Multiplicative Partitions

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Abstract

New formulas for the multiplicative partition function are developed. Besides giving a fast algorithm for generating these partitions, new identities for additive partitions and the Riemann zeta function are also produced.

Keywords: Partitions, Riemann zeta function, sigma function, Gamma function, Selberg formula.

1 Introduction

A phenomenal amount of research has been conducted on the (additive) partition function over the last 100 years, with striking classical results due to Hardy, Ramanujan, and others. In contrast, the topic of multiplicative partitions — sometimes referred to as "factorisatio numerorum" — has received little attention. Counting the number of multiplicative partitions is a natural question since it lies between the two most common questions concerning primes: "Is n prime?" and "What is the prime factorization of n?"

Let a_n denote the number of multiplicative partitions of the natural number n. For example, $a_{12} = 4$ since

$$12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3 = 3 \cdot 4.$$

The sequence $\{a_n\}$ is listed in Sloane A001055. The Dirichlet generating function for this sequence is

$$f(s) = \prod_{k=2}^{\infty} \frac{1}{1 - k^{-s}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{1}$$

For special choices of n, the value of a_n can be determined in closed form. If $n = p^k$ where p is a prime, then $a_n = p(k)$, the number of additive partitions of the number k. If $n = p_1 p_2 \cdots p_k$, a product of k distinct primes, then $a_n = B(k)$, the k^{th} Bell number. Lastly, it is important to note that if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where $p_1, p_2, \cdots, p_k$ are distinct primes, then a_n depends only on $e_1, e_2, \ldots, e_k$. More on multiplicative partitions, including results on bounds and asymptotics of a_n and algorithms for calculating the values, can be found in [5], [7], [9], and [12].

The goal of this paper is to exploit the generating function (1) to determine new identities involving a_n. One of the new formulas can be implemented recursively to perform quick calculations. Moreover, connections are made to additive partitions and the Riemann zeta function.

2 Generating Function

It will be useful to consider the reciprocal of the generating function $f(s)$. Define

$$g(s) = \prod_{k=2}^{\infty} \left(1 - k^{-s}\right) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}.$$

The product $f(s)g(s)$ produces

$$\sum_{d|n} a_d b_{n/d} = \delta_{n,1} \tag{2}$$

where δ is the Kronecker delta function. An important auxiliary sequence is defined as $m_n = \sigma(r)/r$ where $r = \gcd(e_1, e_2, \ldots, e_k)$, $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, and σ is the sum of divisors function. For example, $m_{36} = 3/2$ since $36 = 2^2 \cdot 3^2$. We now offer several theorems which shed light on the generating function $f(s)$.

Theorem 1. *If $\Re s > 1$, then*

$$\log f(s) = \sum_{n=2}^{\infty} \frac{m_n}{n^s}. \tag{3}$$

Proof.

$$\begin{aligned}
\log f(s) &= -\sum_{k=2}^{\infty} \log(1 - k^{-s}) \\
&= \sum_{k=2}^{\infty} \left(k^{-s} + \frac{k^{-2s}}{2} + \frac{k^{-3s}}{3} + \cdots \right) \\
&= \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s}\left(1 + \frac{1}{2}\right) + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} \\
&\qquad + \frac{1}{8^s}\left(1 + \frac{1}{3}\right) + \frac{1}{9^s}\left(1 + \frac{1}{2}\right) + \frac{1}{10^s} + \cdots \\
&= \sum_{n=2}^{\infty} \left(\sum_{d|r} \frac{1}{d} \right) \frac{1}{n^s} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{r} \sum_{d|r} d \right) \frac{1}{n^s} \\
&= \sum_{n=2}^{\infty} \frac{m_n}{n^s}.
\end{aligned}$$

□

Equation (3) can be used to produce a more sophisticated relationship connecting a_n and b_n than equation (2). If p is a prime, let $\nu_p(n)$ denote the number of powers of p in the number n.

