloor \frac{m}{k} \right\rfloor + 1\right)k - \binom{\lfloor \frac{m}{k} \rfloor + 1}{2}k^2. \tag{3.1}$$

It can be seen that in each row of $T_m^{(k)}$ there is exactly one cell of hook length k. Hence $T_m^{(k)}$ is a partition with m k-hooks. The following theorem states that $T_m^{(k)}$ has minimum weight among partitions with m k-hooks.

Theorem 3.1. *For $m \geqslant 0$ and $k \geqslant 1$, if λ is a partition with m k-hooks, then we have $|\lambda| \geqslant \Delta(m,k)$.*

To prove Theorem 3.1, we introduce two operations D_i and P_i defined on Young diagrams. They can also be considered as operations on partitions. We shall show that one can transform a partition λ with m k-hooks into a nearly k-triangular partition $T_m^{(k)}$ by applying the operations D_i and P_i.

The operations D_i and P_i are defined as follows. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition. The operation D_i means to remove the i-th row of the Young diagram of λ. The operation P_i applies to partitions λ for which $\lambda_i > \lambda_{i+1}$. More precisely, $P_i(\lambda)$ is obtained from λ via the following steps. Assume that the cells of hook length k are marked by $*$.

Step 1. Remove the last cell u from the i-th row of the Young diagram of λ, and denote the resulting partition by μ. If the Young diagram of λ contains no marked cell in the column occupied by u, then we set $P_i(\lambda) = \mu$;

Step 2. In this step, there is a cell of hook length k in the column of λ that contains u. Denote this marked cell by w_1 and assume that it is in the j-th row of λ. Evidently, the marked cell w_1 in λ has hook length $k-1$ in μ. There are two cases:

Case 1: $\mu_j = \mu_{j-1}$. Notice that the cell w_1' above w_1 is of hook length k in μ. In other words, w_1' is a marked cell in μ. In this case, we set $P_i(\lambda) = \mu$; see Figure 3.3.

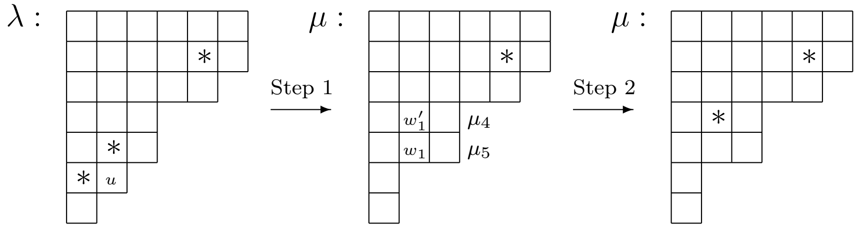

Figure 3.3: The case $\mu_j = \mu_{j-1}$.

Case 2: $\mu_j < \mu_{j-1}$. We add a cell v at the end of the j-th row in μ and denote the resulting partition by ν. Clearly, w_1 is also a marked cell in ν. If the Young diagram of μ contains no marked cell in the $(\mu_j + 1)$-th column, then we set $P_i(\lambda) = \nu$; see Figure 3.4. Otherwise, go to the next step.

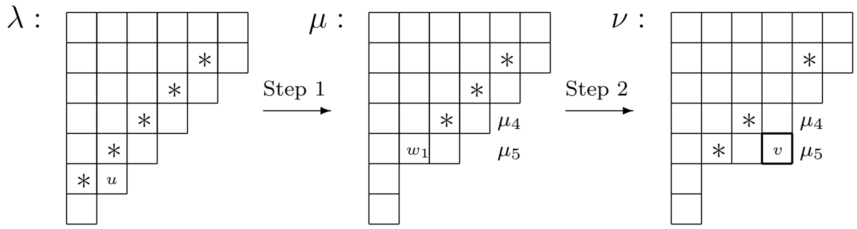

Figure 3.4: The case $\mu_j < \mu_{j-1}$.

