Regularity of join-meet ideals of distributive lattices

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Abstract

Let L be a distributive lattice and $R(L)$ the associated Hibi ring. We compute $\operatorname{reg} R(L)$ when L is a planar lattice and give bounds for $\operatorname{reg} R(L)$ when L is non-planar, in terms of the combinatorial data of L. As a consequence, we characterize the distributive lattices L for which the associated Hibi ring has a linear resolution.

Keywords: Binomial ideals, distributive lattices, regularity

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Introduction

Let L be a finite distributive lattice and $K[L]$ the polynomial ring over a field K. The *join-meet* or *Hibi ideal* of L, denoted I_L, is generated by all the binomials $f_{ab} = ab-(a\vee b)(a\wedge b)$ where $a, b \in L$ are incomparable. The *Hibi ring* of L is $R(L) = K[L]/I_L$. $R(L)$ is a Cohen-Macaulay normal domain as it was shown in [6]. Its properties were investigated in [6], [7], [8]. The Gröbner bases of I_L with respect to various monomial orders have been studied; see, for instance, [1], [5], [6], [11].

Hibi rings are a very natural class of objects in combinatorial commutative algebra, and they have nice connections to representation theory and other fields; see, e.g., [9].

Our aim is to study the regularity of $R(L)$ for a distributive lattice L. When L is a planar lattice, we give the regularity formula in Theorem 4 in terms of the combinatorics of the lattice. For non-planar lattices, we show in Theorem 8 that $\operatorname{reg} R(L)$ is greater than or equal to the maximal number of pairwise incomparable join-irreducible elements minus 1 and smaller than or equal to the number of join-irreducible elements minus 1. These two results enable us to derive that I_L has a 2-linear resolution if and only if L is the divisor lattice of $2 \cdot 3^a$ for some $a \geqslant 1$; see Corollary 10. For other nice properties of this lattice we refer to [5].

Main Results

Let L be a finite distributive lattice of rank $d+1$ where d is a positive integer, and $K[L]$ the polynomial ring over a field K. Let I_L be the join-meet ideal of L and $R(L) = K[L]/I_L$.

Throughout this paper we assume that the lattice L is *simple*, that is, it has no cut edge. By a *cut edge* of L we mean a pair (a, b) of elements of L with $\operatorname{rank}(b) = \operatorname{rank}(a)+1$ such that

$$|\{c \in L\colon\ \operatorname{rank}(c) = \operatorname{rank}(a)\}| = |\{c \in L\colon\ \operatorname{rank}(c) = \operatorname{rank}(b)\}| = 1.$$

In particular, a simple distributive lattice of rank $d + 1$ has at least two elements of rank 1 and at least two elements of rank d.

There is no loss of generality in making this assumption. Let us suppose that L has a cut edge (a, b). Then it is clear that $I_L = I_{L_1} + I_{L_2}$ where L_1 is the sublattice of L consisting of all elements $c \in L$ such that $c \leqslant a$, and L_2 is the sublattice of L consisting of all elements $c \in L$ such that $c \geqslant b$. Since I_{L_1} and I_{L_2} are ideals generated by binomials in disjoint sets of variables, we get $R(L) = R(L_1) \otimes R(L_2)$ which implies that $\operatorname{reg} R(L) = \operatorname{reg} R(L_1) + \operatorname{reg} R(L_2)$.

By Theorem 10.1.3 in [4], we know that the generators of I_L form a Gröbner basis of I_L with respect to the reverse lexicographic order on $K[L]$. Consequently, the initial ideal of I_L is generated by all the squarefree monomials ab where $a, b \in L$ are incomparable elements. This implies that the Hilbert series $H_{R(L)}(t)$ of $R(L)$ coincides with the Hilbert series of the Stanley-Reisner ring $K[\Delta(L)]$ where $\Delta(L)$ is the order complex of L, that is, the simplicial complex whose facets are the maximal chains of L. In particular, $R(L)$ and

