Distinguishing Maps II: General Case

Thomas W. Tucker
Department of Mathematics
Colgate University
Hamilton, NY, U.S.A.

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Abstract

A group A acting faithfully on a set X has distinguishing number k, written $D(A, X) = k$, if there is a coloring of the elements of X with k colors such that no nonidentity element of A is color-preserving, and no such coloring with fewer than k colors exists. Given a map M with vertex set V and automorphism group $Aut(M)$, let $D(M) = D(Aut(M), V)$. If M is orientable, let $D^+(M) = D(Aut^+(M), V)$, where $Aut^+(M)$ is the group of orientation-preserving automorphisms. In a previous paper, the author showed there are four maps M with $D^+(M) > 2$. In this paper, a complete classification is given for the graphs underlying maps with $D(M) > 2$. There are 31 such graphs, 22 having no vertices of valence 1 or 2, and all have at most 10 vertices.

1 Introduction

A group A acting faithfully on a set X has *distinguishing number* k, written $D(A, X) = k$, if there is a coloring of the elements of X with k colors such that no nonidentity element of A is color-preserving, and no such coloring exists with fewer than k colors. The concept was introduced by Albertson and Collins [1] in the context of the automorphism group of a graph acting on the vertex set. It originates in the observation that to destroy any symmetry of a necklace of n beads, one needs beads of three different colors for $n = 3, 4, 5$, but only two colors for $n > 5$; this observation actually plays a role in some of our proofs.

In [6], we considered a variety of questions where X is the vertex set of a map M and A is either the full automorphism group $Aut(M)$ for the map, or the orientation-preserving automorphism group $Aut^+(M)$, with respective distinguishing numbers $D(M)$ and $D^+(M)$. In particular, we showed that there are only four maps with $D^+(M) > 2$. We also showed there are only finitely many maps with $D(M) > 2$ and considered a number of questions for distinguishing chromatic numbers, where vertex colorings are required to be proper.

In this paper, we consider the general case of non-orientable maps and orientation-reversing actions on oriented maps. We find the possible underlying graph G for any map M with $D(M) > 2$ and for each such graph, we give an example of a map M with underlying graph G and $D(M) > 2$. Unlike [6], our list of possible underlying graphs is rather long and complicated, although no graph has more than 10 vertices. This paper is self-contained in terminology, definitions, and methods, but the reader might want to see [6] for a broader discussion of distinguishability and maps, including a much more extensive list of references.

Much of this classification appears in an earlier unpublished preprint (2005). A recent paper by Negami [3] gives a partial classification and relates it to the earlier preprint. We discuss Negami's results at the end of the paper.

This paper is organized as follows. Section 2 gives definitions, terminology, and some background for maps and their automorphisms, including Petrie duality, reflexible and chiral regular maps, and Cayley maps. In Section 3, we give examples of vertex-transitive and intransitive maps with $D(M) > 2$. In Section 4 we give a statement of the classification theorem for maps M with $D(M) > 2$. In Section 4, we prove the classification theorem for the intransitive case. In Section 5, we prove the theorem in the case of regular (reflexible) maps. In Section 6, we complete the proof of the classification theorem for transitive maps using overlays of regular reflexible and chiral maps.

We note that the concept of distinguishability has forced us to develop a variety of new techniques: partial Petrie duality, τ-edges, induced embeddings for subgraphs, overlays, and angle measure. These in turn have led to a much deeper understanding of the symmetries of a map. In particular, angle measure yields an astonishingly simple and short proof that there are no reflexible regular maps with underlying graph $K_n, n \neq 2, 3, 4, 6$, a well-known result obtained at some length by algebraic methods. Angle measure might hint at a geometric explanation of our results depending on the euclidean or hyperbolic structure carried by maps. Distinguishability has also led us to a variety of small maps, some familiar and some not, many with remarkable symmetry properties.

2 Maps, automorphisms, and stabilizers

All our graphs are finite and connected with no multiple edges or loops. A *map* M is an embedding of a graph G, called the *underlying* graph, in a closed surface S, called the *underlying surface*, such that each component, or *face*, of $S - G$ is homeomorphic to an open disc (that is, the embedding is cellular). There are a variety of ways of looking at maps as combinatorial structures: rotation systems or band decompositions [2], permutation groups acting on directed edges (monodromy or dart groups)[4], triples of vertex-edge-face incidences (flags) [5]. The rotation viewpoint, being more intuitive and geometric, serves our purposes best.

An oriented map naturally defines a cyclic ordering of the directed edges beginning at each vertex, usually called the *rotation* at that vertex; the set of all such rotations is called a *rotation system* and can be written as a single permutation of the directed edge

set whose cycles correspond to vertices. Conversely, given a rotation system, one can construct an oriented map by first assigning to each vertex an oriented disk containing a vertex at the center and spokes for edge-ends in the cyclic order given by the rotation system. Then one can join the vertex-disks by edge-bands to form a compact orientable surface with a number of boundary components, a "thickening" of the underlying graph. Finally, one can paste disks to the boundary components to form the faces of a map in a closed orientable surface. The faces can be traced out beginning at any directed edge simply by using the rotation at each vertex to choose the next directed edge in the face.

For non-orientable maps, there are two possible cyclic orderings at a vertex, since there is no orientation present to differentiate "clockwise" from "counterclockwise". We also must specify whether each edge is "flat" or "twisted", depending on whether or not the edge-band can be oriented consistently with its end-point vertex-disks. The collection of rotations and twisting, we call a *general rotation system*. Notice that one can always reverse the rotation at a vertex in exchange for reversing the twisting of all edges incident to the vertex, twisted to flat and flat to twisted; one can use this operation to define an equivalence relation on general rotation systems. Every embedding defines an equivalence class of general rotation systems and every such equivalence class defines an embedding. It can be shown that a general rotation system defines an orientable embedding if and only if every cycle in the graph contains an even number of twisted edges, or equivalently, if and only if there is an equivalent general rotation system for which all edges are flat. Faces are traced out using the rotation for directions at each vertex, but now if uv is twisted we use at v the reverse of what we use at u, and if uv is flat we use the same.

Given a general rotation system for the graph G, if H is a subgraph of G, we can talk about the general rotation system restricted to H, where the cyclic order at a vertex of H is simply the original cyclic order, leaving out all the edges not in H. Thus for any map with underlying graph G, there is an *induced* map for any subgraph H of G. Note, however, that the underlying surface for the induced map may be different.

Given a map M, the map obtained by changing all twisted edges to flat and all flat edges to twisted is called the *Petrie dual*, denoted M^P; see [5] for a definition in terms of monodromy groups and flags. The underlying graph for M^P is the same as that for M, but the faces now correspond to Petrie "right-left" cycles from the original map M. If M is oriented, then M^P is orientable if and only if the underlying graph is bipartite.