Step 3. There is a marked cell in the $(\mu_j + 1)$-th column of μ. Let w_2 denote this marked cell and suppose that it is in the h-th row. Evidently, the cell w_2 has hook length $k + 1$ in ν. There are two cases:

Case 1: $\nu_k = \nu_{k+1}$. Notice that the cell w_2' below w_2 is of hook length k in ν. In this case, we set $P_i(\lambda) = \nu$;

Case 2: $\nu_k > \nu_{k+1}$. We remove the last cell u' in the h-th row of ν and denote the resulting partition by μ'. Now, w_2 is also a marked cell in μ'. If the Young diagram of ν contains no marked cell in the column occupied by u', we set $P_i(\lambda) = \mu'$. Otherwise, we set $u = u'$, $\lambda = \nu$, $\mu = \mu'$ and go back to Step 2.

Figure 3.5 gives an illustration of the operation P_i, where the cells with the symbol $-$ are the removed cells and the cells with the symbol $+$ are the added cells.

The following property of the operation P_i is easy to verify, and hence the proof is omitted. Recall that for given k and for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, there is at most one marked cell in each row of the Young diagram of λ. We use $\alpha_k(\lambda, i)$ to denote the number of marked cells in the i-th row of the Young diagram of λ, which takes the value 0 or 1.

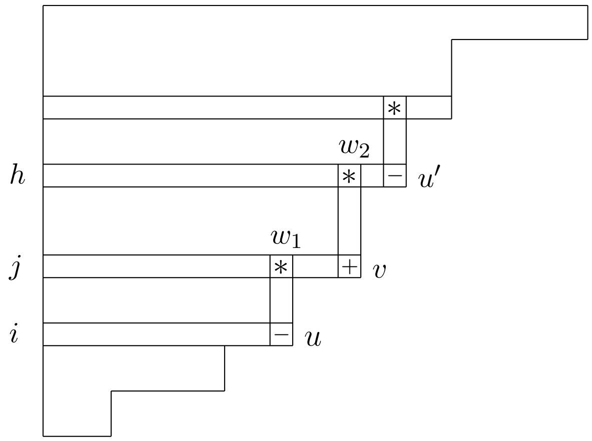

Figure 3.5: The operation P_i.

Lemma 3.2. *Assume that $k \geqslant 1$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition such that $\lambda_i > \lambda_{i+1}$. Then we have $|\lambda| \geqslant |P_i(\lambda)|$ and*

$$\alpha_k(\lambda) - \alpha_k(\lambda, i) \leqslant \alpha_k(P_i(\lambda)) - \alpha_k(P_i(\lambda), i).$$

We are now in a position to present a proof of Theorem 3.1 with the aid of the operations P_i and D_i.

Proof of Theorem 3.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition having m k-hooks. We shall give a procedure to transform λ into $T_s^{(k)}$, where $s \geqslant m$, by using the operations D_i and P_i. Notice that D_i decreases the weight of a partition and P_i either preserves the weight or decreases the weight by one. Moreover, it can be seen that D_i preserves the number of k-hooks and P_i does not decrease the number of k-hooks. Hence we arrive at the conclusion that $|\lambda| \geqslant \Delta(s, k) \geqslant \Delta(m, k)$.

Clearly, we have $r \geqslant m$. We aim to construct a sequence $\beta^{(r)}, \beta^{(r-1)}, \ldots, \beta^{(1)}, \beta^{(0)}$ of nearly k-triangular partitions starting with $T_r^{(k)}$ and ending with $T_s^{(k)}$, where $s \geqslant m$. In the construction of $T_s^{(k)}$ from λ, we denote the intermediate partitions by $\lambda^{(r)}, \lambda^{(r-1)}, \ldots, \lambda^{(1)}, \lambda^{(0)}$ with $\lambda^{(r)} = \lambda$ and $\lambda^{(0)} = T_s^{(k)}$. We compare the partitions $\beta^{(i)}$ and $\lambda^{(i)}$ to construct $\beta^{(i-1)}$ and $\lambda^{(i-1)}$ by the following process, where $\beta^{(i)}$ and $\lambda^{(i)}$ have the same number of parts and $\beta^{(i)}$ and $\lambda^{(i)}$ differ only in the first i rows.

