Balanced vertex decomposable simplicial complexes and their h-vectors

Jennifer Biermann
Department of Mathematics and Statistics
Mt. Holyoke College
South Hadley, MA, USA
jbierman@mtholyoke.edu

Adam Van Tuyl
Department of Mathematical Sciences
Lakehead University
Thunder Bay, ON, Canada
avantuyl@lakeheadu.ca

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Abstract

Given any finite simplicial complex Δ, we show how to construct from a colouring χ of Δ a new simplicial complex Δ_χ that is balanced and vertex decomposable. In addition, the h-vector of Δ_χ is precisely the f-vector of Δ. Our construction generalizes the "whiskering" construction of Villarreal, and Cook and Nagel. We also reverse this construction to prove a special case of a conjecture of Cook and Nagel, and Constantinescu and Varbaro on the h-vectors of flag complexes.

Keywords: simplicial complex, vertex decomposable, flag complex, h-vector

1 Introduction

The work of this paper was inspired by the "whiskering" construction of finite simple graphs found in work of Villarreal [21] and Cook and Nagel [7]. Given a finite graph $G = (V_G, E_G)$ on the vertex set $V_G = \{x_1, \ldots, x_n\}$, Villarreal constructed a new graph, denoted G^W, on the vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ by adjoining the edges $\{x_i, y_i\}$ for every i to the graph G. The new graph has a "whisker" at every vertex of the original graph. As discovered by Villarreal, the edge ideal of the new graph G^W, that is,

$$I(G^W) = \langle w_i w_j \ \mid \{w_i, w_j\} \in E_{G^W} \rangle \subseteq R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

has the property that $R/I(G^W)$ is Cohen-Macaulay. It was later observed by Dochtermann and Engström [8] and Woodroofe [22], and generalized by Cook and Nagel [7], that one could deduce this result by studying the topological properties of the simplicial complex associated to $I(G^W)$ via the Stanley-Reisner correspondence. In particular, Villarreal's construction can be viewed as creating a new independence complex Δ' (sometimes called a flag complex) from the independence complex Δ of G. This new complex Δ' is vertex decomposable (as defined by Provan and Billera [18]), and it is this topological property that implies that $R/I(G^W)$ is Cohen-Macaulay.

Our entry point is to ask whether there is a more general theory that can be applied to all simplicial complexes. Moreover, we want this general theory to specialize to known cases for flag complexes. In Section 3 we will show that a general construction exists using the notion of a colouring χ of a simplicial complex Δ. From the colouring χ and complex Δ, we make a new complex, denoted Δ_χ. Regardless of how one colours Δ, the construction of Δ_χ always results in a balanced vertex decomposable simplicial complex (see Theorem 7). It should be noted that this construction has appeared in several guises over the years (see Remark 4 and Discussion 11).

The consequences of our results are explored in Section 4. In particular, it is shown that the f-vector of a simplicial complex is also the h-vector of a balanced vertex decomposable simplicial complex. Although this result is implicit in work of Björner, Frankl, and Stanley [3], to the best of our knowledge, this fact has not explicitly appeared in the literature (although the case of flag complexes occurs in [7]). We also show that the graded Betti numbers of the Stanley-Reisner ideal of the Alexander dual of Δ_χ can be expressed directly in terms of the f-vector of Δ (see Theorem 13).

Section 5 describes when our construction can be reversed, i.e., starting with a balanced vertex decomposable simplicial complex Δ, we construct another simplicial complex Δ' such that f-vector of Δ' is the same as the h-vector of Δ. We use this procedure to prove that the set of f-vectors of independence complexes of chordal graphs is precisely the set of h-vectors of balanaced vertex decomposable independence complexes of chordal graphs. Our result is a special case of a conjecture of Cook and Nagel [7] and Constantinescu and Varbaro [5] that the set of f-vectors of flag complexes is precisely the set of h-vectors of balanced vertex decomposable flag complexes.

As a final comment, we do not discuss the "whiskering" procedure found in [10] in which whiskers are added to only some of the vertices. This idea is explored in [2].

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2 Background

We work over the polynomial rings $S = k[x_1, \ldots, x_n]$ and $R = k[x_1, \ldots, x_n, y_1, \ldots, y_s]$ where k is any field. Let Δ be a finite simplicial complex on vertex set $\{x_1, \ldots, x_n\}$ of dimension d. We say Δ is *pure* if all its facets (maximal elements) have the same cardinality. An important combinatorial invariant of Δ is its f*-vector*, that is, the vector $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$ where f_i denotes the number of faces of Δ of dimension i.

