Counting k-convex polyominoes*

Anne Micheli

LIAFA UMR 7089, Université Paris Diderot and CNRS,
Case 7014, 75205 Paris, France.

Dominique Rossin

LIX UMR 7161, Ecole Polytechnique and CNRS,
91128 Palaiseau, France.

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Abstract

We compute an asymptotic estimate of a lower bound of the number of k-convex polyominoes of semiperimeter p. This approximation can be written as $\mu(k)p4^p$ where $\mu(k)$ is a rational fraction of k which up to $\mu(k)$ is the asymptotics of convex polyominoes.

A polyomino is a connected set of unit square cells drawn in the plane $\mathcal{Z} \times \mathcal{Z}$ [7]. The size of a polyomino is the number of its cells. A central problem, which proved to be difficult, is to find exactly or even asymptotically $A(n)$, the number a polyominoes of size n. Klarner proves that $\lim_{n\to\infty} \sqrt[n]{A(n)}$ exists and is upper bounded by 4.64 [8]. The lower bound was recently improved up to 3.98 [1].

In order to approximate the number of polyominoes, subclasses have been introduced, one of them being *convex* polyominoes. A horizontal, resp. vertical, convex polyomino is a polyomino for which each row, resp. column, is convex. Figure 1 gives an example of horizontal but not vertical convex polyomino and of a horizontal and vertical convex polyomino called *convex* polyomino. Bender gives an asymptotic estimate fr^{-n} of the number of convex polyominoes with n cells, f and r being numerical constants [2].

Convex polyominoes are often considered according to number of rows and columns, called semiperimeter. Delest and Viennot [5] prove that the number of convex polyominoes of semiperimeter $p+4$ is $f_{p+4} = (2p+11)4^p - 4(2p+1)\binom{2p}{p} \sim 2p4^p$.

In [3], Castiglione and Restivo observe that in a convex polyomino each pair of cells can be joined by a monotone path (monotone means that it contains only two kinds of steps, *e.g.* East $(1,0)$ and North $(0,1)$ or East and South $(0,-1)$). The minimal number

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Figure 1: a horizontal (but not vertical) convex polyomino and a convex polyomino

k of turns in the monotone paths linking two cells gives rise to a parameter which we call the *complexity*, and is the basis of how we shall classify polyominoes here. A polyomino is called *k-convex* if every pair of cells can be connected with a path of complexity at most k and there exists a path of complexity k linking two cells. Examples of k-convex ($k = 0, 1, 2$) polyominoes are given in Figure 2.

1-convex polyominoes are also called L-convex [3], 2-convex polyominoes are also denoted Z-convex. The generating function for L-convex polyominoes where the variable t marks semiperimeter is given by $G(t) = \frac{1-2t+t^2}{1-4t+2t^2}$ [4]. Z-convex polyominoes have been recently studied by Duchi, Rinaldi and Schaeffer [6] who compute the rational generating function and provide an asymptotics in $\frac{p}{24}4^p$ when the semiperimeter is $p + 2$.

In this article we provide a lower bound for the number of k-convex polyominoes for any k and show that the asymptotics is also $\mu(k)p4^p$. The key idea in this article is to transform a random walk into a set of polyominoes. Random walks are chosen with small deviation. Thus the boundary of the obtained polyominoes is delimited by two rectangles and we show that in that case, the polyominoes are k-convex. The first section gives basic definitions and first results on random walks with small deviations and polyominoes. In the second section, we describe the algorithm that tranform one random walk into a set of bounded polyominoes and prove its correctness. Finally, in the last section, we give the asymptotic number of a lower bound of k-convex polyominoes.

1 Small deviation random walks and polyominoes

1.1 Random walks

A random walk of size n is a sequence of n steps in the plane chosen uniformly at random between $(0, 1)$, called **n** step, and $(1, 0)$, called **e** step. Figure 3 illustrates the parameter called the *deviation of a walk*. This is the maximal distance between the walk and the first diagonal. Here we consider only those walks with deviation smaller that $c\sqrt{n}$. This restriction is justified by the following proposition, which is a direct consequence of the central limit theorem:

Proposition 1. *The number of random walks of size n with deviation less than $c\sqrt{n}$ is asymptotically $\lambda 2^n$ where $\lambda = \lambda(c)$.*