First, we compare the last entry of $\lambda^{(r)}$ with the last entry of $\beta^{(r)}$. Recall that $\lambda^{(r)} = \lambda$ and $\beta^{(r)} = T_r^{(k)}$. There are two cases:

Case 1: The r-th entry of $\lambda^{(r)}$ is equal to the r-th entry of $\beta^{(r)}$. Then we set $\beta^{(r-1)} = T_r^{(k)}$ and $\lambda^{(r-1)} = \lambda^{(r)}$.

Case 2: The r-th entry of $\lambda^{(r)}$ is not equal to the r-th entry of $\beta^{(r)}$. There are two subcases:

Case 2.1: $\lambda_r^{(r)} < \beta_r^{(r)} = k$. It is apparent that there are no cells of hook length k in the r-th row of the Young diagram of $\lambda^{(r)}$. Applying the operation P_r to $\lambda^{(r)}$, the r-th entry of $\lambda^{(r)}$ decreases by one. In view of Lemma 3.2, we have $\alpha_k(P_r(\lambda^{(r)})) \geqslant \alpha_k(\lambda)$ and $|P_r(\lambda^{(r)})| \leqslant |\lambda|$. Applying the operation P_r $\lambda_r^{(r)}$ times to $\lambda^{(r)}$, we obtain a partition μ with $r-1$ parts. It is clear that $\alpha_k(\mu) \geqslant \alpha_k(\lambda^{(r)})$ and $|\mu| \leqslant |\lambda^{(r)}|$. We set $\beta^{(r-1)} = T_{r-1}^{(k)}$ and $\lambda^{(r-1)} = \mu$.

Case 2.2: $\lambda_r^{(r)} > \beta_r^{(r)} = k$. Let $d = \lambda_r^{(r)} - k$. Evidently, there is a cell of hook length k in the r-th row of $\lambda^{(r)}$. Applying the operation P_r d times to $\lambda^{(r)}$, we obtain a partition μ. It is easily seen that the r-th entry of μ equals the r-th entry of $\beta^{(r)}$. By Lemma 3.2, we find that $\alpha_k(\mu) \geqslant \alpha_k(\lambda^{(r)})$ and $|\mu| \leqslant |\lambda^{(r)}|$. We set $\beta^{(r-1)} = T_r^{(k)}$ and $\lambda^{(r-1)} = \mu$.

We now proceed to compare the i-th entry of $\lambda^{(i)}$ with the i-th entry of $\beta^{(i)}$ for $r-1 \geqslant i \geqslant 1$, where we assume that $\lambda^{(i)}$ and $\beta^{(i)}$ have been constructed by the above procedure. Assume that $\lambda^{(i)}$ has t parts and $\beta^{(i)} = T_t^{(k)}$. There are two cases:

Case 1: The i-th entry of $\lambda^{(i)}$ is equal to the i-th entry of $\beta^{(i)}$. Then we set $\beta^{(i-1)} = T_t^{(k)}$ and $\lambda^{(i-1)} = \lambda^{(i)}$.

Case 2: The i-th entry of $\lambda^{(i)}$ is not equal to the i-th entry of $\beta^{(i)}$. There are three subcases:

Case 2.1: $i \equiv t \pmod{k}$ and $\lambda_i^{(i)} - \lambda_{i+1}^{(i)} < k$. In this case, let $d = \lambda_i^{(i)} - \lambda_{i+1}^{(i)}$. Clearly, there are no cells of hook length k in the i-th row of the Young diagram of $\lambda^{(i)}$, see Figure 3.6. Applying the operation P_i d times to $\lambda^{(i)}$, we obtain a partition μ with $\mu_i = \mu_{i+1}$, which contains no marked cells in the i-th row. By Lemma 3.2, we see that $\alpha_k(\mu) \geqslant \alpha_k(\lambda^{(i)})$ and $|\mu| \leqslant |\lambda^{(i)}|$.