If $\sigma \in \Delta$, then the *deletion* of σ is the simplicial complex $\Delta \setminus \sigma = \{\tau \in \Delta \mid \sigma \not\subseteq \tau\}$, and the *link* of σ is $\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$. When $\sigma = \{v\}$, we shall abuse notation and write $\Delta \setminus v$ (respectively $\mathrm{link}_\Delta(v)$). With this notation, we introduce a family of complexes due to Provan and Billera [18].

Definition 1. A pure simplicial complex Δ is called *vertex decomposable* if (*i*) Δ is a simplex, or (*ii*) there exits $v \in V$ such that $\Delta \setminus v$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable.

Key to the main construction studied in this paper is the notion of a colouring.

Definition 2. Let Δ be a simplicial complex on the vertex set V with facets $F_1, \ldots, F_t$. An *s-colouring* of Δ is a partition of the vertices $V = V_1 \cup \cdots \cup V_s$ (where the sets V_i are allowed to be empty) such that $|F_i \cap V_j| \leqslant 1$ for all $1 \leqslant i \leqslant t, 1 \leqslant j \leqslant s$. We will sometimes write χ is an s-colouring of Δ to mean χ is a specific partition of V that gives an s-colouring of Δ. If there exists an s-colouring, we say that Δ is *s-colourable*. If Δ has dimension $d-1$, then we say that Δ is *balanced* if it is d-colourable.

We will be interested in how our results specialize to independence complexes of graphs, sometimes called *flag complexes*. Recall that if $G = (V_G, E_G)$ is a finite simple graph with vertex set $V_G = \{x_1, \ldots, x_n\}$ and edge set E_G, then a subset $W \subseteq V_G$ is an *independent set* of a graph G if for every edge $e \in E_G$, we have $e \not\subseteq W$. The *independence complex* of G, denoted $\mathrm{Ind}(G)$, is the simplicial complex defined by

$$\mathrm{Ind}(G) = \{W \subseteq V_G \mid W \text{ is an independent set of } G\}.$$

3 A construction and its properties

Starting with a simplicial complex Δ and an s-colouring χ, we introduce a procedure to construct a new simplicial complex that is pure of dimension $s-1$, balanced, and vertex decomposable.

Construction 3. Let Δ be a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$. Given an s-colouring χ of Δ given by $V = V_1 \cup \cdots \cup V_s$, we define Δ_χ on vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_s\}$ to be the simplicial complex with faces $\sigma \cup \tau$ where σ is a face of Δ and τ is any subset of $\{y_1, \ldots, y_s\}$ such that for all $y_j \in \tau$ we have $\sigma \cap V_j = \emptyset$.

Remark 4. Construction 3 was introduced independently by Frohmader [11, Construction 7.1]. However, the construction appears implicitly in earlier work [3, Section 5], although it is only applied to compressed multicomplexes. Another variation appears in work of Hetyei (see [14, Definition 4.2]). The whiskering constructions found in [7, 21] for flag complexes become special cases of Construction 3. For example, Villarreal's construction in [21] of adding whiskers to every vertex of a graph G, and studying the resulting independence complex corresonds to the colouring $V = \{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_n\}$ and applying Construction 3 to $\mathrm{Ind}(G)$. We point the reader to [11] which makes the connection to Cook and Nagel's clique whiskering more explicit.

Observe that each s-colouring χ of Δ creates a new simplicial complex Δ_χ. Even though these simplicial complexes Δ_χ may be different, they all share some interesting properties.

Theorem 5. *The facets of Δ_χ are in one-to-one correspondence with the faces of the original simplicial complex Δ. In addition, Δ_χ is pure of dimension $s-1$ and balanced.*

Proof. Let $V = V_1 \cup \cdots \cup V_s$ be the colouring of Δ given by χ. From the definition of Δ_χ, the maximal faces are those of the form $\sigma \cup \{y_j \mid V_j \cap \sigma = \emptyset\}$ where σ is a face of Δ. This establishes the one-to-one correspondence.