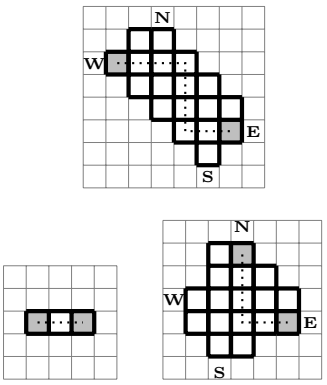

Figure 2: A 0,1,2-convex polyomino

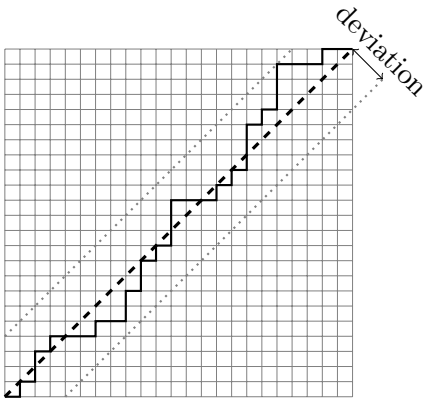

Figure 3: A random walk of size 46

1.2 *k*-convex polyominoes

They are several paths that join two selected cells. Among the paths of minimal complexity we can choose a canonical one as stated in the following lemma.

Lemma 1 (Canonical path of minimal complexity)**.** *For each pair of points P, Q with complexity h, there exists a path joining P and Q with h turns such that each turn (except perhaps the last one) lies on the border of the polyomino.*

Proof. Let $w = u_1 u_2 \dots u_{h+2}$ be a path joining $u_1 = P$ and $u_{h+2} = Q$. Let $i \geqslant 2$ be the smallest integer such that u_i does not belong to the border. If $i = h + 1$ then the Lemma is verified. Otherwise, the idea is to shift the point u_i by 1 along the segment $[u_{i-1}, u_i]$ as shown in Figure 4. Three cases could occur:

- (i) The new path has the same number of turns and point u_i is closer to the border of the polyomino by 1 unit.
- (ii) The new path has $h - 1$ turns, at points $u_2, \dots, u_h$, which contradicts the fact that P and Q are at distance h.
- (iii) The new path has $h - 1$ turns, at points $u_2, \dots, u_i, u_{i+2}, \dots, u_{h+1}$, since point u_{i+1} is not a turn anymore which contradicts the fact that P and Q are at distance h.

Thus, only the first case (i) occurs and repeating the shift, we obtain a new path where u_i is on the border. $u_2, \dots, u_{i-1}$ have not been moved and thus lie on the border. □

Moreover it is straightforward that only pairs of points on the boundary of the polyomino must be taken into account when computing the convexity. More precisely, Figure 2 displays four special points on the boundary which play a key role, the north point N (resp. W, S and E) being the first northern cell of the polyomino preceded by a lower one, encountered by a counterclockwise walk around it.