Next, we apply the operation D_i to μ to generate a partition ν with $t-1$ parts. Since there are no marked cells in the area A of μ, namely, $\{(p,q) : 1 \leqslant p \leqslant i-1, 1 \leqslant q \leqslant \mu_i\}$, see Figure 3.6. The positions of the marked cells of μ stay unchanged in ν with respect to the operation D_i. It follows that $\alpha_k(\nu) = \alpha_k(\mu)$ and $|\nu| < |\mu|$. This implies that $\alpha_k(\nu) \geqslant \alpha_k(\lambda^{(i)})$ and $|\nu| < |\lambda^{(i)}|$. Notice that the partitions ν and $T_{t-1}^{(k)}$ differ only in the first $i-1$ rows. We set $\beta^{(i-1)} = T_{t-1}^{(k)}$ and $\lambda^{(i-1)} = \nu$.

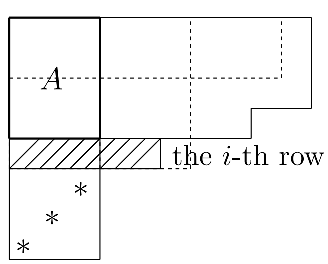

Figure 3.6: The case for $i \equiv t \pmod{k}$ and $\lambda_i^{(i)} - \lambda_{i+1}^{(i)} < k$.

Case 2.2: $i \equiv t \pmod{k}$ and $\lambda_i^{(i)} - \lambda_{i+1}^{(i)} > k$. Let $d = \lambda_i^{(i)} - \lambda_{i+1}^{(i)} - k$. Evidently, there is a cell of hook length k in the i-th row of $\lambda^{(i)}$, see Figure 3.7. Let μ be the partition obtained from $\lambda^{(i)}$ by applying the operation P_i d times. Note that there is also a marked cell in the i-th row of μ. By Lemma 3.2, we see that $\alpha_k(\mu) \geqslant \alpha_k(\lambda^{(i)})$ and $|\mu| \leqslant |\lambda^{(i)}|$. Now the partitions μ and $T_t^{(k)}$ differ only in the first $i-1$ rows. We set $\beta^{(i-1)} = T_t^{(k)}$ and $\lambda^{(i-1)} = \mu$.

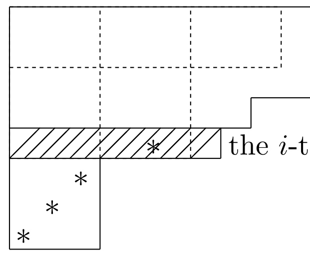

Figure 3.7: The case for $i \equiv t \pmod{k}$ and $\lambda_i^{(i)} - \lambda_{i+1}^{(i)} > k$.

Case 2.3: $i \not\equiv t \pmod{k}$. In this case, we have $\lambda_i^{(i)} - \lambda_{i+1}^{(i)} > 0$, see Figure 3.8. Let $d = \lambda_i^{(i)} - \lambda_{i+1}^{(i)}$. Applying the operation P_i d times to $\lambda^{(i)}$, we obtain a partition μ for which there is a marked cell in the i-th row. By Lemma 3.2, we deduce that $\alpha_k(\mu) \geqslant \alpha_k(\lambda^{(i)})$ and $|\mu| \leqslant |\lambda^{(i)}|$. Now, the partitions μ and $T_t^{(k)}$ differ only in the first $i-1$ rows. We set $\beta^{(i-1)} = T_t^{(k)}$ and $\lambda^{(i-1)} = \mu$.