Theorem 2. *For any prime p and natural number n,*

$$\nu_p(n)m_n = \sum_{d|n} \nu_p(d)a_d b_{n/d}. \tag{4}$$

Proof.

$$\begin{aligned}(\log f(s))' &= f'(s) \cdot \frac{1}{f(s)} \\ &= -\left(\sum_{n=2}^{\infty} \frac{a_n \log n}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{b_n}{n^s}\right) \\ &= -\sum_{n=2}^{\infty} \frac{1}{n^s} \sum_{d|n} (\log d) a_d b_{n/d}.\end{aligned}$$

Integrating with respect to s and noting that the constant of integration must be zero by considering the terms as s tends to infinity, one finds

$$\log f(s) = \sum_{n=2}^{\infty} \frac{1}{n^s} \sum_{d|n} \left(\frac{\log d}{\log n}\right) a_d b_{n/d}.$$

Using equation (3), one has

$$(\log n)m_n = \sum_{d|n} (\log d) a_d b_{n/d}.$$

Writing n as its prime decomposition and applying ν_p, the linear independence of any finite subset of $\{\log(p) : \text{ p prime}\}$ implies

$$\nu_p(n)m_n = \sum_{d|n} \nu_p(d)a_d b_{n/d}.$$

□

There are now two formulas which relate a_n and b_n. For computational reasons, it would be useful to have a formula with only a_n terms.

Theorem 3. *For any prime p, one has*

$$\nu_p(n)a_n = \sum_{d|n} \nu_p(d)m_d a_{n/d}. \tag{5}$$

Proof. The proof involves both convolution and deconvolution. Starting with Theorem 2,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^s} \nu_p(n) m_n &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \nu_p(d) a_d b_{n/d} \\
&= \left(\sum_{n=1}^{\infty} \frac{\nu_p(n) a_n}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{b_n}{n^s} \right) \\
&= \frac{1}{f(s)} \sum_{n=1}^{\infty} \frac{\nu_p(n) a_n}{n^s}.
\end{aligned}$$

Rearranging, one has

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\nu_p(n) a_n}{n^s} &= f(s) \sum_{n=1}^{\infty} \frac{1}{n^s} \nu_p(n) m_n \\
&= \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^s} \nu_p(n) m_n \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \nu_p(d) a_{n/d} m_d.
\end{aligned}$$

Extracting the coefficients in the generating functions implies

$$\nu_p(n) a_n = \sum_{d|n} \nu_p(d) m_d a_{n/d}.$$

□

The generating function $f(s)$, particularly when written in product form, bears an obvious resemblance to the Riemann zeta function. With this in view, it is natural to look for parallels. The following result resembles the Selberg identity (see [2, p.46]).

Theorem 4. *For any natural number n,*

$$m_n (\log n)^2 = \sum_{d|n} a_d b_{n/d} (\log d)^2 - \sum_{d|n} m_d m_{n/d} (\log d)(\log n/d). \tag{6}$$

Proof.

$$\begin{aligned}
(\log f(s))'' &= \frac{f''(s)}{f(s)} - \left(\frac{f'(s)}{f(s)} \right)^2 \\
&= \left(\sum_{n=1}^{\infty} \frac{a_n (\log n)^2}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{b_n}{n^s} \right) - \left(\sum_{n=1}^{\infty} \frac{m_n \log n}{n^s} \right)^2 \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} a_d b_{n/d} (\log d)^2 - \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} m_d m_{n/d} (\log d)(\log n/d).
\end{aligned}$$

Integrating with respect to s, noting the constants of integration equal zero by considering the functions as s approaches infinity, and extracting coefficients in the power series yields the desired result. $\square$

3 Formulas for Additive Partitions

It was noted earlier that if $n = p^k$ for some prime k, then $a_n = p(k)$ where $p(k)$ is the number of additive partitions of the number k. While this paper's fundamental objective is to explore multiplicative partitions, the formulas of the last section enable one to derive identities for the additive partition.