If we partition the vertices of Δ_χ as $\{x_1, \ldots, x_n, y_1, \ldots, y_s\} = V_1' \cup V_2' \cup \cdots \cup V_s'$ where $V_j' = V_j \cup \{y_j\}$, then this partition gives an s-colouring of Δ_χ. We can see from the characterization of the facets of Δ_χ that each facet contains exactly one vertex from each of the sets $V_1', \ldots, V_s'$, and hence Δ_χ is pure of dimension $s-1$ as well as balanced. $\square$

Example 6. Let $\Delta = \langle x_1x_2x_3, x_2x_4, x_3x_4\rangle$ and let χ be the colouring given by $\{x_1, x_4\} \cup \{x_2\} \cup \{x_3\}$. The faces of Δ are $\{\emptyset, x_1, x_2, x_3, x_4, x_1x_2, x_2x_3, x_1x_3, x_2x_4, x_3x_4, x_1x_2x_3\}$. These are in one-to-one correspondence with the facets of Δ_χ:

$$\Delta_\chi = \langle y_1y_2y_3, x_1y_2y_3, x_2y_1y_3, x_3y_1y_2, x_4y_2y_3, x_1x_2y_3, x_2x_3y_1, x_1x_3y_2, x_2x_4y_3, x_3x_4y_2, x_1x_2x_3\rangle.$$

We come to the main result of this section.

Theorem 7. *For any simplicial complex Δ, and any s-colouring χ of Δ, the simplicial complex Δ_χ is vertex decomposable.*

Proof. We proceed by induction on the number of vertices of Δ. If Δ is the simplicial complex consisting of a single vertex x_1, then the only possible colourings of the vertices of Δ are of the form $V = V_1 \cup \cdots \cup V_s$ where $V_1 = \{x_1\}$ and $V_2, \ldots, V_s$ are empty. In this case $\Delta_\chi = \langle x_1y_2 \ldots y_s, y_1y_2 \ldots y_s\rangle$. This complex is vertex decomposable because $\Delta_\chi \setminus x_1 = \langle y_1y_2 \ldots y_s\rangle$ and $\operatorname{link}_{\Delta_\chi}(x_1) = \langle y_2 \ldots y_s\rangle$ are both simplices.

Now suppose that Δ is a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$, and let χ be the s-colouring of Δ given by $V = V_1 \cup \cdots \cup V_s$. We will show that we can decompose Δ_χ by decomposing at any vertex x_i. Let $g_1, \ldots, g_t$ be the faces of Δ and define $g_i' = \{y_j \mid V_j \cap g_i = \emptyset\}$. So $g_1 \cup g_1', \ldots, g_t \cup g_t'$ are the facets of Δ_χ.

We must show that both $\Delta_\chi \setminus x_i$ and $\operatorname{link}_{\Delta_\chi}(x_i)$ are vertex decomposable. First consider $\Delta_\chi \setminus x$. We assume that the facets of Δ_χ are ordered so that the facets $g_1 \cup g_1', \ldots, g_r \cup g_r'$ do not contain the vertex x_i and the facets $g_{r+1} \cup g_{r+1}', \ldots, g_t \cup g_t'$ do contain x_i. So

$$\Delta \setminus x_i = \{\text{faces of } \Delta \text{ which do not contain } x_i\} = \{g_1, \ldots, g_r\}.$$

Note that we are using the fact that $g_1, \ldots, g_r, g_{r+1}, \ldots, g_t$ is a complete list of the faces of Δ by Theorem 5.

Without loss of generality we may assume that $x_i \in V_1$. Then $V \setminus \{x_i\} = (V_1 \setminus \{x_i\}) \cup V_2 \cup \cdots \cup V_s$ is an s-colouring of $\Delta \setminus x_i$. Call this s-colouring χ'. Then $(\Delta \setminus x_i)_{\chi'} =$

$\langle (g_1 \cup g'_1), \ldots, (g_r \cup g'_r) \rangle = \Delta_\chi \setminus x_i$. Since $\Delta \setminus x_i$ is a simplicial complex on fewer than n vertices, $(\Delta \setminus x_i)_{\chi'}$ is vertex decomposable.