An automorphism of a map is an automorphism of the underlying graph that extends (as a homeomorphism of the graph) to a homeomorphism of the surface. In terms of general rotation systems, a map automorphism is an automorphism of the underlying graph that, for some general rotation system of the map, either preserves the rotation at every vertex or reverses the rotation at every vertex and that takes flat edges to flat edges and twisted edges to twisted edges. The set of all automorphisms of a map M forms a group, denoted $Aut(M)$; if the map is orientable, the set of all orientation-preserving automorphisms forms a subgroup of index at most 2 in $Aut(M)$ and is denoted $Aut^+(M)$. The *distinguishing number of a map* M with vertex set V is $D(Aut(M), V)$ and is denoted simply $D(M)$; if M is orientable, the *orientable distinguishing number*, is $D(Aut^+(M), V)$, denoted simply $D^+(M)$.

From our definition of automorphism, it follows that for the Petrie dual M^P, the action of $Aut(M)$ and $Aut^P(M)$ on the underlying graph are the same. Indeed, this is true even for a *partial Petrie dual* obtained by only changing edge-types in a single edge orbit of $Aut(M)$. In fact, if B is a subgroup of $Aut(M)$, we can also change edge-types only in a single edge orbit of B. The resulting map M' will have the same underlying graph G and $Aut(M')$ will contain a subgroup acting on G in the same way as B.

Given an action of the group A on the set X, and given a subset Y of X, the *set-wise stabilizer* of Y is the subgroup of a in A with $a(Y) = Y$. Although this is usually denoted $A_{\{Y\}}$, we will maintain the notation $Stab(Y)$ used in [6]. The *point-wise stabilizer* is the subgroup of all a such that $a(y) = y$ for all y in Y. Again, we use the notation $Fix(Y)$ from [6] rather than the usual A_Y. We say that $Stab(Y)$ or $Fix(Y)$ is trivial if it contains only the identity. The actions in this paper are all faithful, that is for A acting on X, $Fix(X)$ is trivial.

Remarks: Note that $D(A, X) = 1$ if and only if A is the trivial group, and $D(A, X) = 2$ if and only if A is nontrivial but $Stab(Y)$ is trivial for some nonempty subset Y of X: simply color Y white and all all other elements of X black. Also, if $Fix(Y)$ is trivial and Y has k elements, then $D(A, X) \leqslant k + 1$: just color each element of Y with the first k different colors and color the remaining vertices with the last color. Finally, any faithful action of $A = Z_2 \times Z_2$ on a set X has $D(A, X) = 2$: color one element of each orbit black and the rest white.

Unlike graphs, maps have highly restricted set stabilizers. In particular, if uv is an edge, $Fix(u, v)$ has at most one nontrivial element, namely a reflection, which we denote τ_{uv}, that interchanges the faces incident to uv. If the map is oriented, then τ_{uv} is orientation-reversing. Summarizing:

Proposition 2.1. *If uv is an edge in map M, then $Stab(u, v)$ is a subgroup of $Z_2 \times Z_2$. If v is a vertex in M of valence d, then $Stab(v)$ is a subgroup of the dihedral group D_d, acting in the natural way on the cyclic order of vertices adjacent to v given by a rotation for M.*

We call a map *all-τ* if every edge uv has the reflection τ_{uv}, *no-τ* if none do, and *mixed* if some do and some don't. A *regular* map M is one with maximal symmetry, that is, vertex-transitive with vertex stabilizer D_d. In particular, a regular map is all-τ; note, however, that a vertex-transitive all-τ map is not necessarily regular. An oriented map M is *chiral regular* if $Aut^+(M)$ is transitive on directed edges, so that M is vertex-transitive with cyclic vertex stabilizers of order d, but no orientation-reversing automorphism. In particular, a chiral regular map is no-τ. An oriented regular map is also sometimes called *reflexible* regular.

If uv and vw are edges of map M, we call uvw an *angle*. An angle uvw is a *corner* if u and w are consecutive in the rotation at v. If v has valence d, the *measure* of angle uvw is the number $m \leqslant d/2$ of corners between u and w in the rotation at v; notice that angles are not oriented, with a first and second side, so angle measure is independent of the local orientation for the rotation at v. We call an angle *straight* if its measure is $d/2$ and *bent* otherwise. An angle uvw is *closed* if there is an edge uw and *open* otherwise. We

note that if angle uvw is open, then any element of $Stab(u, v, w)$ fixes v. By the dihedral action of $Stab(v)$ on the neighbors of v, there is at most one automorphism fixing v and interchanging neighbors u and w, called an *angle reflection*. We summarize facts about angle stabilizers:

Proposition 2.2. *Given a bent angle uvw, then $Fix(u, v, w)$ is trivial. In particular, if uvw is open, $|Stab(u, v, w)| \leqslant 2$. If uvw is instead straight, then $|Fix(u, v, w)| = |Fix(u, v)| \leqslant 2$. In particular, if uvw is open, then $Stab(u, v, w)$ is a subgroup of $Z_2 \times Z_2$.*

We will need to describe some fairly complicated small maps. The easiest way to do this is with Cayley maps [4, 5]. Given a group A with generating set S, the associated *Cayley graph* $C(A, S)$ is the directed, labeled graph with vertex set A and directed edge labeled s from a to as for each $a \in A$ and $s \in S$. Left multiplication by A of vertex labels gives a regular (transitive and free) action of A by automorphisms of the graph $C(A, S)$. If we also assign a cyclic order ρ to the elements of $S \cup S^{-1}$, the associated *Cayley map* $CM(A, \rho)$ is the oriented map with underlying graph $C(A, S)$ and vertex rotations given by ρ. Again, left multiplication by A gives a regular action by map automorphisms.

In addition to the natural regular action of A on $CM(A, \rho)$ there may be other symmetries fixing a vertex. In particular, if f is an automorphism of A with $f(\rho(s)) = \rho(f(s))$ for all $s \in S$ (i.e. f "respects" the rotation), then f also defines an orientation-preserving automorphism of the map $CM(A, \rho)$, fixing the identity vertex. If instead $f(\rho(s)) = \rho^{-1}(f(s))$ for all $s \in S$ (i.e. f "reverses" the rotation), then f defines an orientation-reversing automorphism of the map $CM(A, \rho)$, fixing the identity vertex.

A Cayley map $CM(A, \rho)$ is *balanced* if $\rho(s^{-1}) = \rho(s)^{-1}$, for all $s \in S$; that is, if $s \neq s^{-1}$, then they are antipodal in the rotation. Note this implies that if one element of X is an involution, then all are. The natural action of A on $CM(A, \rho)$, as a subgroup of $Aut(CM(A, \rho))$, is normal if and only if $CM(A, \rho)$ is balanced [5]. In the case where the elements of S are d non-involutions, we give only the first half of the cycle for ρ and abbreviate $(s_1, \cdots, s_d)^b = (s_1, \ldots, s_d, s_1^{-1}, \ldots, s_d^{-1})$.