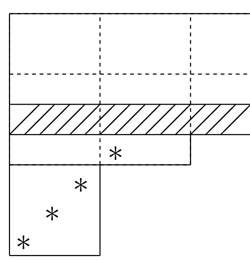

Figure 3.8: The case for $i \not\equiv t \pmod{k}$.

Repeating the above process, we eventually obtain a nearly k-triangular partition $\lambda^{(0)} = \beta^{(0)} = T_s^{(k)}$. From the construction of $\lambda^{(0)}$ from $\lambda^{(r)}$, we deduce that

$$m = \alpha_k(\lambda^{(r)}) \leqslant \cdots \leqslant \alpha_k(\lambda^{(i)}) \leqslant \alpha_k(\lambda^{(i-1)}) \leqslant \cdots \leqslant \alpha_k(\lambda^{(0)}) = s$$

and

$$|\lambda| = |\lambda^{(r)}| \geqslant \cdots \geqslant |\lambda^{(i)}| \geqslant |\lambda^{(i-1)}| \geqslant \cdots \geqslant |\lambda^{(0)}| = \Delta(s,k).$$

Since $s \geqslant m$, we have $|\lambda| \geqslant |\lambda^{(0)}| = \Delta(s,k) \geqslant \Delta(m,k)$. This completes the proof. $\square$

As a consequence of Theorem 3.1, we obtain the following upper bound on $b(n,k)$. Together with the lower bound given in the next section, we can determine the range of n for which $b(n,k) = m$.

Corollary 3.3. *Assume $m \geqslant 0$ and $k > 0$, If n is a nonnegative integer such that $n < \Delta(m+1,k)$, then $b(n,k) \leqslant m$.*

Figure 3.9 illustrates the transformation from $\lambda = (10,7,4,3,3,3,3)$ to a nearly 3-triangular partition $T_5^{(3)} = (6,6,3,3,3)$. It can be checked that both λ and $T_5^{(3)}$ have five 3-hooks and $|\lambda| > \Delta(5,3) = 21$.

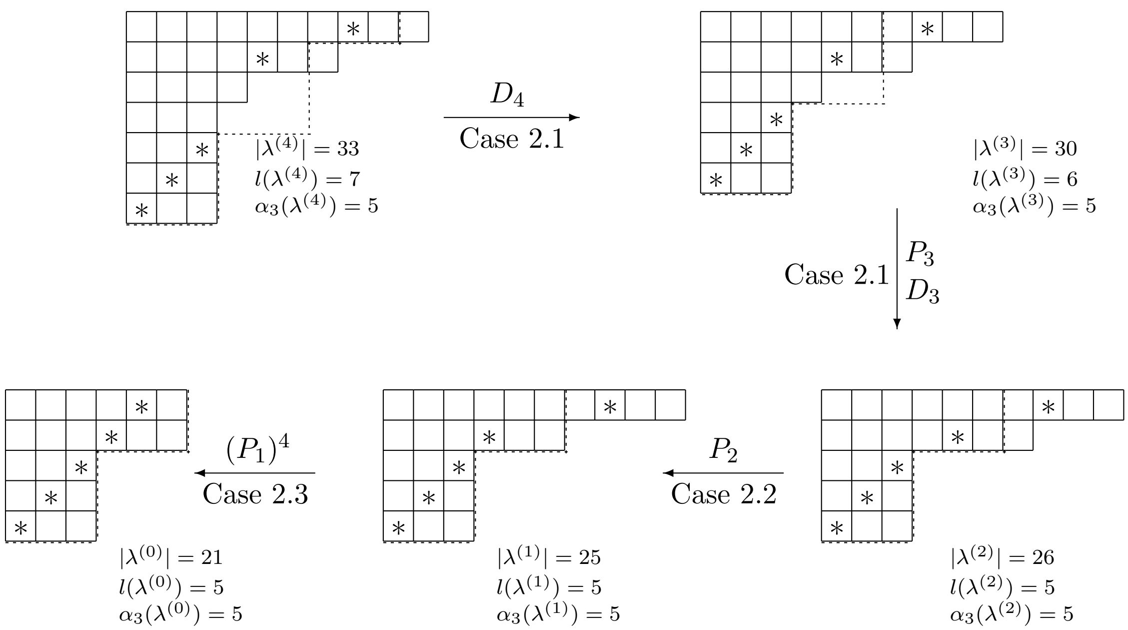

Figure 3.9: $\lambda = (10,7,4,3,3,3,3)$ and $T_5^{(3)} = (6,6,3,3,3)$.