$K[\Delta(L)]$ have the same h-vector $h_{R(L)}$. Since $R(L)$ is Cohen-Macaulay, we may choose in $R(L)$ a regular sequence of linear forms, $\mathbf{u} = u_1, \ldots, u_{\dim R(L)}$. Then $R(L)$ and $R(L)/\mathbf{u}R(L)$ have the same h–vector. By [10, Theorem 20.2], we have $\operatorname{reg} R(L) = \operatorname{reg}(R(L)/\mathbf{u}R(L))$, and, since $\dim(R(L)/\mathbf{u}R(L)) = 0$, the regularity of $R(L)/\mathbf{u}R(L)$ is given by the degree of its h–vector [3, Exercise 20.18]. Consequently, $\operatorname{reg} R(L) = \deg h_{R(L)}$.

The coefficients of $h_{R(L)} = h_{K[\Delta(L)]}$ have a nice combinatorial interpretation which we are going to recall below.

Let P be the subposet of L of the join-irreducible elements. By Birkoff's Theorem, L equals the distributive lattice $\mathcal{I}(P)$ of all poset ideals of P. If $|P| = d+1$ for some positive integer d, then $\operatorname{rank} L = d+1$ and $\dim(R(L)) = d+2$.

By [2] or [12, Section 2], we have

$$h_{K[\Delta(L)]}(t) = \sum_{S \subset [d]} \beta(S) t^{|S|} \tag{1}$$

where $\beta(S)$ is the number of the linear extensions of the poset P whose descent set is S. We recall that if $\pi = (a_1, \ldots, a_{d+1})$ is a permutation of $[d+1]$, then the descent set of π is defined by $\mathcal{D}(\pi) = \{i : \ a_i > a_{i+1}\}$.

By [2, Section 2], the number $\beta(S)$ may be also interpreted as follows. Let λ be an edge-labeling of L. This means that each edge $x \to y$ in the Hasse diagram of L has a label $\lambda(x \to y)$. Here $x \to y$ means that y covers x in L. Then each chain in L, say $x_0 \to x_1 \to x_2 \to \cdots \to x_k$, is labeled by the k-tuple $(\lambda(x_0 \to x_1), \ldots, \lambda(x_{k-1} \to x_k))$. We compare two such k-tuples, say $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$, lexicographically, that is, $(a_1, \ldots, a_k) >_{\text{lex}} (b_1, \ldots, b_k)$ if the most-left nonzero component of the vector $(a_1 - b_1, \ldots, a_k - b_k)$ is positive.

Definition 1 ([2]). *The edge-labeling λ of L is called an EL-labeling if for every interval $[x, y]$ in L:*

- (i) *there is a unique chain $\mathfrak{c} : x = x_0 \to x_1 \to \cdots \to x_k = y$ such that $\lambda(x_0 \to x_1) \leqslant \lambda(x_1 \to x_2) \leqslant \ldots \leqslant \lambda(x_{k-1} \to x_k)$;*
- (ii) *for every other chain $\mathbf{b} : x = y_0 \to y_1 \to \cdots \to y_k = y$ we have $\lambda(\mathbf{b}) >_{\text{lex}} \lambda(\mathfrak{c})$.*

For a maximal chain $\mathfrak{c} : \min L = x_0 \to x_1 \to \cdots \to x_{d+1} = \max L$ in L, we define the *descent set* $\mathcal{D}(\mathfrak{c}) = \{i \in [d] : \lambda(x_{i-1} \to x_i) > \lambda(x_i \to x_{i+1})\}$.

We recall now Theorem 2.2 in [2].

Theorem 2. *[2] Let L be a graded poset of rank $d+1$. For $S \subset [d]$, $\beta(S)$ equals the number of maximal chains $\mathfrak{c}$ in L such that $\mathcal{D}(\mathfrak{c}) = S$.*

Planar distributive lattices

Let $\mathbb{N}^2$ be the infinite distributive lattice of all the pairs (i, j) where i, j are nonnegative integers. The partial order is defined as $(i, j) \leqslant (k, \ell)$ if $i \leqslant k$ and $j \leqslant \ell$. A *planar distributive lattice* is a finite sublattice L of $\mathbb{N}^2$ with $(0, 0) \in L$ which has the following property: for any $(i,j),(k,\ell) \in L$ there exists a chain $\mathfrak{c}$ in L of the form $\mathfrak{c}: x_0 < x_1 < \cdots < x_t$ with $x_s = (i_s, j_s)$ for $0 \leqslant s \leqslant t$, $(i_0, j_0) = (i,j)$, and $(i_t, j_t) = (k,\ell)$, such that $i_{s+1} + j_{s+1} = i_s + j_s + 1$ for all s.