Our graph notation and terminology are as follows. If the graph G has edge uv, then we say u and v are *adjacent* or u and v are *neighbors*. The subgraph of G induced by all the neighbors of u is the *link* of v, denoted $Link(u)$. The *distance* between vertices u and v is the length of the shortest path between them. The *diameter* of G is the greatest distance between any two vertices. The complete graph on n vertices is denoted K_n, the complete bipartite graph on m and n vertices is $K_{m,n}$, the cycle of length n is C_n, and the graph obtained from K_{2n}, for $n > 1$, by removing n disjoint edges is the octahedral graph O_{2n} (note $O_4 = C_4$) The graph obtained by adding k independent vertices to G and joining them by edges to all vertices of G is the *k-fold suspension* of G and denoted $S_k(G)$. Note that $S_1(K_n) = K_{n+1}$ and $S_2(O_{2n}) = O_{2n+2}$. Alternatively, $S_k(G)$ can be written as the join $\bar{K}_k * G$ of G with the complement of K_k.

For groups, we let Z_n denote the cyclic group of order n and D_n the dihedral group of order $2n$. The direct product $A \times A \times \cdots \times A$ of k copies of the group A is denoted A^k.

3 Examples of maps with $D(M) > 2$

We begin by giving two examples to illustrate the role of Petrie duality.

Example 3.1. *Let M the map on the sphere obtained by joining an m-cycle, $m = 3, 4, 5$, along the equator with the north and south poles; its underlying graph is $S_2(C_m)$. The action of the dihedral group $B = D_m$ on this map, leaving fixed the north and south poles, has distinguishing number 3, by the original necklace problem. The action of B has three edge orbits, so we can take a variety of partial Petrie duals with respect to B, one of which will be the Petrie dual. Each of these maps has $D(M) = 3$.*

Example 3.2. *Let M be the tetrahedron. Its Petrie dual M^P has three faces, all of size 4 and is a map in the projective plane, since the Euler characteristic is $4 - 6 + 3 = 1$. Notice that for any vertices u, v, w, each of uvw,vwu, and wuv is a corner of the map and $Stab(u, v, w) = D_3$. One might expect, as in the orientable case, that this means there is a triangular face of the embedding whose boundary is the 3-cycle uvw. But the embedding has no triangular faces. In addition, we can place a new vertex inside each face and join to the original four vertices to get a map M with underlying graph $S_3(K_4)$ and $D(M) = 3$.*

The following example has underlying graph $K_{m,n}$.

Example 3.3. *Consider the Cayley map $M(m, n) = CM(A, (x, x^{-1}, y, y^{-1}))$, where $A = Z_m \times Z_n$ generated by x, y of order m, n. Each face of the map is bounded by a cycle in the Cayley graph corresponding to $x^m = 1, y^n = 1$ or $(xy)^{lcm(m,n)} = 1$. Let $B(m, n)$ be the map obtained by placing a new vertex at the center of each of the faces corresponding to $x^m = 1$ or $y^n = 1$ and then joining the new vertices by edges through the vertices of the original graph $M(m, n)$ (and discarding all the original vertices and edges). This makes $M(m, n)$ the medial graph of $B(m, n)$ [5]. The graph underlying $B(m, n)$ is $K_{m,n}$. In $M(m, n)$, the stabilizer of the face F corresponding to $x^m = 1$ is D_m: multiplication by x is a map automorphism rotating F and the group automorphism inverting both x and y reverses the rotation providing a map automorphism reflecting F. Similarly, the stabilizer of a $y^n = 1$ face is D_n. Thus vertex stabilizers in $B(m, n)$ are D_m or D_n. In addition, if $m = n$, then the group automorphism interchanging x and y is also a map automorphism for $M(n, n)$, making $B(n, n)$ vertex-transitive.*

We now give examples of maps M with $D(M) = 3$. First, we have:

Theorem 3.1. *[6] If M has a vertex of valence 1 or 2 and $D(M) > 2$, then the underlying graph is $C_n, K_{1,n}$ or $K_{2,n}$, for $n = 3, 4, 5$.*

Theorem 3.2. *[6] If M has a no-τ group action A with $D(A, V) = 3$ and all vertices of valence $d > 2$, then the underlying graph is K_4, K_5, K_7, O_6 or O_8.*

The following table provides examples of vertex-transitive, oriented maps M with $D(M) = 3, D^+(M) = 2$, and no vertices of valence 1 or 2. Here G stands for the underlying graph, d for valence, *Stab* for vertex stabilizer, g for genus. The type of each map is regular ("reg"), not regular but all-τ ("all"), no-τ ("no") or mixed ("mix").

ID	Name	G	d	$Stab$	g	Type
T1	$CM(Z_4,(1,-1,2))$	K_4	3	D_1	1	mix
T2	$CM(Z_5,(1,-1,2,-2))$	K_5	4	D_1	2	no
T3	$CM(Z_2^3,(x,y,z))$	cube	3	D_3	1	reg
T4	$CM(Z_2^3,(x,x+y,y,y+z,\dots))$	O_8	6	D_3	7	all
T5	$CM(Q,(i,j,k)^b)$	O_8	6	D_3	5	no
T6	$CM(Z_3^2,(x,y)^b)$	$C_3\times C_3$	4	D_4	1	reg
T7	$CM(Z_3^2,(x,y,-x+y)^b)$	$K_{3,3,3}$	6	D_6	1	reg
T8	$CM(Z_3^2,(x,x+y,y,y-x)^b)$	K_9	8	D_4	10	all
T9	$B(3,3)$	$K_{3,3}$	3	D_3	1	reg
T10	$B(4,4)$	$K_{4,4}$	4	D_4	3	reg
T11	$B(5,5)$	$K_{5,5}$	5	D_5	6	reg

Table 1: Vertex-transitive maps with $D(M)=3$ but $D^+(M)=2$.

The proofs that each of these maps have $D(M)>2$, that is $Stab(Y)$ is nontrivial for any set Y of vertices, we leave to the reader; only when $|Y|=4,5$ is there much to check. We note that maps $T1, T2$ are just the necklace problem for D_4 and D_5. The cube T3 is well-known [1]. The action of $Aut(T4)$ on its vertex set is the same as that of $Aut(T3)$. T5 is discussed in [6]. The action of $Aut(T8)$ on its vertex set is the same as that of $Aut(T6)$. The maps T9-T11 are variations of the necklace problem.