4 Proof of Theorem 1.2

In this section, we show that the number $b(n,k)$ can be determined by the number $\Delta(m,k)$, namely, the weight of the nearly k-triangular partition $T_m^{(k)}$. In the previous section, we have obtained an upper bound on $b(n,k)$. To determine $b(n,k)$, we give a lower bound on $b(n,k)$.

Theorem 4.1. *Assume that $m \geqslant 0$ and $k > 0$. If n is an integer such that $n \geqslant \Delta(m,k)$, then we have $b(n,k) \geqslant m$.*

To prove Theorem 4.1, we introduce an operation Q_j defined on Young diagrams. In fact, Q_j differs from the operation P_j only in the first step. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition. The operation Q_j applies to partitions λ for which $\lambda_{j-1} > \lambda_j$. More precisely, $Q_j(\lambda)$ is constructed via the following steps.

Step 1. Add a cell v at the end of the j-th row of the Young diagram of λ, and denote the resulting partition by μ. If the Young diagram of λ contains no marked cells in the $(\lambda_j + 1)$-th column of λ, then we set $Q_j(\lambda) = \mu$;

Step 2. In this step, there is one cell of hook length k in the $(\lambda_j + 1)$-th column of λ. Denote this marked cell by w and assume that it is in the h-th row of λ. Note that the marked cell w in λ is of hook length $k + 1$ in μ. There are two cases:

Case 1: $\mu_h = \mu_{h+1}$. Now that the cell w' below w is of hook length k in μ, we set $Q_j(\lambda) = \mu$.

Case 2: $\mu_h > \mu_{h+1}$. We apply the operation P_h to μ and denote the resulting partition by ν. It is easily seen that the h-th entry of μ decreases by one. Consequently, w has hook length k in ν. We set $Q_j(\lambda) = \nu$.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we regard the $(r + 1)$-th entry as 0 when we apply Q_{r+1} to λ. Under this convention, Q_{r+1} increases the number of parts of λ by one.

The following property of the operation Q_j is similar to Lemma 3.2. The proof is omitted.

Lemma 4.2. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition such that $\lambda_j < \lambda_{j-1}$. Then we have*

$$\alpha_k(\lambda) - \alpha_k(\lambda, j) \leqslant \alpha_k(Q_j(\lambda)) - \alpha_k(Q_j(\lambda), j) \tag{4.1}$$

and

$$|\lambda| \leqslant |Q_j(\lambda)| \leqslant |\lambda| + 1. \tag{4.2}$$

We are now ready to prove Theorem 4.1 by using the operation Q_j.

Proof of Theorem 4.1. Assume that $n \geqslant \Delta(m, k)$. It suffices to show that there exists a partition λ of n with at least m k-hooks. We proceed by induction on n.

First, when $n = \Delta(m, k)$, the nearly k-triangular partition $T_m^{(k)}$ is a partition of $\Delta(m, k)$ with m k-hooks. So the theorem holds for $n = \Delta(m, k)$.

We now assume that there is a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of N with s k-hooks, where $N \geqslant \Delta(m, k)$ and $s \geqslant m$. The following procedure gives the construction of a partition of $N + 1$ with at least s k-hooks.