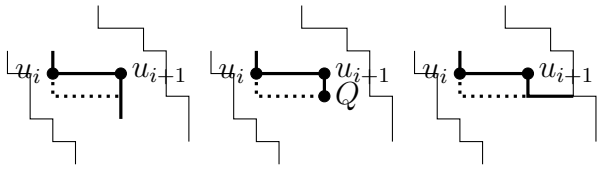

Figure 4: Cases $(i), (ii), (iii)$ for a path joining P and Q

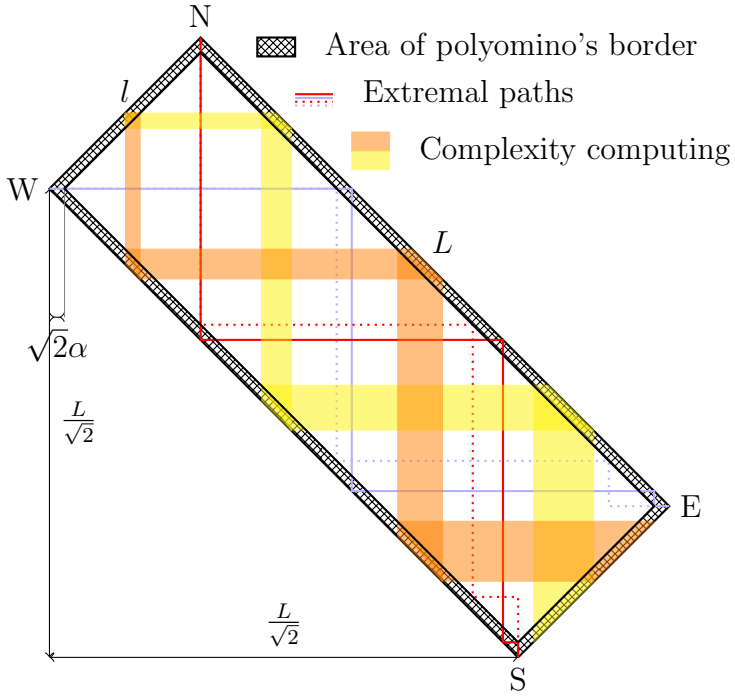

Figure 5: Definition of k-bounded polyominoes ($k = 4$)

1.3 *k*-bounded polyominoes

Typically, the shape of random k-convex polyominoes resembles a rectangle rotated by $\pi/4$, such that the ratio of height to length is a function of k. We introduce a subclass of k-convex polyominoes that we call *k-bounded polyominoes*. More precisely, a k-bounded polyomino is a convex polyomino whose boundary is at a small distance from such a rectangle.

Formally, there exists $l > 0$ such that the boundary lies within a rectangular strip of width $\alpha = \frac{l}{4k}$ delimited by two rectangles, a $L \times l$ one with $L = (k-1)l + \frac{l}{2k}$, and a smaller one at distance α (see Figure 5).

The values have been chosen such that the plain, resp. dashed, line which represents the monotone path linking extremal cells (N/S or E/W if k is even and N/E or S/W if k is odd) of the outer, resp. inner, rectangle has k turns. Indeed, the choice of L, l is such that every polyomino that is k-bounded, is also k-convex as stated in the following theorem:

Theorem 1. *A k-bounded polyomino is k-convex.*

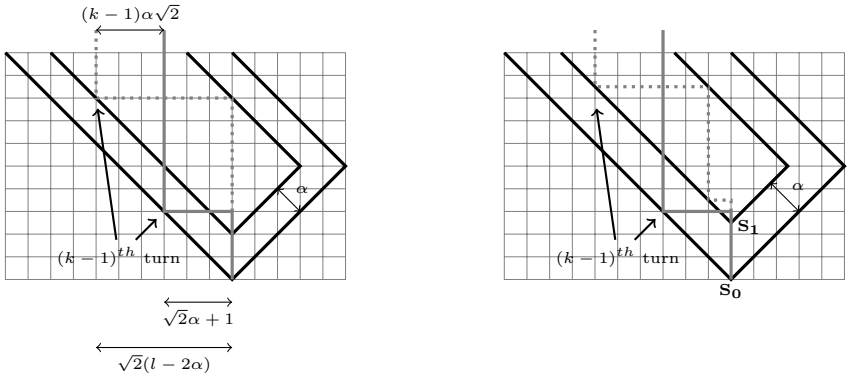

Figure 6: End of the internal/external paths

Proof. By definition of k-convexity, the complexity of every path linking a pair of points in the polyomino must be less than k and at least one pair must be of complexity k.

To ensure the k-convexity for every k-bounded polyomino, it is enough to check this result for some specific paths. Similarly to the definition of point N of a polyomino, we denote by N_0, resp. N_1, the northest point of the outer, resp. inner, rectangle, the same goes for the other directions S, E, W (Figure 6 displays point S_0 and S_1).

We call external (resp. internal) path, the monotone path linking extremal points of the outer $L \times l$ (resp. inner $(L-2\alpha) \times (l-2\alpha)$) rectangle (Figure 5 displays the external, resp. internal, path in plain, resp. dashed, line).

Remark that the external (resp. internal) path linking N_0 (resp. N_1) to S_0 (resp. S_1) has an even number of turns whereas the external (resp. internal) path linking N_0 (resp. N_1) to E_0 (resp. E_1) has an odd number of turns. In the following, we will consider an even k, meaning that we will work with the extremal path linking N_0 (resp. N_1) to S_0 (resp. S_1), but the same holds for an odd k, considering then the extremal path linking N_0 (resp. N_1) to E_0 (resp. E_1).

$L = (k-1)l + 2\alpha$ ensures that the external path ends up as shown in Figure 6, thus has complexity k, *i.e.* the complexity of the monotone path linking any point on segment $[N_0N_1]$ and any point on segment $[S_0S_1]$ is at least k. Thus, k-bounded polyominoes are at least k-convex.