We have not given the group structure of $Aut(M)$ for each of the maps in Table 1. Since all the maps except T9-T11 are balanced Cayley maps for some group A, making A normal, $Aut(M)$ is a semi-direct product of A with $Stab(v)$. The action of $Stab(v)$ by conjugation on A can be inferred from $Stab(v)$, the presence of τ edges, and the given generating set for A. For example, for map T5, since $Stab(v)=D_3$ and there are no τ edges, there must be only corner reflections. Thus a typical involution in $Stab(v)$ must be a group automorphism of the quaternions interchanging i and j, and interchanging k and $-k$. As another example, for T7, the reflection fixing edges labeled x is the group automorphism $f(x)=x, f(y)=-y$. The groups in T9-T11 are best understood in terms of the original Cayley map $CM(Z_m\times Z_m,(x,y,-x,-y))$, which is the semi-direct product of $Z_m\times Z_n$ by D_2 generated by the group automorphism inverting x and y and the group automorphism interchanging x and y.

We now give examples of intransitive maps with $D(M)=3$. All except $B(m,n)$ are obtained by the following construction. Let M be a map on $n<6$ vertices having a face of size n incident to all n vertices and suppose that the stabilizer of the face is D_n, so that $D(M)=3$. Let M^r be the *radial* map obtained by adding a new vertex at the center of the face and joining it to all the original vertices of M. Then $Aut(M^r)=D_n$ so $D(M^r)=3$. The underlying graph for $Aut(M^r)$ is $S_1(G)$ where G is the underlying graph of M. If M has two such faces, as in map T2, the process can be repeated to get a 2-radial map with underlying graph $S_2(G)$.

Examples for intransitive maps are given in Table 2. For the radial types, the column headed by "map" give the map to which the radial construction is applied. Columns V

and E are the number of vertex and edge orbits, respectively. For map N9, we abbreviate the Petrie dual of the tetrahedron by K_4^P. We note that by partial Petrie duality, each map in the table can give rise to many other maps. For example, map N10 has 16 possible partial Petrie duals. We do not give intransitive examples with underlying graph $K(n,n), n = 3,4,5$. We will show later that there are such maps only for $n = 4$.

ID	Type	map	G	surface	V	E
N1-N3	1 rad	$C_n, n = 3,4,5$	S_1C_n	sphere	2	2
N4	1 rad	T1	S_1K_4	torus	2	3
N5	1 rad	T2	S_1K_5	$g = 2$	2	3
N6-N8	2 rad	$C_n, n = 3,4,5$	S_2C_n	sphere	3	3
N9	3 rad	K_4^P	S_3K_4	proj	2	2
N10	2 rad	T2	S_2K_5	$g = 2$	3	4
N11-13	bipart	$B(m,n), 2 < m < n < 6$	$K_{m,n}$	$g = 3,4,6$	2	1

Table 2: Intransitive maps with $D(M) = 3$.

We have only claimed each of these maps has $D(M) > 2$. We must also show that none have $D(M) = 4$.

Theorem 3.3. *If $D(M) = 4$, then M is the tetrahedron or its Petrie dual.*

Proof. Suppose that $D(M) = 4$. Let uv be any edge. If $Stab(u,v) \subset Z_2 \times Z_2$ acts on the remaining vertices faithfully, then its distinguishing number in that action is 2, so $D(M) = 3$, a conatradiction. Thus there is $f \in Stab(u,v)$ fixing all other vertices; since it cannot also fix u and v, it must interchange them.

It is easily proved by induction that a set E of transpositions in the symmetric group S_n generates S_n if and only if the graph on n vertices having E as edges is connected. We have shown for each edge uv in the map M, the transposition (u,v), as a permutation of the vertices, is an automorphism. Since the map is connected, $Aut(M) = S_n$, where n is the number of vertices. But $|Aut(M)| \leqslant 4[n(n-1)/2] = 2n(n-1)$, since each edge stabilizer has size at most 4. Thus $n! \leqslant 2n(n-1)$, so $n \leqslant 4$. Since M has no vertices of valence 2, we have $n = 4$ and $Aut(M) = S_4$, so M is the tetrahedron or its Petrie dual. □

4 The Classification Theorem

Theorem 4.1. *(Classification Theorem) Suppose that $D(M) = 3$ and no vertex of M has valence 1 or 2. If M is vertex-transitive, the underlying graph is one of the following: K_n for $n = 4,5,6,7,9$; $K_{n,n}$ for $n = 3,4,5$; the octahedral graphs O_6 and O_8; the cube; $K_{3,3,3}$; or $C_3 \times C_3$. If M is not vertex-transitive, then the underlying graph for M is one of the following: S_kC_n for $k = 1,2$ and $n = 3,4,5$; S_kK_4 for $k = 1,3$; S_kK_5 for $k = 1,2$; or $K_{m,n}$ for $3 < m < n < 6$ or $K_{4,4}$. For each underlying graph, Tables 1 and 2 give an*

example of the map M, except for K_6 and an intransitive map for $K_{4,4}$. The only map M with $D(M) = 4$ is the tetrahedron or its Petrie dual.

Before proceeding, we first show all our maps are small.

Theorem 4.2. *The only map M with $D(M) = 3$ and diameter greater than 2 is the cube or its Petrie dual.*

Proof. We note at the outset that by Theorem 3.1, every vertex has valence at least 3. Also, since the action of $Aut(M)$ is not necessarily transitive, we write $u \sim v$ if u and v are in the same orbit of the action of $Aut(M)$.

Suppose that y has distance 3 from v and let $vwxy$ be a path from v to y and let $Y = \{v, w, x, y\}$. In particular, angles vwx and wxy are open. Note that any nontrivial element f of $Stab(Y)$ either fixes the path or reverses it. In either case, if vwx is straight, so is wxy. If vwx is straight, there is a z adjacent to y such that xyz is bent. Since z cannot be adjacent to v (or else y has distance 2 from v), it cannot make a straight angle with any of the edges induced by Y. Thus, any $f \in Stab(Y \cup \{z\})$ fixes z, and hence x and hence the other vertices in Y, then fixing the bent angle xyz, a contradiction. We conclude that vwx is bent. In particular, $f \in Stab(Y)$ cannot fix the path so $f(v) = y$. Thus $v \sim y$. Also, $v \sim x$ and $w \sim y$ since angles vwx and xwy are bent and open. Therefore $v \sim w \sim x \sim y$.