Now consider the link. Since $(g_{r+1} \cup g'_{r+1}), \ldots, (g_t \cup g'_t)$ are the facets of Δ_χ which contain x_i,

$$\begin{aligned}\operatorname{link}_{\Delta_\chi}(x_i) &= \langle (g_{r+1} \cup g'_{r+1}) \setminus \{x_i\}, \ldots, (g_t \cup g'_t) \setminus \{x_i\} \rangle \\ &= \langle ((g_{r+1} \setminus \{x_i\}) \cup g'_{r+1}), \ldots, ((g_t \setminus \{x_i\}) \cup g'_t) \rangle .\end{aligned}$$

For each $1 \leqslant j \leqslant s$, set $W_j = \{x_\ell \in V_j \mid x_\ell \in \operatorname{link}_\Delta(x_i)\}$. Note that some of these sets may be empty. Then $W = W_1 \cup \cdots \cup W_s$ is an s-colouring of $\operatorname{link}_\Delta(x_i)$. We call this s-colouring χ''. Then

$$(\operatorname{link}_\Delta(x_i))_{\chi''} = \operatorname{link}_{\Delta_\chi}(x_i)$$

and by induction $(\operatorname{link}_\Delta(x_i))_{\chi''}$ is vertex decomposable. □

The fact that Δ_χ is vertex decomposable has the following consequence.

Definition 8. A pure simplicial complex Δ is *shellable* if there is an ordering $F_1, \ldots, F_s$ on the facets of Δ such that for all $1 \leqslant i < j \leqslant s$ there exists some $v \in F_j \setminus F_i$ and some $\ell \in \{1, \ldots, j-1\}$ with $F_j \setminus F_\ell = \{v\}$. Such an ordering on the facets is called a *shelling order.*

Corollary 9. *For any simplicial complex Δ, and any s-colouring χ of Δ, the complex Δ_χ is shellable (and Cohen-Macaulay). Also, any order of the facets of Δ_χ which refines the order given by ordering the faces of Δ by increasing dimension is a shelling order.*

Proof. By Theorem 7, Δ_χ is vertex decomposable, so by [18, Corollary 2.9] it is also shellable, and consequently, Cohen-Macaulay (e.g, see [15, Theorem 8.2.6]).

For the rest, let $F_1, \ldots F_s$ be the facets of Δ_χ. By Theorem 5, each $F_i = g_i \cup g'_i$ where g_i is a face of Δ and $g'_i = \{y_j \mid V_j \cap g_i = \emptyset\}$. We order the facets $F_1, \ldots F_s$ so that $\dim g_i \leqslant \dim g_j$ if $i < j$. We now show that this is a shelling order.

Let F_i, F_j be any two distinct facets of Δ_χ with $i < j$. Since $i < j$, we have $\dim g_i \leqslant \dim g_j$ and so there is some $x_u \in F_j \setminus F_i$. Since $g_j \setminus \{x_u\}$ is a face of Δ we have $g_j \setminus \{x_u\} = g_\ell$ for some ℓ, and since $\dim g_\ell < \dim g_j$ we have $\ell < j$. Since $g_\ell = g_j \setminus \{x_u\}$ we must have $F_\ell = g_\ell \cup g'_\ell \in \Delta_\chi$ where $g'_\ell = g'_j \cup \{y_w\}$ where $x_u \in V_w$. Then

$$F_j \setminus F_\ell = (g_j \cup g'_j) \setminus (g_\ell \cup g'_\ell) = (g_j \cup g'_j) \setminus ((g_j \setminus \{x_u\}) \cup (g'_j \cup \{y_w\})) = \{x_u\} .$$

Thus our ordering is a shelling order. □

4 Consequences: h-vectors and Betti numbers

In this section, we explore some consequences of Theorem 7 for h-vectors. The h-vector $(h_0, h_1, \ldots, h_{d+1})$ of a d-dimensional simplicial complex Δ, denoted $h(\Delta)$, is defined in terms of the f-vector $f(\Delta) = (f_{-1}, f_0, \ldots, f_d)$ as follows

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Delta) .$$

We use Theorem 7 to obtain a new proof of one implication result of [3]. Note that for brevity, we have omitted the definition of a colour-shifted simplicial complex. See [12] for the definition and the discussion after the proof for some additional comments.

Theorem 10. *Let $m = (m_1, \dots, m_t) \in \mathbb{Z}^t_+$. Then the following are equivalent:*

(i) *m is the f-vector of a simplicial complex.*

(ii) *m is the f-vector of a colour-shifted simplicial complex.*

(iii) *m is the h-vector of a balanced, vertex decomposable simplicial complex.*

(iv) *m is the h-vector of a balanced, shellable simplicial complex.*

(v) *m is the h-vector of a balanced, Cohen-Macaulay simplicial complex.*

Proof. The equivalence of (i), (ii), (iv), and (v) appear in [3]. A alternative proof can be found in [1]. The implication of $(iii) \Rightarrow (iv)$ follows from the fact that vertex decomposability implies shellabilty. Our contribution is $(i) \Rightarrow (iii)$. Let $f(\Delta)$ be the f-vector of a simplicial complex Δ. For any s-colouring χ of Δ, Δ_χ is a balanced vertex decomposable simplicial complex by Theorems 5 and 7. Then we have $h(\Delta_\chi) = f(\Delta)$. Indeed, one uses the standard technique of using the shelling of Δ_χ (as given by Corollary 9) to find $h(\Delta_\chi)$ using [20, Proposition III.2.3]. One then uses the correspondence of Theorem 5 between the faces of Δ and the facets of Δ_χ to relate the h-vector back to $f(\Delta)$. In essence, the proof of [7, Theorem 3.8] for clique-whiskering generalizes in the natural way. □