In the planar case, we may compute the regularity of $R(L)$ in terms of the cyclic sublattices of L. A sublattice of L is called *cyclic* if it looks like in Figure 1 with some possible cut edges in between the squares. By a *square* in L we mean a sublattice with elements a,b,c,d such that $a \to b \to d$, $a \to c \to d$, and b,c are incomparable.

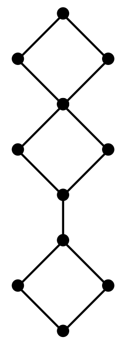

Figure 1: Cyclic sublattice

Lemma 3. *Let C be a cyclic lattice with r squares. Then* $\operatorname{reg} R(C) = r$.

Proof. I_C is generated by a regular sequence of length r since $\operatorname{in}_{\text{rev}}(I_C)$ is generated by a regular sequence of monomials. Therefore, the Koszul complex of the generators of I_C is the minimal free resolution of $R(C)$ over $K[C]$ and, hence, $\operatorname{reg} R(C) = r$. □

Theorem 4. *Let L be a planar distributive lattice. Then* $\operatorname{reg} R(L)$ *equals the maximal number of squares in a cyclic sublattice of L.*

In order to prove this theorem, we need some preparatory results.

Let L be a simple planar distributive lattice of rank $d+1$. Let $\mathfrak{c}_0 : x_0 < x_1 < \cdots < x_d < x_{d+1}$ be the chain of L with $x_t = (i_t, j_t)$ for all $0 \leqslant t \leqslant d+1$ and $(i_0, j_0) = (0,0)$, $(i_{d+1}, j_{d+1}) = \max L$, having the following property: for any $(k,\ell) \in L$ with $k = i_t$ for some t, we have $\ell \leqslant j_t$. In other words, $\mathfrak{c}_0$ is the "most upper" chain of L. We label the edges of $\mathfrak{c}_0$ by $\lambda(x_t \to x_{t+1}) = t+1$ for $0 \leqslant t \leqslant d$. Next, we label all the edges in the Hasse diagram of L as follows. If $i_{t+1} = i_t + 1$, in other words $x_t \to x_{t+1}$ is an horizontal edge, then we label by $t+1$ all the edges of L of the form $(i_t, j) \to (i_{t+1}, j)$. If $j_{t+1} = j_t + 1$, that is, $x_t \to x_{t+1}$ is a vertical edge, then we label by $t+1$ all the edges of L of the form $(i, j_t) \to (i, j_{t+1})$.

Lemma 5. *Let* $\mathfrak{c} : \min L = y_0 < y_1 < \cdots < y_{d+1} = \max L$ *be an arbitrary maximal chain in L, $\mathfrak{c} \neq \mathfrak{c}_0$. Then:*

(i) $\lambda(\mathfrak{c}) >_{\text{lex}} \lambda(\mathfrak{c}_0)$.

(ii) there exists q such that $\lambda(y_{q-1} \to y_q) > \lambda(y_q \to y_{q+1})$.

Proof. (i) Since $\mathfrak{c} \neq \mathfrak{c}_0$, we may choose $s = \min\{t : x_t \neq y_t\}$. Let $x_t = (i_t, j_t)$ and $y_t = (k_t, \ell_t)$ for all t. Assume that $i_{s-1} = i_s$. The case $j_s = j_{s-1}$ can be treated in a similar way. Since $x_s \neq y_s$, we must have $k_s = i_{s-1} + 1$. Let $r = \max\{t : t > s-1, i_t = i_{s-1}\}$. Then $\lambda(y_{s-1} \to y_s) = \lambda(x_r \to x_{r+1}) > \lambda(x_{s-1} \to x_s)$, which implies that $\lambda(\mathfrak{c}) >_{\text{lex}} \lambda(\mathfrak{c}_0)$.