Suppose that two neighbors z_1, z_2 of y are not adjacent to w. Let $Y = \{v, w, x, y, z_1, z_2\}$ and let H be the subgraph induced by Y. Then any nontrivial $f \in Stab(Y)$ must fix the edge vw, since it is the only edge in H joining a vertex of valence 1 and a vertex of valence 2. But then f fixes x, the only vertex in H adjacent to w, thus fixing the bent angle vwx, a contradiction. Suppose that z is a neighbor of y that is adjacent to w. Let $Y = \{v, w, x, y, z\}$ and let H be the subgraph induced by Y. Then any nontrivial $f \in Stab(Y)$ must fix v (the only vertex of valence 1 in H) and hence w, so $f = \tau_{vw}$. Then f fixes y (the only vertex in H a distance 3 from v) and interchanges z and x (since it cannot fix angle wxy) so it also functions as the angle reflector for zyx. Thus, there is only one possibility for z other than x.

Since y has valence at least 3 with only one vertex not adjacent to w and at most 1 vertex other than x adjacent to w, we conclude that y has valence 3. Moreover, since the roles of v and y are interchangeable and $Stab(v)$ has τ_{vw} fixing y, then $Stab(y)$ also has an edge reflection fixing v. We already have the angle reflector for zyx, fixing v, so $Stab(y)$ is D_3 and $Stab(v) = Stab(y)$. Since w has two neighbors in $Link(y)$, so do all vertices in $Link(v)$. Since $v \sim y \sim w$, all vertices in $Link(v)$ and $Link(w)$ have valence 3 and each is incident to two edges between the links. Thus the map M has 8 vertices. Since it has valence 3 and is regular (it is vertex-transitive and all vertex stabilizers are D_3), and has diameter 3, it is the cube (as a map) or its Petrie dual. □

We recall that we know from [6] that the problem is a finite one:

Theorem 4.3. *There are only finitely many maps M with $D(M) > 2$.*

Moreover, in the following sections, we easily bound the valence $d \leqslant 8$. At that point, with diameter at most 2 and small valence, maps with $D(M) > 2$ are very small, at most $1 + 8 + 8 \cdot 7 = 65$ vertices. Indeed, with not much more work one can get the number of vertices to be smaller still. In particular, the Classification Theorem can be proved simply with a computer search. On the other hand, although our overall proof involves some case-by-case analysis, the number of cases is not that large and the proofs are not long.

5 The intransitive case

We note that the orbits under $Aut(M)$ form a kind of coloring, so any question about distinguishability depends on the relative valences of the different orbits. This coloring leads to the banning of certain angles (assume P, Q, R are different orbits):
There can be no bent PQR angles.
There can be no bent open PQQ angles

If P and Q are different orbits and u is in P, we let d_{PQ} be the number of neighbors of u in Q and d_{PP} the number of neighbors of u in P.

Lemma 5.1. *Let P and Q be different orbits. Then*
a) $d_{PQ} \geqslant d_{QQ}$ and if $d_{QP} > 1$, then $d_{PQ} \geqslant d_{QQ} + 1$;
b) if $d_{PQ} > 0$, then it divides d_{PP}; in particular, if $d_{PP} > 0$, then $d_{PQ} \leqslant d_{PP}$;
c) $d_{PQ} < 6$;
d) if $d_{PP} > 0$ and $d_{PQ} > 0$, then $d_{QQ} = 0$;
Note that by symmetry, the same statements hold with P and Q interchanged.

Proof. We let u be a vertex in P and v a vertex in Q adjacent to u.
a) Let w be any vertex in Q adjacent to v. Then unless uvw is straight, there must be an edge uw since there are no open bent PQQ angles. Thus u is adjacent to at least $d_{QQ}-1$ of the neighbors of v in Q; including the adjacency of u to v, we have $d_{PQ} \geqslant d_{QQ}$. If $d_{QP} > 1$, we claim that u must also be adjacent to w even if uvw is straight, so $d_{PQ} \geqslant d_{QQ} + 1$. Suppose not. Since $d_{QP} > 1$, then there is another vertex x from P adjacent to v. Let $Y = \{u, v, w, x\}$, let H be the subgraph induced by Y, and let $f \in Stab(u, v, w, x)$ be nontrivial. Since $v, w \in Q$ and $u, x \in P$, f fixes v since it has valence 3 in H and w does not. Then f also fixes w, but that is impossible since uvw is straight and xvw is not.

b) For any PQ edge uv, there is a reflection τ_{uv}. Thus $Stab(u)$ acts transitively on the QPQ "corners" at u, that is angles vuv' where v and v' are consecutive Q vertices in the rotation at u (there may be intervening vertices from other orbits). Then any $w \in P$ adjacent to u must lie in some QPQ corner and hence has at least d_{PQ} images under $Stab(u)$. Thus d_{PQ} divides d_{PP}.

c) This follows from the original necklace problem.

d) Suppose instead that $d_{QQ} > 0$. From (a) and (b), we have

$$d_{PQ} \geqslant d_{QQ} \geqslant d_{QP}.$$

Reversing the roles of P and Q, we get $d_{QP} \geqslant d_{PQ}$. Thus $d_{PQ} = d_{QP} = d_{QQ}$. Since $d_{QQ} > 1$ implies $d_{PQ} > d_{QQ}$, the common value must be 1. Since there are no vertices of valence 2, we have $d_{QR} > 0$ for some third orbit R, but then there must be a bent PQR angle or an open bent PQQ angle, a contradiction. □

Lemma 5.2. *If there are three or more vertex orbits, then the underlying graph G for M is S_2C_n, for $n < 6$, or S_2K_5.*

Proof. By connectivity, there must be an angle uvw with the vertices in different orbits P, Q, R. Since there can be no bent PQR angles, u and w are the only neighbors of v not in Q and uvw is straight. Since v does not have valence 2, we have $d_{QQ} > 0$, so by part (d) of Lemma 5.1, $d_{PP} = d_{RR} = 0$. Moreover, u cannot be adjacent to a vertex not in Q, since by the same argument as we used for v, we would have $d_{PP} > 0$. We conclude that both u and w only have neighbors in Q. In particular, P, Q, R are the only orbits and $P = \{u\}, R = \{w\}$, since $d_{QP} = d_{QR} = 1$. By Lemma 5.1, we also have $d_{PQ} < 6$ and $d_{PQ} \geqslant d_{QQ}$, so $|Q| < 6$. Since every vertex in Q is like u, every vertex in Q has a P and Q neighbor, and since $|P| = |R| = 1$, every vertex in Q is adjacent to both u and w.

Thus $G = S_2H$, where H is the subgraph induced by Q. Since the PQR angle uvw is straight, $Stab(uvw)$ includes τ_{uv}, which pairs the Q neighbors of v, so d_{QQ} is even. Since H is also vertex-transitive, because of the action of $Stab(u)$ on its Q neighbors, and since $|H| < 6$, the only possibilities for H are C_n, for $n = 3, 4, 5$ and K_5. □

The exceptions appear in our list of intransitive maps with $D(M) = 3$.