Discussion 11. Theorem 10 is stated as a theorem about simplicial complexes, but [3] addressed the more general case of multi-complexes. Furthermore, in [3], colour-shifted complexes are called compressed complexes. The strategy behind the orginal proof of Theorem 10 is to show that the f-vector of a simplicial complex is also the f-vector of some colour-shifted simplicial complex (the procedure for building this new colour-shifted complex is iterative in nature). Then, from this colour-shifted simplicial complex, a construction like Construction 3 is used to build a simplicial complex that is balanced and shellable whose h-vector is the same as the f-vector of the colour-shifted complex. One can show that if one applies Construction 3 to a colour-shifted simplicial complex, one has a vertex decomposable simplicial complex. Therefore, the implication $(ii) \Rightarrow (iii)$ is implicit in Theorem 10. [3, Comment 6.5] states, without proof, that one can prove $(i) \Rightarrow (iv)$ for simplicial complexes without having to pass through colour-shifted complexes. One assumes that the omitted proof uses something like Construction 3. What we have made explicit is that one can prove $(i) \Rightarrow (iii)$ directly without the need of colour-shifted complexes, and at the same time, identify the f-vectors of a simplicial complexes with a smaller subset of h-vectors of balanced complexes.

It has been asked whether the above statements of Theorem 10 still hold if we restrict to the class of flag complexes. In particular, Cook and Nagel [7], and Constantinescu and Varbaro [5] have posited the following conjecture (the conjecture of Cook and Nagel does not include the word balanced):

Conjecture 12. The following equality of sets holds:

$$\left\{ \begin{array}{c} f\text{-vectors of} \\ \text{flag complexes} \end{array} \right\} = \left\{ \begin{array}{c} h\text{-vectors of balanced} \\ \text{vertex decomposable flag complexes} \end{array} \right\}.$$

The set of f-vectors of flag complexes has been shown to be a subset of the set of h-vectors of balanced vertex decomposable flag complexes. We omit the proof here, but instead point the reader to the proofs of [7, Corollary 3.10] and [5, Proposition 4.1]. The second proof is interesting since the authors use basically the same construction as Construction 3, but in the special case that the colouring is given by the partition $V = \{x_1\} \cup \cdots \cup \{x_n\}$. In some special cases, e.g., bipartite graphs (see [7]), the conjecture has been proved. We add additional evidence for Conjecture 12 in the next section.

We conclude this section by showing how to use Theorem 10 to find the graded Betti numbers of the Alexander dual of the Stanley-Reisner ideal associated to I_{Δ_χ}. Recall that the *Stanley-Reisner ideal* of Δ is the monomial ideal

$$I_\Delta = \langle x_{i_1} x_{i_2} \cdots x_{i_s} \mid \{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\} \notin \Delta \rangle \subseteq S = k[x_1, \ldots, x_n].$$

The *Alexander dual* of a Δ, denoted $\Delta^\vee$, is the simplicial complex $\Delta^\vee = \{\overline{\sigma} \mid \sigma \notin \Delta\}$. Here, given $\sigma \subseteq \{x_1, \ldots, x_n\}$, we let $\overline{\sigma} = \{x_1, \ldots, x_n\} \setminus \sigma$. We then have:

Theorem 13. *Let $(f_{-1}, f_0, \ldots, f_d)$ be the f-vector of a d-dimensional simplicial complex Δ on $V = \{x_1, \ldots, x_n\}$, and let χ be any s-colouring of Δ. The graded Betti numbers of $I_{\Delta_\chi^\vee}$ in R are given by the formula*

$$\beta_{i,i+n}(I_{\Delta_\chi^\vee}) = \sum_{j=i}^{d+1} \binom{j}{i} f_{j-1}(\Delta).$$

In particular, $\text{proj-dim}(I_{\Delta_\chi^\vee}) = \text{reg}(R/I_{\Delta_\chi}) = d+1$.