For proving (ii), we consider again the case $i_{s-1} = i_s$ and keep the notation of (i). Let $q = \max\{t : t > s-1, \ell_t = \ell_{s-1}\}$. Then we get

$$\lambda(y_q \to y_{q+1}) = \lambda(x_{s-1} \to x_s) < \lambda(x_r \to x_{r+1}) = \lambda(y_{s-1} \to y_s) \leqslant \lambda(y_{q-1} \to y_q).$$

The case $j_s = j_{s-1}$ can be done similarly.

□

Proposition 6. *The above defined edge-labeling of L is an EL-labeling.*

Proof. Let $[x, y]$ be an interval of L. We first prove condition (i) in Definition 1. In the first step, we show that, starting with an arbitrary chain $\mathfrak{c}$ from x to y, we may find a chain γ whose successive edges are labeled in increasing order. This shows the existence of the chain in (i). In the second step we show the uniqueness.

For an arbitrary chain $\mathfrak{c} : x = x_0 = (i_0, j_0) \to x_1 = (i_1, j_1) \to \cdots \to x_k = (i_k, j_k) = y$, we say that x_t is an *upper corner* of $\mathfrak{c}$ if $j_t = j_{t-1} + 1$ and $i_{t+1} = i_t + 1$. Similarly, x_t is a *lower corner* of $\mathfrak{c}$ if $i_t = i_{t-1} + 1$ and $j_{t+1} = j_t + 1$. It is almost obvious that if x_t is not a corner or is an upper corner, than $\lambda(x_{t-1} \to x_t) < \lambda(x_t \to x_{t+1})$. Indeed, if x_t is not a corner, then the edges $x_{t-1} \to x_t$ and $x_t \to x_{t+1}$ are both either horizontal or vertical and, by the chosen labeling, we get $\lambda(x_{t-1} \to x_t) < \lambda(x_t \to x_{t+1})$. Let now x_t be an upper corner. We look at the edges $(i_t, k) \to (i_{t+1}, k)$ and $(\ell, j_{t-1}) \to (\ell, j_t)$ in the chain $\mathfrak{c}_0$. By the choice of $\mathfrak{c}_0$, we have $\ell \leqslant i_t$ and $k \geqslant j_t$ which implies that $(\ell, j_t) \leqslant (i_t, j_t) \leqslant (i_t, k)$. Consequently, we get

$$\lambda(x_{t-1} \to x_t) = \lambda((\ell, j_{t-1}) \to (\ell, j_t)) < \lambda((i_t, k) \to (i_{t+1}, k)) = \lambda(x_t \to x_{t+1}).$$

Let now x_t be a lower corner of $\mathfrak{c}$ with $\lambda(x_{t-1} \to x_t) > \lambda(x_t \to x_{t+1})$. We will replace x_t in $\mathfrak{c}$ by $x_t' = (i_t', j_t')$ where $i_t' = i_{t-1}$ and $j_t' = j_{t+1}$. Now we need to explain that the edges $x_{t-1} \to x_t'$ and $x_t' \to x_{t+1}$ do appear in the Hasse diagram of L. Let $(i_{t-1}, j) \to (i_t, j)$ and $(i, j_t) \to (i, j_{t+1})$ be the edges of $\mathfrak{c}_0$ with the same labels as $x_{t-1} \to x_t$ and $x_t \to x_{t+1}$, respectively. As $\lambda(x_{t-1} \to x_t) > \lambda(x_t \to x_{t+1})$, by the choice of $\mathfrak{c}_0$, we must have $i \leqslant i_{t-1}$ and $j_{t+1} \leqslant j$. Hence $x_{t-1} \to x_t'$ and $x_t' \to x_{t+1}$ are edges in L.