Lemma 5.3. *Suppose that $d_{PP} = d_{QQ} = 0$. Then the underlying graph G for the M is a complete bipartite graph $K_{m,n}$ where $m = 3, 4, 5$ and $n = 3, 4, 5$.*

Proof. The graph G is bipartite, with colors P and Q. By Lemma 4.2, the diameter is 2, so the distance between any vertex in P and any in Q, which must be odd, can only be 1.

The limits on m and n follow from (d) of the Lemma 5.1. □

We observe that our list of intransitive examples includes all the values $2 < m < n < 6$ for $K_{m,n}$, but not the values where $m = n$. We already have transitive examples when $m = n$ and our goal is simply to classify underlying graphs, but this does raise the question of whether there are intransitive examples with $m = n$.

Lemma 5.4. *There are intransitive maps M with $D(M) = 3$ and underlying graph $K_{n,n}$ only for $n = 4$.*

Proof. To show there is no such map for $n = 3, 5$, we reverse the process of turning the map $M(m, n)$ into $B(m, n)$. Suppose that M is a map with underlying graph $K_{3,3}$ and $D(M) = 3$, but there is no automorphism switching the parts; in particular, for any edge uv, $|Stab(u, v)| \leqslant 2$ so $Aut(M) \leqslant 18$. If $Stab(v) \neq D_3$ for some vertex v, then $D(M) = 2$ by the necklace problem and the fact that v is not interchanged with any of its neighbors by an automorphism. If $Stab(v) = D_3$ for every vertex, then the map is edge-transitive,

so either all edges are twisted or none are. But then since $K_{3,3}$ is bipartite, the map is oriented. Then $Aut^+(M)$ acts regularly on the edges, so the medial map is a Cayley map for a group A of order 9, with generating set consisting of two elements of order 3, both bounding faces. The only possibility is $M(3,3)$ so $M = B(3,3)$. The same proof works for $n = 5$.

For $n = 4$, however, there are more possibilities for A. In particular, if $A = \langle s, t : s^4 = t^4 = 1, tst^{-1} = s^{-1} \rangle$, the Cayley map $CM(A, (s, s^{-1}, t, t^{-1})$ has a reflection inverting s and t, but none interchanging s and t. Thus, when we put vertices at the center of s^4 and t^4 faces, the resulting map is not vertex-transitive, but still $Stab(v) = D_4$; moreover the underlying graph is bipartite (by construction) and hence is $K_{4,4}$. □

Lemma 5.5. *Suppose that $d_{PP} = 0$ and $d_{QQ} > 0$. Then the underlying graph G for M is S_kC_n where $k = 1, 2$ and $n = 3, 4, 5$; S_kK_4 where $k = 1, 3$; S_kK_5 where $k = 1, 2$.*

Proof. Let u be any vertex in P. We claim all vertices in Q are adjacent to u. If $P = \{u\}$, then $d_{QP} = 1$, so all vertices in Q are adjacent to u. Suppose instead $u' \in P$. Because the diameter is at most 2, there must be a PQP path from u to u', since $d_{PP} = 0$. Thus $d_{QP} > 1$. Suppose that $w \in Q$. Again, there must be a path uvw with $v \in Q$. Since $d_{QP} > 1$, by the proof of part (a) of Lemma 5.1, w must be adjacent to u.

Let $2 < d < 6$ be the valence of u, and let $H = Link(u)$, which is vertex-transitive (the vertices are just Q). If $d_{QQ} > 1$, the only possibilities for H are C_n, for $n = 3, 4, 5$ or K_4 or K_5. If $d_{QQ} = 1$, the only possibility is that H is the disjoint union of 2 edges. Let $v, w \in Q$ be endpoints of one of those edges. Then the only nontrivial element of $Stab(v)$ is τ_{vw}. Thus if $|P| > 1$, some element $z \in P$ is moved by τ_{vw}, so $Stab(v, z)$ is trivial. If $|P| = 1$ instead, then v has valence 2, a contradiction. Thus $d_{QQ} > 1$ and H is one of the required graphs.

It remains to show $|P| < 3$ in all cases, except for K_4, where we want $|P| = 1, 3$. But by Lemma 5.1, we have d_{QQ} is divisible by $d_{QP} = |P|$. This restricts $|P|$ to the required values in all cases, except for the possibility $|P| = 4$ for $H = K_5$. For this last case, suppose $u, v \in Q$. Then $Stab(u, v) \subset Z_2 \times Z_2$ acts on P. Since any faithful action of $Z_2 \times Z_2$ on a set with 4 elements has distinguishing number 2, some nontrivial $f \in Stab(u, v)$ fixes all vertices in P. But f also fixes an element of Q since f is an involution and $|Q| = 5$, so f fixes a bent PQP angle. □

Again, we have provided examples of maps with the given underlying graph G.

6 The vertex-transitive case: regular maps

Our basic plan for the vertex-transitive case is to factor the vertex-transitive map M with $D(M) = 3$ into an all-τ map M_1 and a no-τ map M_2; either factor may be disconnected. Since the collection of τ-edges is invariant under the action of $Aut(M)$, the map obtained by restricting the general rotation system only to the edges in M_1 or M_2 is vertex-transitive and $D(M_1) = D(M_2) = 3$. In [6], we found the five graphs underlying no-τ maps M with

$D(M) = 3$. Thus, we need to classify the all-τ maps M with $D(M) = 3$, and then see how all-τ and no-τ maps can be overlaid with each other.

If M is an all-τ map, then vertex stabilizers are D_d if d is odd and D_d or $D_{d/2}$ if d is even. In the D_d case we have a regular (reflexible) map. In the $D_{d/2}$ case, there are two edge orbits and we can again restrict M to one of those orbits to obtain a regular map M'. Thus our first task is to classify reflexible regular maps M with $D(M) = 3$.

Throughout this section, we let M denote a regular (reflexible) map with $D(M) = 3$ and underlying graph G of valence $d > 2$, and diameter at most 2.

Theorem 6.1. *The graph G is K_n for $n = 4, 6$, $K_{n,n}$ for $n = 3, 4, 5$; $K_{3,3,3}$; $C_3 \times C_3$; or O_6.*

The proof follows from a number of lemmas. First we handle K_n. It is well-known [5] that the only reflexible regular maps with underlying graph K_n are for $n = 3, 4, 6$. The following proof, however, is astonishingly simple and illustrates the power of angle measure.