Proof. The projective dimension follows directly from our formula, and for the regularity, we use the identity (e.g., see [15, Proposition 8.1.10]) that $\text{proj-dim}(I_{\Delta^\vee}) = \text{reg}(R/I_\Delta)$.

Because Δ_χ is pure and vertex decomposable (and thus shellable), [9, Corollary 5] gives

$$\sum_{i \geqslant 1} \beta_i(R/I_{\Delta_\chi^\vee}) t^{i-1} = \sum_{i \geqslant 0} h_i(\Delta_\chi)(t+1)^i. \quad (1)$$

Note that in [9], the authors are taking the resolution of $R/I_{\Delta_\chi^\vee}$, so $\beta_i(R/I_{\Delta_\chi^\vee}) = \beta_{i-1}(I_{\Delta_\chi^\vee})$. Furthermore, although the formula of [9] is expressed in terms of total graded Betti numbers, the resolution of $I_{\Delta_\chi^\vee}$ is linear (this is because Δ_χ is shellable and pure of dimension $s-1$, and hence $I_{\Delta_\chi^\vee}$ is generated in degree n and is componentwise linear, which implies the ideal has a linear resolution). We therefore have $\beta_{i-1}(I_{\Delta_\chi^\vee}) = \beta_{i-1,n+i-1}(I_{\Delta_\chi^\vee})$.

To finish the proof, Theorem 10 allows us to replace $h_i(\Delta_\chi)$ with $f_{i-1}(\Delta)$ in the formula (1), thus giving the desired formula for $\beta_{i-1,n+i-1}(I_{\Delta_\chi^\vee})$. □

Remark 14. For any valid f-vector $f(\Delta) = (f_0, \ldots, f_d)$, the sequence

$$\left(\sum_{j=0}^{d+1}\binom{j}{0}f_{j-1}(\Delta), \sum_{j=1}^{d+1}\binom{j}{1}f_{j-1}(\Delta), \ldots, \sum_{j=d+1}^{d+1}\binom{j}{d+1}f_{j-1}(\Delta)\right) \tag{2}$$

is a valid sequence of Betti numbers for an ideal with a linear resolution by Theorem 13. Herzog, Sharifan, and Varbaro [17] classified all valid sequences of Betti numbers for an ideal with a linear resolution. While one can deduce that (2) is valid sequence of Betti numbers from [17], our Theorem 13 highlights how to start with a simplicial complex with a given f-vector, and find a square-free monomial ideal whose graded linear resolution has Betti sequence given by (2). This contrasts with [17] since the ideal they construct with Betti sequence (2) need not be a square-free monomial ideal.

5 Application: independence complexes of chordal graphs

In this section we provide new evidence for Conjecture 12. We give a condition to "reverse" Construction 3, and then apply it to independence complexes of chordal graphs.

The *restriction of* Δ to $W \subseteq V$ is the subcomplex $\Delta|_W = \{F \in \Delta \mid F \subseteq W\}$. Our criterion for "reversing" the process of the last section requires the following terminology.

Definition 15. Suppose $\Delta = \langle F_1, \ldots, F_s\rangle$ is a simplicial complex on the vertex set V. We say that Δ has a *facet restriction with respect to* F if F is a facet of Δ such that

$$\Delta|_{V\setminus F} = \{F_1 \setminus F, \ldots, F_s \setminus F\}.$$

Note that the inclusion $\Delta|_{V\setminus F} \supseteq \{F_1 \setminus F, \ldots, F_s \setminus F\}$ always holds; however, in general the two sets may not be equal as we see in the following example.

Example 16. Let $\Delta = \langle 123, 234, 345, 456\rangle$ (to simplify notation, we are writing i for x_i). By considering each facet of Δ, we can show it has no facet restriction. Let F be the facet 123. Then

$$\Delta|_{V\setminus F} = \{\emptyset, 4, 5, 6, 45, 56, 46, 456\} \neq \{123 \setminus F, 234 \setminus F, 345 \setminus F, 456 \setminus F\} = \{\emptyset, 4, 45, 456\}.$$

Similarly, if we consider the facet 234 we see that

$$\Delta|_{V\setminus 234} = \Delta|_{156} = \{\emptyset, 1, 5, 6, 56\} \neq \{123\setminus 234, 234\setminus 234, 345\setminus 234, 456\setminus 234\} = \{1, \emptyset, 5, 56\}.$$

By symmetry, the facets 345 and 456 also fail to give a facet restriction. Therefore the simplicial complex Δ has no facet restriction.

