Embedding cycles in finite planes

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Abstract

We define and study embeddings of cycles in finite affine and projective planes. We show that for all k, $3 \leqslant k \leqslant q^2$, a k-cycle can be embedded in any affine plane of order q. We also prove a similar result for finite projective planes: for all k, $3 \leqslant k \leqslant q^2+q+1$, a k-cycle can be embedded in any projective plane of order q.

Keywords: Graph embeddings, finite affine plane, finite projective plane, cycle, hamiltonian cycle, pancyclic graph

1 Introduction

Our work concerns substructures in finite affine and projective planes. In order to explain the questions we consider, we will need the following definitions and notations.

Any graph-theoretic notion not defined here may be found in Bollobás [1]. All of our graphs are finite, simple and undirected. If $G = (V, E) = (V(G), E(G))$ is a graph, then the *order* of G is $v(G) = |V|$, the number of vertices of G, and the *size* of G is $e(G) = |E|$, the number of edges in G. Each edge of G is thought as a 2-subset of V. An edge $\{x, y\}$ will be denoted by xy or yx. A vertex v is *incident* with an edge e if $v \in e$. We say that a graph $G' = (V', E')$ is a *subgraph of* G, and denote it by $G' \subset G$, if $V' \subset V$ and $E' \subset E$. If $G' \subset G$, we will also say that G *contains* G'. For a vertex $v \in V$, $N(v) = N_G(v) = \{u \in V : uv \in E\}$ denotes the *neighborhood* of v, and

$deg(v) = deg_G(v) = |N(v)|$, the *degree* of v. If the degrees of all vertices of G are equal to d, then G is called *d-regular*. For a graph F, we say that G is *F-free* if G contains no subgraph isomorphic to F.

For $k \geqslant 2$, any graph isomorphic to the graph with a vertex-set $\{x_1, \ldots, x_k\}$ and an edge-set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k\}$ is called an x_1x_k-path, or a *k-path*, and we denote it by $\mathcal{P}_k$. The length of a path is its number of edges. For $k \geqslant 3$, the graph with a vertex-set $\{x_1, \ldots, x_k\}$ and edge-set $\{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1\}$ is called a *k-cycle*, and it is often denoted by $\mathcal{C}$ or $\mathcal{C}_k$. Any subgraph of G isomorphic to a k-cycle is called a *k-cycle in G*. The *girth* of a graph G containing cycles, denoted by $g = g(G)$, is the length of a shortest cycle in G. Let $V(G) = A \cup B$ be a partition of $V(G)$, and let every edge of G have one endpoint in A and another in B. Then G is called *bipartite* and we denote it by $G(A, B; E)$. If $|A| = m$ and $|B| = n$, then we refer to G as an (m, n)-bipartite graph.

All notions of incidence geometry not defined here may be found in [2]. A *partial plane* $\pi = (\mathcal{P}, \mathcal{L}; \mathcal{I})$ is an incidence structure with a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$, and a symmetric binary relation of incidence $I \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ such that any two distinct points are on at most one line, and every line contains at least two points (note that we have used $\mathcal{P}$ for two different objects as of now: to denote a path and to denote the points on a partial plane. The usage of this symbol should be clear from the context). The definition implies that any two lines share at most one point. We will often identify lines with the sets of points on them. We say that a partial plane $\pi' = (\mathcal{P}', \mathcal{L}'; \mathcal{I}')$ is a *subplane* of π, denoted $\pi' \subset \pi$, if $\mathcal{P}' \subset \mathcal{P}, \mathcal{L}' \subset \mathcal{L}$, and $\mathcal{I}' \subset \mathcal{I}$. If there is a line containing two distinct points X and Y, we denote it by XY or YX. For $k \geqslant 3$, we define a k-gon as a partial plane with k distinct points $\{P_1, P_2, \ldots P_k\}$, with k distinct lines $\{P_1P_2, P_2P_3, \ldots, P_{k-1}P_k, P_kP_1\}$. Note that for each $k \geqslant 5$, a line of a k-gon may contain more than two of its vertices. A subplane of π isomorphic to a k-gon is called a *k-gon in π*. The *Levi graph* of a partial plane π is its point-line bipartite incidence graph $Levi(\pi) = Levi(\mathcal{P}, \mathcal{L}; E)$, where $Pl \in E$ if and only if point P is on line l. The Levi graph of any partial plane is 4-cycle-free. Clearly, there exists a bijection between the set of all k-gons in π and the set of $2k$-cycles in $Levi(\pi)$.