Now we look at the chain $\mathfrak{c}'$ obtained from $\mathfrak{c}$ by replacing x_t with x_t'. If it still has a lower corner, say y_t, with $\lambda(y_{t-1} \to y_t) > \lambda(y_t \to y_{t+1})$, we replace y_t by y_t' as we have done before in the chain $\mathfrak{c}$. In this way, after finitely many such successive replacements, we get a new chain, say γ, from x to y, whose edges are labeled in increasing order.

For uniqueness, we proceed as follows. By Lemma 5, $\mathfrak{c}_0$ is the unique maximal chain of L with the property that its edges are labeled in increasing order. Let us assume that we have γ_1 and γ_2 chains from x to y whose edges are labeled in increasing order. We extend these two chains to maximal chains in L, say Γ_1 and Γ_2. By suitable replacements of "bad" lower corners in Γ_1 and Γ_2 we reach the same maximal chain $\mathfrak{c}_0$. But these replacements do not affect γ_1 and γ_2, which implies that $\gamma_1 = \gamma_2$.

Condition (ii) in Definition 1 may be checked as in the proof of Lemma 5 (ii). □

Proof of Theorem 4. Let L be endowed with the above defined edge labeling and assume that the maximum number of squares in a cyclic sublattice of L is r. By Theorem 2 and equation (1), we have to show that

$$r = \max\{|S| : \text{ there exists a maximal chain } \mathfrak{c} \text{ in } L \text{ with } \mathcal{D}(\mathfrak{c}) = S\}.$$

Let $\mathfrak{c} : \min L = x_0 < x_1 < \cdots < x_{d+1} = \max L$ be a maximal chain in L with $\mathcal{D}(\mathfrak{c}) = \{i_1, \ldots, i_m\}$. This means that for every $1 \leqslant j \leqslant m$, we have

$$\lambda(x_{i_j-1} \to x_{i_j}) > \lambda(x_{i_j} \to x_{i_j+1}).$$

As we have already seen in the proof of Proposition 6, $x_{i_1}, \ldots .x_{i_m}$ must be lower corners of $\mathfrak{c}$ for which there exists $x'_{i_1}, \ldots, x'_{i_m} \in L$ such that, for every $1 \leqslant j \leqslant m$, $x_{i_j-1} \to x'_{i_j}$ and $x'_{i_j} \to x_{i_j+1}$ are edges in the Hasse diagram of L. Therefore, we get a sublattice L' of L whose elements are the vertices of $\mathfrak{c}$ together with $x'_{i_1}, \ldots, x'_{i_m}$ which is a cyclic sublattice with m squares. Consequently, it follows that

$$r \geqslant \max\{|S| : \text{ there exists a maximal chain } \mathfrak{c} \text{ in } L \text{ with } \mathcal{D}(\mathfrak{c}) = S\}.$$

For the other inequality, let L' be a cyclic sublattice of L which contains r squares; see Figure 2.

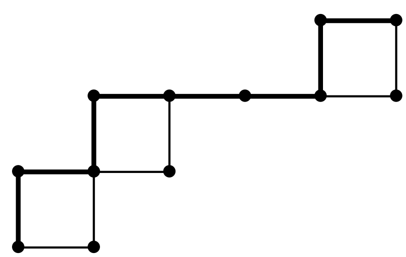

Figure 2: The sublattice L'

Let $\mathbf{b}$ be the upper chain (drawn by the fat line in Figure 2) in L' and $\mathfrak{c}$ the lower chain. Every lower corner in a square is a lower corner in $\mathfrak{c}$ which gives an element in the descent set $\mathcal{D}(\mathfrak{c})$. Hence,

$$r \leqslant \mathcal{D}(\mathfrak{c}) \leqslant \max\{|S| : \text{ there exists a maximal chain } \mathfrak{c} \text{ in } L \text{ with } \mathcal{D}(\mathfrak{c}) = S\}.$$

□

Non-planar distributive lattices

In the case of non-planar distributive lattices we give only bounds for the regularity of the Hibi ring.