Example 17. Let Δ be the simplicial complex $\langle 124, 245, 235, 456\rangle$. Then Δ has a facet restriction with respect to the facet 245 since

$$\Delta|_{V\setminus 245} = \Delta|_{136} = \{\emptyset, 1, 3, 6\} = \{124 \setminus 245, 245 \setminus 245, 235 \setminus 245, 456 \setminus 245\}.$$

The existence of a facet restriction allows us to "reverse" our proof of Theorem 10 $(i) \Rightarrow (iii)$. In particular, we show that if Δ has facet restriction, we can make a new simplicial complex whose f-vector is the same of $h(\Delta)$.

Theorem 18. *Let $\Delta = \langle F_1, \ldots, F_t \rangle$ be a pure, balanced simplicial complex such that Δ has a facet restriction with respect to the facet F. Then $\Delta = (\Delta|_{V \setminus F})_\chi$ where χ is the colouring induced from the colouring of Δ. In particular, Δ is vertex decomposable and $h(\Delta) = f(\Delta|_{V \setminus F})$.*

Proof. Let $d-1$ be the dimension of Δ. Because Δ is pure and balanced, the colouring χ is given by a partition $V = V_1 \cup V_2 \cup \cdots \cup V_d$ such that $|F_j \cap V_i| = 1$ for all $1 \leqslant j \leqslant t$ and $1 \leqslant i \leqslant d$. After relabelling, we can assume that F_1 is the facet that gives the facet restriction. Note that $\Delta|_{V \setminus F_1}$ is a simplicial complex on $Y = V \setminus F_1$, and is d-colourable since $\Delta|_{V \setminus F_1}$ inherits a colouring from χ given by:

$$Y = V \setminus F_1 = (V_1 \setminus F_1) \cup (V_2 \setminus F_1) \cup \cdots \cup (V_d \setminus F_1).$$

Abusing notation, let χ denote this new colouring. Then $(\Delta|_{V \setminus F_1})_\chi$ is a balanced vertex decomposable simplicial complex such that $h((\Delta|_{V \setminus F_1})_\chi) = f(\Delta|_{V \setminus F_1})$ by Theorem 10.

To complete the proof, it suffices to show that $(\Delta|_{V \setminus F_1})_\chi$ and Δ are the same simplicial complexes, but with a different labelling of the vertices. By Theorem 5, the facets of $(\Delta|_{V \setminus F_1})_\chi$ are in one-to-one correspondence with the faces of $\Delta|_{V \setminus F_1}$. But we also have that the facets of Δ are in one-to-one correspondence with the faces of $\Delta|_{V \setminus F_1}$ via the map $F_i \mapsto F_i \setminus F_1$. Indeed, this map is clearly onto by our assumption that Δ has a facet restriction with respect to F_1. It suffices to show that this map is one-to-one. So, suppose $F_i \setminus F_1 = F_j \setminus F_1$, but $F_i \neq F_j$. This means that there is a vertex $x \in F_i \setminus F_j$ because the simplicial complex is pure. Since Δ is balanced, there is a vertex $y \in F_j \setminus F_i$ with the same colour as x. Because $F_i \setminus F_1 = F_j \setminus F_1$, we must have x and y in F_1. But this contradicts the colouring of Δ. By combining these two one-to-one correspondences, we get the desired bijection between the facets of Δ and $(\Delta|_{V \setminus F_1})_\chi$. ☐

Example 19. In Example 17 we saw that $\Delta = \langle 124, 245, 235, 456 \rangle$ has a facet restriction with respect to the facet 245. Since $\Delta|_{136} = \{\emptyset, 1, 3, 6\}$, the f-vector of $\Delta|_{136}$ is $f(\Delta|_{136}) = (1, 3)$. Therefore the h-vector of Δ is $h(\Delta) = f(\Delta|_{136}) = (1, 3)$.

Recall that a graph G is *chordal* if every induced cycle of G of length $\geqslant 4$ has a chord. We will prove the following fact about the independence complexes of chordal graphs.

Lemma 20. *Let $\Delta = \mathrm{Ind}(G)$ be the independence complex of a chordal graph G. If Δ is also pure, then Δ has a facet restriction.*

Theorem 18 and Lemma 20 combine to prove a special case of Conjecture 12.