Theorem 6.2. *The only complete graphs $K_n, n > 3$, underlying a reflexible regular map are K_4 and K_6.*

Proof. We note that all triangles are equiangular, since any angle uvw has a reflection fixing v and interchanging u and w. We will show that if M has closed angles of measure 1 and 2, then the valence $d = 5$. Indeed, if angle uvw and wvx have measure 1 and uvx has measure 2, then the K_4 subgraph induced by u, v, w, x has two triangles with common angle measure 1 (uvw and wvx) and two with common angle measure 2 (uvx and uwx). But then at vertex u, the three incident edges make two angles of measure 2 and one of measure 1, which can only happen if $d = 2 \cdot 2 + 1 = 5$. If the graph underlying M is K_n, all angles are closed, so the only possibilities are K_6 for $d = 5$ and K_4, since it has angles only of measure 1. □

We have not yet given a vertex-transitive map M with $D(M) = 3$ and underlying graph K_6. There is one, up to Petrie duality: the quotient of the icosahedron under the antipodal map is a regular map triangulating the projective plane by K_6. Since vertex-stabilizers are D_5, we have $D(M) > 2$.

Next, for diameter 2, we bound the valence.

Lemma 6.1. *If the diameter of M is 2, then $d \leqslant 6$.*

Proof. Let v be any vertex and let its neighbors be $u_1, u_2, \ldots, u_d$, in cyclic order. Let Y be all the neighbors of v except u_1, u_2, u_4 and let H be the subgraph induced by Y and v. As long as $d - 4 \geqslant d/2$, given any angle measure $a \leqslant d/2$ and any vertex $u \neq u_3$ in Y, there is a vertex u' in Y such that angle uvu' has measure a. For u_3 the only missing measure is 1. Thus unless the only open angle measure is 1, the only vertex in H of valence $d - 3$ is v. Thus any automorphism stabilizing H fixes v, and hence must be trivial, by the necklace problem. If the only open angle measure is 1, throw out instead the neighbors u_1, u_4, u_5. Again, every vertex in Y has valence less than $d - 3$, and again the stabilizer of H is trivial, as long as $d > 7$; for $d = 7$, unfortunately $\tau_{u_1 v}$ stabilizes H.

For $d = 7$, we again have trivial stabilizer for H, looking at open angles as before, unless the only open angle measure is 1 or the only open angle measure is 3. But by the previous lemma, we cannot have both measure 1 and 2 closed. Thus the only possibility is for measures 2 and 3 to be closed. But then as in the previous lemma, we have a K_4 subgraph with two triangles having common measure 2 and two having common measure 3. At one vertex of this K_4 we have two angles of measure 3 and one of measure 2, so $2 \cdot 3 + 2 = 8 \neq 7$, a contradiction. □

Now we handle the remaining valences.

Lemma 6.2. *For $d = 6$, we have $G = K_{3,3,3}$.*

Proof. Let v be any vertex and let the vertices of $L = Link(v)$ be, in cyclic order, $u_1, \dots u_6$. Suppose there are two open angle measures. Let H be the graph induced by v, u_1, u_2, u_4. If the valence of v is larger than the other vertices, any automorphism stabilizing H fixes v, contradicting the necklace problem for $d = 6$. By the proof of Lemma 6.1 applied to $d = 6$, we cannot have angles of measure 1 and 2 both closed, or measures 2 and 3. We conclude that the only open angles have measure 2.

Since angles of measures 1 and 3 are closed, u_1 is adjacent to u_2, u_6 and u_4, in addition to v. Thus u_1 is adjacent to two other vertices w, w' that have distance 2 from v. Since all triangles are equiangular, angles u_1u_2v, u_1u_6v have measure 1 and angle vu_1u_4 is straight. Thus the cyclic order at u_1 is v, u_2, w, u_4, w', u_6. Repeating this argument at vertices $u_2, \cdots u_6$, we conclude that w and w' are adjacent to each of $u_1, \dots, u_6$. Thus $G = K_{3,3,3}$ with parts $\{v, w, w'\}, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}$. □

Next we finish the cases $d = 3, 5$.

Lemma 6.3. *For $d = 3, 5$, the possibilities are $G = K_{3,3}$ and $G = K_{5,5}$.*

Proof. For $d = 3$, since the diameter is 2, all angles are open, so $G = K_{3,3}$. For $d = 5$, if all angles are open then $G = K_{5,5}$.

It remains to show for $d = 5$ that it is impossible to have one of the angle measures $1, 2$ open and one closed (they cannot both be closed since the diameter is 2). Let H be the subgraph induced by the vertices a distance 2 from v and let $L = Link(v)$. If $u \in L$, then u is adjacent to v and two vertices in L, so u is adjacent to two vertices w, w' in H, which must be interchanged by τ_{uv}. Thus there are exactly $5 \cdot 2 = 10$ edges between L and H. Since one angle measure at u is closed, w is adjacent to two vertices in $Link(u)$. At least one of those two vertices must also be in L, since the other neighbors of u are w, w', v and w is not adjacent to v. Thus each vertex in H has at least two edges to L, so H has at most 5 vertices.

Let $f \in Stab(v)$ have order 5. If $f(w) = w$, then all edges from w lead to L; the same holds for w' since τ_{vu} interchanges w and w'. Thus H consists of two nonadjacent vertices, so G has 8 vertices. But then the complement is vertex-transitive, with 8 vertices of valence 2 and a triangle vww', a contradiction. If instead $f(w) \neq w$ then the orbit of w under f has size 5 and is all of H. In particular, w has exactly two edges to L, so H has 5 vertices, all of valence 3, which is impossible. □

Finally, we handle $d = 4$.

Lemma 6.4. *If $d = 4$, then G is $K_{4,4}, O_6$ or $C_3 \times C_3$.*

Proof. If all angles are open, then $G = K_{4,4}$, as in the other cases. Since the diameter is 2, there must be both open and closed angles. Since there are only two possible angle measures, either bent angles are open and straight are closed, or vice versa. Let v be any vertex, $L = Link(v)$, and H the subgraph induced by vertices a distance 2 from v. If all bent angles are closed and straight angles are open, $G = O_6$, since there are four edges from L to H and vertices a distance 2 apart (for example, at the end of straight angle at v) have at least 3 common neighbors, forcing H to be a single vertex.

Suppose instead that bent angles are open and straight angles closed. Let $u_1, \ldots, u_4$ be the vertices of L in cyclic order around v. Then u_1vu_3 and u_2vu_4 form triangles T and R through v in which every angle is straight; it is best to think of these as perpendicular lines. Since M is regular, there is an automorphism f of order 3 leaving T invariant; think of f as a translation along the line T taking v to u_1 to u_3 to v. Then f also leaves invariant the straight triangle/line T' through u_2 disjoint (or "parallel") to T. Similarly, f takes R to the parallel line R' through u_1. This forces a point of intersection $f(u_2)$ of R' and T'. Similarly, $f^2(u_2), f(u_4), f^2(u_4)$ give other points of intersection of lines through u_1, u_2, u_3, u_4, giving 9 vertices in all on two perpendicular families of three parallel lines. The result is $C_3 \times C_3$. In addition, the automorphism f, together with a similar automorphism for the straight triangle u_2vu_4, make M a Cayley map for $Z_3 \times Z_3$. □

7 The vertex-transitive case: completing the classification

It remains to consider all-τ nonregular maps and mixed maps.