A *projective plane of order $q \geqslant 2$*, denoted π_q, is a partial plane with every point on exactly $q + 1$ lines, every line containing exactly $q + 1$ points, and having four points such that no three of them are collinear. It is easy to argue that π_q contains $q^2 + q + 1$ points and $q^2 + q + 1$ lines. Let $n_q = q^2 + q + 1$. It is easy to show that a partial plane is a projective plane of order q if and only if its Levi graph is a $(q + 1)$-regular graph of girth 6 and diameter 3. Projective planes π_q are known to exist only when the order q is a prime power. If $q \geqslant 9$ is a prime power but not a prime, there are always non-isomorphic planes of order q, and their number grows fast with q. Let $PG(2, q)$ denote the *classical* projective plane of prime power order q which can be modeled as follows: points of $PG(2, q)$ are 1-dimensional subspaces in the 3-dimensional vector space over the finite field of q elements, lines of $PG(2, q)$ are 2-dimensional subspaces of the vector space, and a point is incident to a line if it is a subspace of it.

Removing a line from a projective plane, and removing its points from all other lines, yields a partial plane known as an *affine plane*. The line removed is often referred to as

the *line at infinity*, and it is denoted by l_∞. Conversely, a projective plane of order q can be obtained from an affine plane of order q (i.e. having $q+1$ lines through each point) by adding a line at infinity to it, which can be thought of as a set of $q+1$ new points, called *points at infinity*, which are in bijective correspondence with the set of parallel classes (also called the set of all slopes) of lines in the affine plane. We will use π_q to denote a projective plane of order q, and α_q for affine planes of order q.

The following problem, stated in terms of set systems, appears in Erdős [5]:

Problem 1. *Is every finite partial linear space embedded in a finite projective plane?*

It is possible that the question was asked before, as it was well known that every partial linear space embeds in some infinite projective plane, by a process of free closure due to Hall [10]. For recent results related to the question, see Moorhouse and Williford [16]. Rephrased in terms of graphs, Problem 1 is the following:

Problem 1*. *Is every finite bipartite graph without 4-cycles a subgraph of the Levi graph of a finite projective plane?*

Thinking about cycles in Levi graphs of projective planes, we introduced the following notion of embedding of a graph into a partial plane, and found it useful. Let G be a graph and let $\pi = (\mathcal{P}, \mathcal{L}; \mathcal{I})$ be a partial plane. Let

$$f : V(G) \cup E(G) \to \mathcal{P} \cup \mathcal{L}$$

be an injective map such that $f(V(G)) \subset \mathcal{P}$, $f(E(G)) \subset \mathcal{L}$, and for every vertex x and edge e of G, their incidence in G implies the incidence of point $f(x)$ and line $f(e)$ in π. We call such a map f an *embedding of G in π*, and if it exists we say that G *embeds in* π and write $G \hookrightarrow \pi$. If $G \hookrightarrow \pi$, then adjacent vertices of G are mapped to collinear points of π. Note that if $G \hookrightarrow \pi_q$, then $v(G) \leqslant n_q$, $e(G) \leqslant n_q$, and $deg_G(x) \leqslant q+1$ for all $x \in V(G)$.