Lemma 5. *If $n = p^k$ for some prime p, then $b_n = t_k$ where t_k is the coefficient of x^k in the function $\sum_{j=-\infty}^{\infty}(-1)^j x^{j(3j+1)/2} = \prod_{j=1}^{\infty}(1-x^j)$.*

Proof. Equation (2) can be written as

$$0 = \sum_{j=0}^{k} p(k-j) b_{p^j}.$$

A basic result of additive partitions [1] states

$$0 = \sum_{j=0}^{\infty} p(k-j) t_j.$$

An inductive argument using these two equations proves that $b_{p^j} = t_j$. $\square$

Theorem 6. *The following equations hold for all natural numbers k:*

$$k \cdot p(k) = \sum_{j=1}^{k} \sigma(j) p(k-j) \tag{7}$$

$$\sigma(k) = \sum_{j=1}^{k} j \cdot p(j) t_{k-j} \tag{8}$$

$$k \cdot \sigma(k) = \sum_{j=1}^{k} j^2 p(j) t_{k-j} - \sum_{j=1}^{k-1} \sigma(j)\sigma(k-j) \tag{9}$$

Proof. Let $n = p^k$ for some prime p. This forces $\nu_p(n) = k$ and $m_n = \sigma(k)/k$. Since the divisors of n take the form $d = p^j$, $j = 0, \ldots, k$, one has $\nu_p(d) = j$, $a_d = p(j)$, and Lemma 5 implies $b_{n/d} = t_{k-j}$. Substituting these values into equations (4), (5) and (6), one produces the three desired equations. $\square$

Equation (7) is well-known in the theory of partitions, equation (8) is essentially derived in [13], while equation (9) appears to be new.

4 Connections to the Riemann zeta function

The relationship between the generating functions $f(s)$ and $\zeta(s)$ is more than just a meta comparison. Let $\mu(n)$ denote the Möbius function.

Theorem 7. *If $\Re s > 1$, then*

$$\zeta(s) = 1 + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log f(ns). \tag{10}$$

Proof. Starting with work seen in the proof of equation (2), one has

$$\begin{aligned} \log f(s) &= \sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n \cdot (k^s)^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n} (\zeta(ns) - 1). \end{aligned}$$

Applying Möbius inversion concludes the proof. □

This new formula converges quickly. For large s,

$$\log f(ns) = \log \left(1 + \frac{1}{2^{ns}} + \frac{1}{3^{ns}} + \cdots \right) = \frac{1}{2^{ns}} + \cdots$$

Comparing successive terms in equation (10) — assuming the Möbius function is non-zero — their ratio approaches $1/2^s$ as $s \to \infty$. This implies that larger choices of s require fewer terms in the series to achieve the same accuracy. The formulas usually used to compute approximations of $\zeta(s)$ when s is odd (see [6]) are due to Ramanujan and the terms in the series have a ratio of $1/e^{2\pi}$. The ratio from equation (10) is smaller for $s > 10$. For an extensive article on computing $\zeta(s)$, see [4].

Using equation (3), one can construct a formula for $\zeta(s)$ which is independent of the multiplicative partitions:

$$\begin{aligned} \zeta(s) &= 1 + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log f(ns) \\ &= 1 + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=2}^{\infty} \frac{m_k}{k^{ns}} \\ &= 1 + \sum_{k=2}^{\infty} m_k \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{k^{ns}}. \end{aligned}$$

When $s > 1$ is an integer, a closed form expression for $f(ns)$ can be found.

Theorem 8. *If $n > 1$ is a natural number, then*

$$f(n) = \prod_{j=1}^{n-1} \Gamma(2 - \omega^j)$$

where $\Gamma(z)$ is the Gamma function and $\omega = \exp(2\pi i/n)$.

Proof. Starting with equation (1), repeated use of $\Gamma(x+1) = x\Gamma(x)$ produces

$$\begin{aligned} f(n) &= \lim_{N\to\infty} \prod_{k=2}^{N} \frac{k^n}{k^n - 1} \\ &= \lim_{N\to\infty} \prod_{k=2}^{N} \prod_{j=1}^{n} \frac{k}{k - \omega^j} \\ &= \lim_{N\to\infty} \prod_{j=1}^{n} \frac{\Gamma(N+1)\Gamma(2-\omega^j)}{\Gamma(N+1-\omega^j)} \\ &= \prod_{j=1}^{n-1} \Gamma(2-\omega^j) \lim_{N\to\infty} \prod_{j=1}^{n} \frac{\Gamma(N+1)}{\Gamma(N+1-\omega^j)} \end{aligned}$$