The complexity of the internal path between N_1 and S_1 is at least the same as the complexity of the external one, but it can be greater. Figure 6 illustrates the increase by $\alpha\sqrt{2}$ in the distance between the internal and external path at each turn. Thus, after $k-1$ turns the distance between them is $(k-1)\alpha\sqrt{2}$. But this distance is also constrained by the fact that the internal path must not reach point S_1 in more than k turns. This implies that (see Figure 6):

$$(k-1)\sqrt{2}\alpha + (\sqrt{2}\alpha + 1) \leqslant \sqrt{2}(l-2\alpha)$$

A simple calculation shows that $\alpha \leqslant \frac{l-1/\sqrt{2}}{k+2}$ is required.

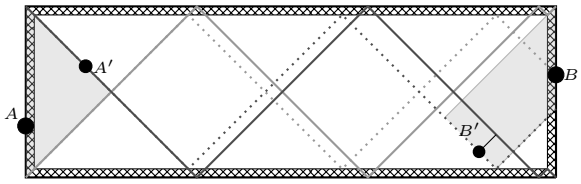

Figure 7: Paths with k turns in a k-convex polyomino.

In order to obtain k-convexity, we determine the value of α such that the complexity of the monotone path linking any pair of points (A, B) is at most k. We call a segment the border of a polyomino between two specified points A and B, turning clockwise from A to B. For $k \geqslant 2$, we can restrict positions of A (resp. B) to be on the segment WN (resp. segment ES).

The first step of any path from A to B goes from A to A', which belongs to one of the extremal paths, and follows it as shown in Figure 7 until reaching B' (which is the point on the $(k-1)^{\text{th}}$ segment of the internal path at the same ordinate than B) and then goes to B. This path has exactly k turns. Each internal path, the one linking N_1 to S_1 and the one linking W_1 to E_1, allows to reach a part of the segment ES. To cover all this segment, the two parts must intersect. This situation occurs when the internal path starting from N_1 (resp. W_1) ends on segment E_1S_1 on the half part containing S_1 (resp. E_1). It gives the following inequalities :

$$\begin{aligned} & L + l - 2\alpha - k(l-2\alpha) \leqslant \tfrac{l}{2} \qquad \left(\text{resp. } k(l-2\alpha) \geqslant L - 2\alpha + \frac{l}{2}\right) \\ \Leftrightarrow \quad & ((k-1)l + 2\alpha) + l - 2\alpha - k(l-2\alpha) \leqslant \tfrac{l}{2} \\ \Leftrightarrow \quad & \alpha \leqslant \tfrac{l}{4k} \end{aligned}$$

Thus $\alpha = \frac{l}{4k}$ refines the previous value if $l > 1$ and $k \geqslant 2$. □

2 An algorithm deriving a set of k-bounded polyominoes from a random walk

2.1 Description of the algorithm

Let $w = w_1 w_2 \dots w_{2p-6}$ be a random walk made of an odd number of East steps ($\mathbf{e}$) and an odd number of North steps ($\mathbf{n}$). We give an algorithm to transform this walk into a set S_w of k-convex polyominoes of perimeter $2p$. The proof of the correctness of our algorithm will be given in the next section. In this algorithm, we cut the path into different pieces and apply reflections on each piece. For example, $\overleftarrow{w_i}$ is the symmetry along the vertical axis and can be seen as the transformation of $\mathbf{e}$ (resp. $\mathbf{w}$) step into $\mathbf{w}$ (resp. $\mathbf{e}$) step. Similarly, $\downarrow w_i$ is the reflection along the horizontal axis (exchange of $\mathbf{n}$ and $\mathbf{s}$ steps).