Theorem 7.1. *The possible nonregular all-τ maps M with $D(M) = 3$ have underlying graph either $G = K_9$, as an overlay of two copies of a regular map for $C_3 \times C_3$, or $G = O_8$, as an overlay of the cube with two tetrahedrons.*

Proof. Since M is all-τ but not regular, there are two edge orbits inducing regular maps M_1 and M_2, whose edges alternate at each vertex; the maps need not be connected. Let uvw be a corner. Since there is no automorphism taking edge uv to uw, the angle must be closed, so there is an edge uw. Without loss of generality, we can assume that uw is in M_1. Then all edges closing the corners must be in M_1 by the transitive action of $Stab(v)$ on the corners at v. In particular, if uv is in M_2, there is a path of length 2 in M_1 from u to v, so M_1 is connected and hence contains all vertices of M. In addition, M_1 contains the cycle C_d with alternate vertices joined to v (there may be other edges between vertices in L that are also in M_1, but no other edges to v).

Checking our list of regular maps with $D(M) = 3$ for the structure of M_1, we find the only possibilities are $C_3 \times C_3$ and the cube. In the first case, the graph G underlying M is K_9 so the graph underlying M_2 is the complement of $C_3 \times C_3$ which is again $C_3 \times C_3$.

For the cube, $G = O_8$ since the valence is $d = 3 + 3 = 6$ and there are 8 vertices. There are various ways to see that the complement in O_8 of any cube subgraph is two copies of K_4. For example, O_8 is K_8 with four disjoint edges removed, which we can consider as the four diagonals through the center of the cube; viewing the cube as a bipartite graph, the complement in O_8 is then all the edges within the two parts. □

Theorem 7.2. *The only mixed map M with $D(M) = 3$ is $CM(Z_4, (1, -1, 2))$ and its three partial or full Petrie duals.*

Proof. Let M_1 be the all-τ map induced by τ edges and M_2 be the no-τ map induced by the remaining edges; both maps could be disconnected. Denote their valences by d_1 and d_2, respectively. Whether M_1 is regular or not, $Aut(M)$ acts transitively on the corners of M_1, so d_1 divides d_2. Also, if uv is in M_1, then τ_{uv} fixes no neighbor of u in M_2, so d_2 is even.

We claim that M_2 is connected. It suffices to show that for any edge uv in M_1, there are edges uw and wv in M_2. Suppose not. If w is any neighbor of u in M_2, then the angle vuw must be bent since uv is a τ edge but uw is not. Since there is no automorphism fixing u and interchanging edges uv and uw, the angle vuw must be closed, so there is an edge vw. By our assumption, vw is in M_1. But then for each of d_2 neighbors w of u in M_2, there is an edge vw in M_1, which implies $d_1 > d_2$, a contradiction. Thus M_2 is connected and its underlying graph spans M.

Suppose that $d_2 > 2$. Then $Aut(M)$ acts faithfully on M_2, so $D(M_2) = 3$. In this case, since d_2 is even, we must have by Theorem 3.2 that the graph underlying M_2 is K_5, K_7, O_6 or O_8. It cannot be K_5 or K_7, since that would make M_1 empty. If it is either O_6 or O_8, then there is only one edge from M_1 incident to any vertex v. But then the vertex stabilizer in $Aut(M)$ has order at most two, which is not the case for any group acting with distinguishing number 3 on a map with underlying graph O_6 or O_8.

We conclude that $d_2 = 2$ so the graph underlying M_2, is C_n for $n = 3, 4, 5$. Since M_1 is not empty, $n \neq 3$. For $n = 5$, the graph underlying M_2 is also C_5 and for any edge uv in M_1, we have τ_{uv} fixing exactly two of five vertices in C_5, which is impossible.

For $n = 4$, the underlying graph is K_4, so we can simply enumerate all transitive subgroups of the symmetric group S_4 with blocks 13 and 24 and containing the involutions (13) and (24) (all permutations in S_4 can be realized by automorphisms of the standard tetrahedron map). The only such subgroup is generated by (1234) and (13) and is therefore the same as $Aut(CM(Z_4, (1, -1, 2)))$. □

8 Comments

We have not classified maps M with $D(M) > 2$; rather we have classified the underlying graphs. A careful count, eliminating duplications such as $S_1(K_4) = K_5$, gives 22 different graphs having no vertex of valence 1 or 2.

If instead we want to count maps, then things get very complicated. For example, the graph $S_2(K_5)$ underlying $N10$ (the double radial map of $T2$) has 4 edge orbits $E_1, \ldots, E_4$

under a D_5 action, which would appear to lead to $2^4 = 16$ different maps using all partial and full Petrie duals. On the other hand, there is also an automorphism f of T2 interchanging the edge orbits in pairs E_1 with E_2 and E_3 with E_4, so, for example, the Petrie dual twisting only edges in E_1 is isomorphic to the one twisting edges only in E_2. In this way, f is an automorphism of 4 of the 16 maps and pairs the remaining 12, so there are only 10 isomorphism classes of maps, not 16. As another example, the graph K_5 underlies three very different maps with $D(M) = 3$: a chiral regular embedding in the torus (which gives two maps because of the orientation), the transitive but non-regular map $T2$, and the intransitive map N_4; moreover, each of these maps have various partial Petrie duals.

In the transitive case, many of our arguments classify the map up to Petrie duality (see for example, Lemma 6.4 in the case $C_3 \times C_3$). In [6], we also classify the maps M with $D^+(M) = 3$. We suspect that with more effort, we could also classify the maps in the intransitive case.

Finally, our results should be compared with Theorem 6 of Negami [3], which gives a partial classification of graphs underlying polyhedral maps M with $D(M) = 3$. A map is *polyhedral* if every face boundary is a cycle and the intersection of two face boundaries is either empty, a single vertex, or a single edge. This is a major restriction eliminating all the maps of Theorem 3.1 and 3.2 except those for $K_{2,n}, K_4, K_7, O_6$, and all the vertex-transitive maps from Table 1 except T3,T6, T7. It also eliminates the intransitive maps N4, N5, N10, N11-N13. In addition, Negami leaves open the possibility that there are maps with $D > 2$ for some or all of the complete graphs $K_8, K_n, n \geqslant 10$ and for the complement of $K_6 \times K_2$. As we have shown, there are none. Negami's methods are completely different from the methods in this paper.

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