Theorem 21. *We have the the following equivalence of sets:*

$$\left\{ \begin{array}{c} f\text{-vectors of independence} \\ \text{complexes of chordal graphs} \end{array} \right\} = \left\{ \begin{array}{c} h\text{-vectors of balanced, vertex decomposable} \\ \text{independence complexes of chordal graphs} \end{array} \right\}.$$

Proof. If $f(\Delta)$ is the f-vector of $\Delta = \text{Ind}(G)$ when G is chordal, then for any colouring χ of $\Delta = \text{Ind}(G)$, the simplicial complex Δ_χ is balanced and vertex decomposable by Theorem 7, and $f(\Delta) = h(\Delta_\chi)$ by Theorem 10. It remains to explain why Δ_χ is the independence complex of a chordal graph. As noted in Remark 4, in the case of independence complexes, Construction 3 is the same as the clique wiskering construction of Cook and Nagel [7]. Furthermore, the clique wiskering construction of a chordal graph produces a new chordal graph, so Δ_χ is the independence complex of a chordal graph.

To show the reverse containment, let G be any chordal graph such that $\Delta = \text{Ind}(G)$ is balanced and vertex decomposable. Because Δ is vertex decomposable, and thus pure, by Lemma 20, Δ has a facet restriction with respect to some facet F. Then by Theorem 18, we have $h(\Delta) = f(\Delta|_{V\setminus F})$. To complete the argument, we note that

$$\Delta|_{V\setminus F} = \text{Ind}(G)|_{V\setminus F} = \text{Ind}(G|_{V\setminus F}).$$

The graph $G|_{V\setminus F}$ is an induced subgraph of a chordal graph, and so is a chordal graph. So $h(\Delta) = f(\text{Ind}(G|_{V\setminus F}))$, thus completing the proof. □

To prove Lemma 20 we will require a result of Herzog, Hibi, and Zheng about the clique complex. For any finite simple graph $G = (V_G, E_G)$ the *clique complex* of G is the simplicial complex $Cl(G) = \{C \subseteq V \mid G|_C \text{ is a clique}\}$. Below, a vertex $v \in V$ is a *free vertex* if v is contained in exactly one facet of Δ.

Theorem 22 ([16, Theorem 2.1]). *Let G be a chordal graph and let $C_1, \ldots, C_t$ be all the facets of $Cl(G)$ that contain a free vertex. The following are equivalent:*

- *(a) G is unmixed, i.e., all maximal independent sets have the same cardinality.*
- *(b) $V = C_1 \cup C_2 \cup \cdots \cup C_t$ is a partition of the vertices of G.*

We are now ready to prove Lemma 20.

Proof. (of Lemma 20) Let $\Delta = \text{Ind}(G)$ be the independence complex of a chordal graph, and furthermore, assume Δ is pure. Let $C_1, \ldots, C_t$ be the facets of $Cl(G)$ which contain a free vertex. Since Δ is pure, we know that G is unmixed. Thus by Theorem 22, we have the partition $V = C_1 \cup \cdots \cup C_t$. For $1 \leqslant i \leqslant t$, let y_i be a free vertex of $Cl(G)$ contained in C_i. Set $F = \{y_1, \ldots, y_t\}$. We will show that F is a facet of Δ and that Δ has a facet restriction with respect to F.

It is clear that $F = \{y_1, \ldots, y_t\}$ is an independent set since each y_i is in a unique maximal clique C_i and an edge $\{y_i, y_j\}$ would constitute a clique of size 2. Further, F is a maximal independent set since every vertex $x \notin F$ is in some C_i and therefore adjacent to some y_i. Since Δ is assumed to be pure, this means that every facet has size t.

Next let $F_1, \ldots, F_s$ be the facets of Δ. To finish the proof we will show that

$$\Delta|_{V\setminus F} = \{F_1 \setminus F, \ldots, F_s \setminus F\}.$$

We simply need to show $\Delta|_{V\setminus F} \subseteq \{F_1 \setminus F, \ldots, F_s \setminus F\}$. Let $H \in \Delta|_{V\setminus F}$, and define $H' = H \cup \{y_i \mid C_i \cap H = \emptyset\}$. Then H' is independent since the neighbours of y_i are the elements of $C_i \setminus \{y_i\}$. Since H' has cardinality t, it is a facet of Δ. Therefore $H = H' \setminus F$ which proves that $\Delta|_{V\setminus F} = \{F_1 \setminus F, \ldots, F_s \setminus F\}$. □

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