Algorithm 1: Function `BendWalk`

input : $w = w_1\, w_2\, \dots\, w_{2p-6}$ a random path made of North and East steps, $i \in \{1 \dots 2p-6\}$

output: A convex polyomino w' of perimeter $2p$

1 Let j maximal such that $|w_1, w_2, \dots, w_{j-1}|_{\mathbf{n}} = |w_j, w_{j+1}, \dots, w_{2p-6}|_{\mathbf{n}} + 1$;

2 Let k maximal such that $|w_1, w_2, \dots, w_{i-1}|_{\mathbf{e}} + |w_k, w_{k+2}, \dots, w_{2p-6}|_{\mathbf{e}} = |w_i, w_{i+1}, \dots, w_{k-1}|_{\mathbf{e}} + 1$;

3 $w' \leftarrow \mathbf{e} w_1\, w_2\, \dots\, w_{i-1}\, \mathbf{nw}\, \overleftarrow{w_i} \dots \overleftarrow{w_{j-1}}\, \mathbf{w}\, \lfloor\overleftarrow{w_j} \dots \lfloor\overleftarrow{w_{k-1}}\mathbf{s}\, \lfloor w_k \dots \lfloor w_{2p-6}\mathbf{s}$;

Algorithm 2: Function `Polyominoes`

input : $c \in [0,1]$, $w = w_1\, w_2\, \dots\, w_{2p-6}$ a path with deviation less than $c\sqrt{p}$, k the expected convexity

output: E_w a set of k-convex polyominoes of perimeter $2p$

1 $\alpha \leftarrow \frac{p}{4k^2+2}$, $l \leftarrow 4k\alpha$;

2 **foreach** i *in* $[\sqrt{2}(l - \alpha - \frac{\alpha}{6}), \sqrt{2}(l - \alpha + \frac{\alpha}{6})]$ **do** `BendWalk(`w`,`i`)`;

2.2 Correctness

The correctness of the algorithm is done in two steps. First, we prove that it outputs a set of polyominoes. Then, we prove that those polyominoes are asymptotically k-bounded, hence k-convex by Theorem 1.

Notice first that the obtained geometric shape is closed. This is a direct consequence of the form of w' in step 3 of function `BendWalk` and the conditions described in step 1 and 2. Moreover it is a polyomino as w' is a closed self-avoiding walk in the plane. Recalling that $w' \leftarrow \mathbf{se} w_1\, w_2\, \dots\, w_{i-1}\, \mathbf{nw}\, \overleftarrow{w_i} \dots \overleftarrow{w_{j-1}}\, \mathbf{w}\, \lfloor\overleftarrow{w_j} \dots \lfloor\overleftarrow{w_{k-1}}\mathbf{s}\, \lfloor w_k \dots \lfloor w_{2p-6}$, this defines four different segments, the first one beeing made of **e** and **n** steps, the next ones consisting in **n** and **w** steps, then **w** and **s** steps and finally **s** and **e** steps. For the path to intersect itself, the only possibilities are in each change of directions or between two opposite segments. The first possibility is forbidden by the insertion of some steps between each segment (printed in bold face). The second possibility cannot happen since the initial walk has a deviation bounded by $c\sqrt{p}$, which is less than half the distance (recall $l = \mathcal{O}(p)$) between opposite segments.

The second step is to prove that the obtained polyominoes are asymptotically k-bounded. In fact, we can prove this result for all output polyominoes, nevertheless, we only need an asymptotic behaviour which is easier to prove.

We consider the strip $\mathcal{S}$ of width α and outer rectangle of size $L \times l$ where $L = (k-1) + \frac{l}{2k}$. The starting point of our polyomino is S, the south point. Notice first that if $\hat{w}$ is the *alternating path* $(\mathbf{ne})^+$ and $i = l - \alpha$, the obtained polyomino is the closest to the median rectangle. Since the path has small deviation, this implies the following statistics on :

- the number n of steps between the south point, *i.e.* the origin, and each corner,
- the difference d between the number of **n** and **e** steps at the same points,

	S	E	N	W
n	0	$i \in [l - \alpha \pm \frac{\alpha}{6}]$	$j \in [p \pm \mathcal{O}(\sqrt{p})]$	$k \in [p + i \pm \mathcal{O}(\sqrt{p})]$
$\lvert d \rvert$	0	$\leqslant c\sqrt{i}$	$\mathcal{O}(\sqrt{p})$	$\mathcal{O}(\sqrt{p})$

For example, the point N separates the path into two parts, the first one having one more north step than the second one. Since the deviation is bounded by $c\sqrt{p}$, $w_1 \dots w_p$ contains nearly half of the north steps. This implies the result for j and for d. The same argument holds for point W.