Lemma 7. *Let B_n be the Boolean lattice of rank n. Then* $\operatorname{reg} R(B_n) = n-1$.

Proof. Let $P = \{p_1, \ldots, p_n\}$ be the join-irreducible elements of B_n. P is an antichain, that is, p_i is incomparable to p_j for any $i \neq j$. By using equation (1), it follows that $\operatorname{reg} R(B_n) = \max\{|S| :$ there exists a linear extension of the poset P whose descent set is $S\}$. As P is an antichain, it follows that this maximum is $n-1$, corresponding to the permutation π of P given by $\pi(p_i) = p_{n+1-i}$ for $1 \leqslant i \leqslant n$. Thus, $\operatorname{reg} R(B_n) = n-1$. □

Theorem 8. *Let $L = \mathcal{I}(P)$ be a non-planar distributive lattice. Then*

$$|P| - 1 \geqslant \operatorname{reg} R(L) \geqslant \max\{|Q| : Q \textit{ is a set of pairwise incomparable join-irreducible elements of } L\} - 1.$$

Proof. The first inequality is trivially true since, by equation (1), $\deg h_{R(L)} \leqslant |P| - 1$. Let us prove the second inequality.

Let $Q = \{p_1, \ldots, p_r\}$ be a maximal set of pairwise incomparable join-irreducible elements of L. It follows that for any other join-irreducible element $p \in P$ we have either $p < p_i$ for some i or $p > p_j$ for some j. On the set P of join-irreducible elements of L we consider a new order, $\prec$, defined as follows: $\prec$ is a linear order on the set $P' = \{p \in P : p < p_i \text{ for some } i\}$ and on the set $P'' = \{p \in P : p > p_j \text{ for some } j\}$ which extends the original order on P, that is, $p < q$ implies $p \prec q$. Moreover, we set $\max_\prec P' \prec p_i \prec \min_\prec P''$ for all $1 \leqslant i \leqslant r$. By the definition of $\prec$, it follows that, for any $p, q \in P$, if $p \leqslant q$, then $p \preceq q$. By using [13, Proposition 15.4], we get $\beta_{(P,\leqslant)}(S) \geqslant \beta_{(P,\preceq)}(S)$ for any $S \subset [d]$. Together with equation (1), this implies that

$$\operatorname{reg} R(L) = \deg h_{K[\Delta(L)]} \geqslant \deg h_{K[\Delta(L')]} = \operatorname{reg} R(L'), \tag{2}$$

where L' is the distributive lattice of the poset ideals of $(P, \preceq)$. It is obvious by the definition of $\prec$ that the regularity of $R(L')$ is equal to the regularity of $R(B_r)$ where B_r is the Boolean lattice of rank r. Therefore, Lemma 7 and inequality (2) lead to the desired inequality. □

The next example shows that both inequalities in Theorem 8 may be strict.

Example 9. Let $P = \{p_1, p_2, p_3, p_4, p_5\}$ be the poset with $p_1 < p_4, p_2 < p_4, p_2 < p_5, p_3 < p_5$ and $L = \mathcal{I}(P)$. Then $\operatorname{reg} R(L) = 3$ and the maximal number of pairwise incomparable elements of P is equal to 3.

As a corollary of the above theorems, we may characterize the distributive lattices L with the property that the Hibi ring $R(L)$ has a linear resolution over the polynomial ring $K[L]$.

Corollary 10. *Let L be a distributive lattice. Then $R(L)$ has a linear resolution if and only if L is the divisor lattice of $2 \cdot 3^a$ for some $a \geqslant 0$.*

Proof. It is well known that if L is the divisor lattice of $2 \cdot 3^a$ for some $a \geqslant 0$, then $R(L)$ has a linear resolution. Let now L be a distributive lattice such that $R(L)$ has a linear resolution. If L is non-planar, then it has at least three pairwise incomparable join-irreducible elements, thus $\operatorname{reg} R(L) \geqslant 2$, which is a contradiction to our hypothesis. Therefore, L must be planar. In this case, the conclusion follows immediately as a consequence of Theorem 4. □

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