A cycle containing all vertices of a graph is called a *hamiltonian cycle* of the graph, and if such exists, the graph is called a *hamiltonian* graph. Similarly, if π_q contains an n_q-gon, we call it *hamiltonian*. A graph G containing k-cycles of all possible lengths, $3 \leqslant k \leqslant v(G)$, is called *pancyclic*. Similarly, we say that π_q is *pangonal*, if it contains k-gons for all $3 \leqslant k \leqslant n_q$. The latter is equivalent to $Levi(\pi_q)$ containing $2k$-cycles for all $3 \leqslant k \leqslant n_q$. Clearly, if $G \hookrightarrow \pi_q$, a k-cycle in G corresponds to a k-gon in π_q, which, in turn, corresponds to a $2k$-cycle in $Levi(\pi_q)$. From now on we choose to be less pedantic, and will feel free to use graph theoretic and geometric terms interchangeably. For example, we will say 'point' for a vertex of a graph, 'vertex' for a point of a partial plane, and we will speak about 'path' and 'cycle' in a plane, etc.

Determining whether a graph is hamiltonian, or, more generally, understanding what cycles it contains, is one of the central problems in graph theory, and it has been a subject of active research for many years. The existence of hamiltonian cycles in π_q (or $Levi(\pi_q)$), or its pancyclicity, was addressed by several researchers. The presence of k-gons of some small lengths in π_q is easy to establish. In [14], the authors presented explicit formuli for the numbers of distinct k-gons in every projective plane of order q for $k = 3, 4, 5, 6$. Very

recently, and in a very impressive way, Voropaev [21] extended this list to $k = 7, 8, 9, 10$. The existence of very special hamiltonian cycles in $PG(2, q)$ is a celebrated result of Singer [19]. These cycles are often referred to as the *Singer cycles* in $PG(2, q)$. For $q = p$ (prime) Schmeichel [18] showed by explicit constructions that $PG(2, p)$ is pancyclic, and that the hamiltonian cycles he constructed were different from Singer cycles. DeMarco and Lazebnik [4] constructed a hamiltonian cycle in a Hall plane of order p^2. Most of the known sufficient conditions for the existence of hamiltonian cycles in graphs are effective for rather dense graphs: graphs of order n and size greater that cn^2 for some positive constant c (see a survey by Gould [9]). Levi graphs of projective planes are much sparser; being $(q + 1)$-regular, their size is $(1/(2\sqrt{2}) + o(1))n^{3/2}$ for $n \to \infty$, and that is why most techniques of proving hamiltonicity of graphs do not apply to them. For the same reason, upper bounds on the Turán number of a $2k$-cycle, see, e.g., Pikhurko [17] and and references therein, are not effective for proving the existence of $2k$-cycles in Levi graphs of projective planes for most values of k (as k may depend on q).

A new approach for establishing hamiltonicity and the existence of shorter cycles came from probabilistic techniques and studies of cycles in random and pseudo-random graphs (we omit the definition). See, e.g., Thomasson [20], Chung, Graham and Wilson [3], Frieze [7], and Frieze and Krivelevich [8].

In [12], Krivelevich and Sudakov explored relations between pseudo-randomness and hamiltonicity in regular non-bipartite graphs. Some other results related to hamiltonicity and pancyclicity appeared in recent publications by Keevash and Sudakov [11], Krivelevich, Lee and Sudakov [13], and Lee and Sudakov [15].

It is likely that proofs in these papers can be modified to give results for (bipartite) Levi graphs of projective planes, but the requirement on the order of the graph to be sufficiently large (as is the case in the aforementioned papers) will remain. In this paper we establish the pancyclicity of π_q and α_q, for all q, and our proof is constructive.

Our main results follow.

Theorem 1. *Let α_q be an affine plane of order $q \geqslant 2$. Then $C_k \hookrightarrow \alpha_q$ for all k, $3 \leqslant k \leqslant q^2$.*

Theorem 2. *Let π_q be a projective plane of order $q \geqslant 2$, and $n_q = q^2 + q + 1$. Then $C_k \hookrightarrow \pi_q$ for all k, $3 \leqslant k \leqslant n_q$.*

We now proceed to give a construction for paths and cycles in *any* finite affine or projective plane. We start with a remark that will be very useful later on.