Apply Q_{r+1} to λ and denote the resulting partition by $\mu^{(1)}$. By Lemma 4.2, we see that $\alpha_k(\mu^{(1)}) \geqslant s$ and $|\lambda| \leqslant |\mu^{(1)}| \leqslant |\lambda| + 1$. There are two cases:

Case 1: $|\mu^{(1)}| = |\lambda| + 1 = N + 1$. Then $\mu^{(1)}$ is a partition of $N + 1$ with at least s k-hooks.

Case 2: $|\mu^{(1)}| = |\lambda| = N$. We continue to construct a sequence of partitions

$$\mu^{(2)} = Q_{r+2}(\mu^{(1)}), \ldots, \mu^{(i+1)} = Q_{r+i+1}(\mu^{(i)}), \ldots, \mu^{(k-1)} = Q_{r+k-1}(\mu^{(k-2)}).$$

It is clear that $\mu^{(i)}$ has $r+i$ parts and contains at least i parts equal to one. It follows from (4.1) that $\alpha_k(\mu^{(i)}) \geqslant s$ for $2 \leqslant i \leqslant k-1$. There are two subcases:

Case 2.1: There exists a partition $\mu^{(i)}$ such that $|\mu^{(i)}| = N+1$, where $2 \leqslant i \leqslant k-1$. Then $\mu^{(i)}$ is a partition of $N+1$ with at least s k-hooks.

Case 2.2: $|\mu^{(i)}| = N$ for $2 \leqslant i \leqslant k-1$. Now, we construct a partition $\mu^{(k)}$ from $\mu^{(k-1)}$. Recall that $\mu^{(k-1)}$ contains at least $k-1$ parts equal to one and it has at least s k-hooks. Set $\mu^{(k)}$ to be the partition obtained from $\mu^{(k-1)}$ by adding 1 as a new part. Now, we have $|\mu^{(k)}| = N+1$. Moreover, there are at least k parts equal one and there is a cell of hook length k in the first column of $\mu^{(k)}$. Meanwhile, the positions of the marked cells in other columns of $\mu^{(k-1)}$ stay unchanged in $\mu^{(k)}$. This implies that $\alpha_k(\mu^{(k)}) \geqslant \alpha_k(\mu^{(k-1)}) \geqslant s$. Thus $\mu^{(k)}$ is a partition of $N+1$ with at least s k-hooks. This completes the proof. $\square$

Combining Corollary 3.3 and Theorem 4.1, we obtain the following Theorem which can be used to compute $b(n,k)$.

Theorem 4.3. *Assume $m \geqslant 0$ and $k \geqslant 1$. If n is an integer such that $\Delta(m,k) \leqslant n \leqslant \Delta(m+1,k) - 1$, then we have $b(n,k) = m$.*

Using the construction in Theorem 4.1, for any integer n and fixed k, we can transform a suitable nearly k-triangular partition into a partition λ of n. Theorem 4.3 implies that the partition λ attains the maximum number of k-hooks among partitions of n. Figure 4.10 gives an illustration of the construction of a partition of 12 with three 3-hooks from the nearly 3-triangular partition with three parts.

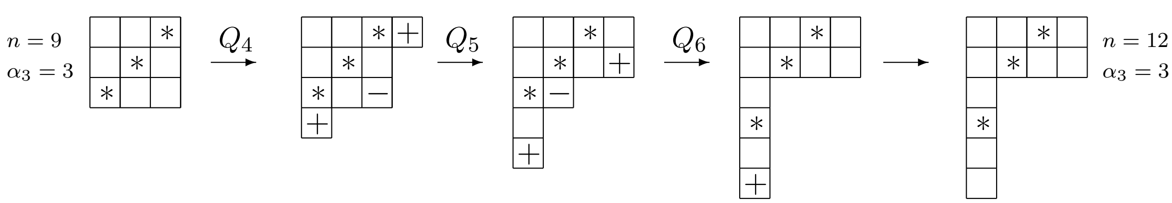

Figure 4.10: The construction of a partition of 12 with three 3-hooks.

Note that Theorem 4.3 reduces to Theorem 2.1 when $k = 1$. We conclude this paper with a proof of Theorem 1.2 on the generating function of $b(n,k)$.