Considering the alternating path $\hat{w}$ and a given i, it is straightforward to prove that the obtained polyomino $P(\hat{w}, i)$ stays within the strip and moreover the distance between this polyomino and the border of the strip is bounded by $\frac{\alpha}{6}$. To conclude our proof, given another path and a value i, the distance between the obtained polyomino and $P(\hat{w}, i)$ is bounded by the sum of all possible deviations at each corner that is $\mathcal{O}(\sqrt{p})$. Since $\alpha = \mathcal{O}(p)$, the polyomino lies asymptotically in the strip.

Theorem 2. *Algorithm* `BendWalk` *defines an injection between the set of random walks with a distinguished step and the set of convex polyominoes.*

Proof. In fact, given the output polyomino by function `BendWalk`(w, i), it is straightforward to recover the arguments w and i. To do so, notice that the added steps in the algorithm can be deduced from the points E, N, W, S of the polyomino. □

2.3 Enumeration of k-bounded polyominoes

As shown in Section 2.2, Algorithm `BendWalk` outputs a subset of k-bounded polyominoes. Hence we have:

Theorem 3. *The number of k-bounded, hence k-convex, polyominoes is asymptotically at least $\mu \frac{p}{k^2} 4^p$, where μ is a constant. Furthermore, the number of k-convex polyominoes is asymptotically $\mathcal{O}(p4^p)$.*

Proof. By Theorem 2, function `BendWalk` is injective and outputs one polyomino. The number of calls for a given walk w is exactly $\sqrt{2}\frac{\alpha}{3}$. The number of possible walks of length $2p - 6$ with deviation bounded by $c\sqrt{p}$ is $\lambda(c)2^{2p-6}$ by central-limit theorem. Taking the expression of α in p, the number of k-bounded polyominoes is asymptotically at least $\mu \frac{p}{k^2} 4^p$. □

3 Conclusion

The result provided is an asymptotic estimate of a lower bound of the number of k-convex polyominoes. The estimate is $\mathcal{O}(p4^p)$ for every value of k. A more precise result including the function $\mu(k)$ would be relevant but unfortunately our approach cannot approximate the correlation with k.

Nevertheless, the two graphs below seem to point out that k-convex polyominoes of semiperimeter p have an asymtotics in $\mathcal{O}(\frac{1}{k^2}(p-4)4^{p-4})$.

The following figures depict experimental results obtained by uniform sampling of convex polyominoes of semiperimeter $p = 302$. For each polyomino we compute its k-convexity and in the first diagram we draw the first ratios P_k/P_{k+1} between two consecutive numbers of k-convex polyominoes. The plot seems to have a finite limit when k increases. The second diagram plots $F(k)$ the number of k-convex polyominoes divided by $(p-4)4^{p-4}$ with respect to k. It seems to show that the asymptotic number of k-convex polyominoes is also in $\mathcal{O}(\frac{1}{k^2})$.

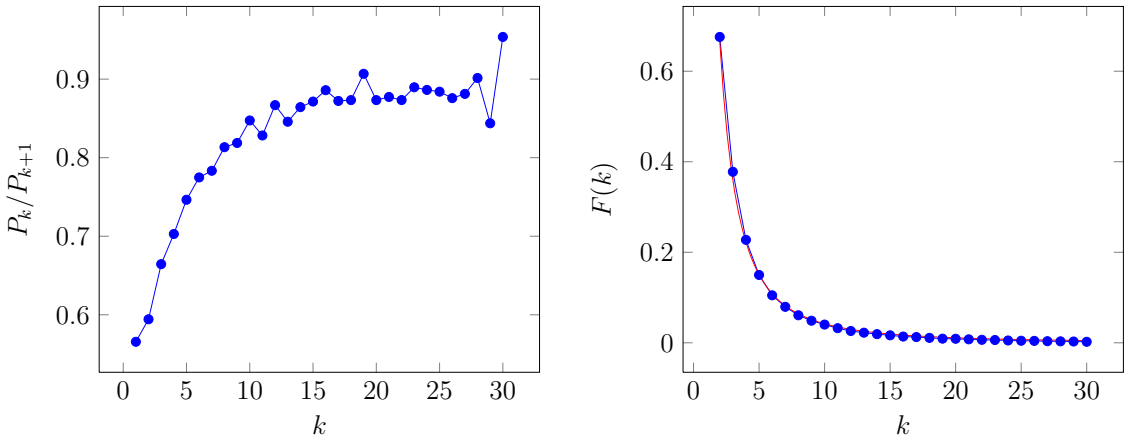

Figure 8: Experimental results on the asymptotic number of k-convex polyominoes

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