To evaluate the limit, one uses the lesser-known identity [3]

$$\lim_{n\to\infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1.$$

Then

$$\begin{aligned} \lim_{N\to\infty} \prod_{j=1}^{n} \frac{\Gamma(N+1)}{\Gamma(N+1-\omega^j)} &= \lim_{N\to\infty} \left(\prod_{j=1}^{n} \frac{1}{N^{-\omega^j}} \right) \left(\prod_{j=1}^{n} \frac{N^{-\omega^j}\Gamma(N+1)}{\Gamma(N+1-\omega^j)} \right) \\ &= \lim_{N\to\infty} \frac{1}{N^{-\omega-\omega^2-\cdots-\omega^{n-1}-1}} \cdot 1 \\ &= 1. \end{aligned}$$

□

When $s > 1$ is an integer, Theorem 8 may be used in conjunction with equation (10) to yield

$$\zeta(s) = 1 + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{j=1}^{ns-1} \log\Gamma(2-\omega^j). \tag{11}$$

where $\omega = \exp(2\pi i/ns)$.

5 Numerical Explorations

Values of a_n for small n can be found in Sloane A001055. Hughs and Shallit [8] built a recursive formula for a_n which employs the finer multiplicative partitions $a_{n,m}$, the number of partitions of n with all elements bounded by m. This approach was programmed by Knopfmacher and Mays [11] in Mathematica.

Using equation 5, we implemented a recursive algorithm which appears significantly faster than the previous approach. Two important features in the implementation should be mentioned. First, the recursive nature of this approach implies that the procedure to calculate a_n may be called many times for the same value of n, particularly if n is small. To avoid this duplication in calculations, the procedures which calculate a_n and m_n employed Maple's `remember` option which caches values and will not redo a calculation. Indeed, instead of using Maple's built-in function for $\nu_p(n)$, we constructed our own which also caches the values. Secondly, we took advantage of the observation that a_n depends only on the exponents in the prime decomposition of n. To calculate a_n, the value of n was replaced by the smallest number which had the same exponents (for some n, the new value is no smaller). These two features worked in tandem to blitz through the calculations: using Maple 14 on a Mac Book Pro, all the values of a_n for $n < 2,000,000$ were calculated in 141 CPU seconds.

As is evident from earlier formulas, an understanding of the sequence $\{a_n\}$ is bolstered by more insight into the sequence $\{b_n\}$. Continuing the metaphorical comparison of $f(s)$ to the Riemann zeta function, one compares b_n to the Möbius function $\mu(n)$. Initial calculations yield the following values for b_n:

Form of n	1	p_1	p_1^2	p_1p_2	p_1^3	$p_1^2p_2$	$p_1p_2p_3$	p_1^4	$p_1^3p_2$	$p_1^2p_2^2$	$p_1^2p_2p_3$
b_n	1	-1	-1	0	0	1	1	0	1	1	1

Just as the Möbius function takes only the values $\{-1, 0, 1\}$, the table suggests the values of b_n are similarly restricted. Lemma 5 extends this list to infinitely many cases, all with $b_n = \pm 1$. One more piece of evidence is of a combinatorial nature. Let d_n^e denote the number of multiplicative partitions of n into an even number of distinct parts and d_n^o the number of multiplicative partitions of n into an odd number of distinct parts. For example, since $12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3 = 3 \cdot 4$, $d_{12}^e = 2$ and $d_{12}^0 = 1$.

Theorem 9. *For all natural numbers n, $b_n = d_n^e - d_n^o$.*

Proof.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{b_n}{n^s} &= \prod_{k=2}^{\infty} \left(1 - k^{-s}\right) \\ &= \left(1 - 2^{-s}\right)\left(1 - 3^{-s}\right)\left(1 - 4^{-s}\right)\cdots \\ &= \sum_{n=1}^{\infty} \frac{d_n^e}{n^s} - \sum_{n=1}^{\infty} \frac{d_n^o}{n^s} \end{aligned}$$

□

Since one would expect a balance between the even and odd partitions, having $b_n \in \{-1, 0, 1\}$ seems reasonable. However, one finds (see [10]) that $b_{360} = -2$. Indeed, using the two million values of a_n which were computed with equation (2) found wilder extremes: the minimum value of b_n is -29 and the maximum value is 87. Alas, it seems the tight structure of the Möbius function does not extend to b_n.

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