Proof of Theorem 1.2. First, write (1.2) in the following form

$$(1-q)\sum_{n\geqslant 0} b(n,k)q^n = \sum_{t\geqslant 1} q^{\binom{t}{2}k^2}\frac{1-q^{tk^2}}{1-q^{tk}} - 1. \tag{4.3}$$

The left hand side of (4.3) can be rewritten as

$$(1-q)\sum_{n\geqslant 0} b(n,k)q^n = \sum_{n\geqslant 0}\left(b(n+1,k)-b(n,k)\right)q^{n+1}. \tag{4.4}$$

To compute $b(n+1,k)-b(n,k)$, we denote the interval $[\Delta(m,k),\Delta(m+1,k)-1]$ by $I_m^{(k)}$. By Theorem 4.3, we see that $b(n,k)$ is determined by the interval containing n. We consider two cases:

Case 1: n and $n+1$ belong to the same interval $I_m^{(k)}$. By Theorem 4.3, we have $b(n+1,k) = b(n,k) = m$, and so $b(n+1,k)-b(n,k)=0$.

Case 2: n and $n+1$ lie in two consecutive intervals $I_m^{(k)}$ and $I_{m+1}^{(k)}$. It follows that $n = \Delta(m+1,k)-1$. By Theorem 4.3, we obtain that $b(n,k)=m$ and $b(n+1,k)=m+1$. So we have $b(n+1,k)-b(n,k)=1$.

Combining the above two cases, we deduce that

$$\sum_{n\geqslant 0}(b(n+1,k)-b(n,k))q^{n+1} = \sum_{m\geqslant 0} q^{\Delta(m+1,k)} = \sum_{m\geqslant 0} q^{\Delta(m,k)} - 1.$$

Consequently,

$$(1-q)\sum_{n\geqslant 0} b(n,k)q^n = \sum_{m\geqslant 0} q^{\Delta(m,k)} - 1. \tag{4.5}$$

Recall that

$$\Delta(m,k) = m\left(\left\lfloor \frac{m}{k}\right\rfloor + 1\right)k - \binom{\lfloor \frac{m}{k}\rfloor + 1}{2}k^2.$$

Write $m = sk + r$, where $s \geqslant 0$ and $0 \leqslant r \leqslant k-1$. Then we have

$$\begin{aligned}\Delta(sk+r,k) &= (sk+r)\,(s+1)\,k - \binom{s+1}{2}k^2 \\ &= \binom{s+1}{2}k^2 + r(s+1)k.\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), we find that

$$\begin{aligned}\sum_{m\geqslant 0} q^{\Delta(m,k)} - 1 &= \sum_{s\geqslant 0}\sum_{r=0}^{k-1} q^{\binom{s+1}{2}k^2 + r(s+1)k} - 1 \\ &= \sum_{s\geqslant 0} q^{\binom{s+1}{2}k^2}\frac{1-q^{(s+1)k^2}}{1-q^{(s+1)k}} - 1.\end{aligned}$$

In view of (4.5), we obtain that

$$(1-q)\sum_{n\geqslant 0} b(n,k)q^n = \sum_{t\geqslant 1} q^{\binom{t}{2}k^2}\frac{1-q^{tk^2}}{1-q^{tk}} - 1.$$

This completes our proof. $\square$

Using the Jacobi triple product identity [4, Eq. 1.6.1], we may express the generating function $\sum_{n\geqslant 0} b(n,k)q^n$ in the following form:

$$\frac{1}{1-q}\left(\frac{(q^{2k^2};q^{2k^2})_\infty}{(q^{k^2};q^{2k^2})_\infty}+\frac{1}{2}\sum_{r=1}^{k-1}(-q^{rk};q^{k^2})_\infty(-q^{k^2-rk};q^{k^2})_\infty(q^{k^2};q^{k^2})_\infty-\frac{k+1}{2}\right).$$

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