Remark 3. Let $P_1 \to P_2 \to \cdots \to P_k$ and $Q_1 \to Q_2 \to \cdots \to Q_n$ be two disjoint (in terms of points and lines) paths embedded in π_q or α_q. Then, if the line $\ell = P_kQ_n$ has not been used in these embeddings, we can create the following embedding for a path on $n + k$ vertices:

$$P_1 \to P_2 \to \cdots \to P_k \xrightarrow{\ell} Q_n \to Q_{n-1} \to \cdots Q_2 \to Q_1.$$

Here, the symbol $P_k \xrightarrow{\ell} Q_n$ indicates that the line ℓ joins the points P_k and Q_n. Moreover, if the line $m = Q_1P_1$ is still available, then we get a cycle of length $k+n$ embedded in π_q

(or α_q).

Our main technique in the next two sections will be to construct paths that can be combined using Remark 3 to create cycles of any length.

2 Cycles in Affine Planes

Let α_q be an affine plane of order q, and let O be any point of the plane. We label the $q+1$ lines through O by $l_0, l_1, \ldots, l_q$. For any given point $Q \in \alpha_q$, we use $l_i + Q$ to denote the line parallel to l_i that passes through Q. Let $a \mod q+1$ denote the remainder of the division of a by $q+1$.

Pick any point P_0 on l_0, different from O. Let P_1 be the point of intersection of $l_2 + P_0$ and l_1. Let P_2 be the point of intersection of $l_3 + P_1$ and l_2, etc. In general, let P_i be the point of intersection of $l_{i+1 \bmod q+1} + P_{i-1}$ and l_i, for all $i = 1, 2, \ldots, q$. Since $O \neq P_i \in l_i$, for all $i = 1, 2, \ldots, q$, then all these points are distinct. Similarly, the lines $P_{i-1}P_i$ are in different parallel classes, for all $i = 1, 2, \ldots, q$. It follows that by joining the points P_{i-1} and P_i, for all $i = 1, 2, \ldots, q$, we obtain a path on $q+1$ vertices. Denote this path by $\mathcal{P}_{P_0}$.

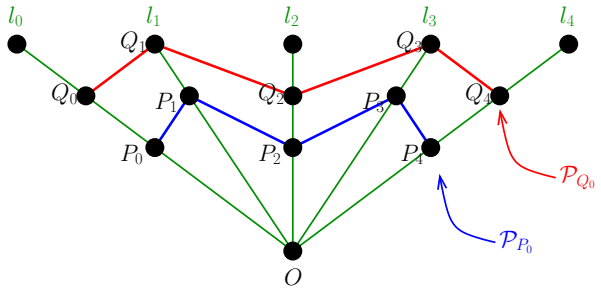

Figure 1: Two vertex/edge disjoint paths, $\mathcal{P}_{P_0}$ and $\mathcal{P}_{Q_0}$, for $q = 4$.

Lemma 4. *Let $P_0 \neq Q_0 \in l_0$. Then the paths $\mathcal{P}_{P_0}$ and $\mathcal{P}_{Q_0}$ share neither points nor lines.*

Proof. Let

$$\mathcal{P}_{P_0} : \ P_0 \to P_1 \to \cdots \to P_q \qquad\qquad \mathcal{P}_{Q_0} : \ Q_0 \to Q_1 \to \cdots Q_q$$

Clearly $P_i \neq Q_j$, for $i \neq j$. We also know that $P_0 \neq Q_0$. So, assume that $P_i = Q_i$, for some $i = 1, \ldots, q$, so that $P_j \neq Q_j$, for all $0 \leqslant j < i$. It follows that

$$(l_{i+1 \bmod q+1} + P_{i-1}) \cap l_i = P_i = Q_i = (l_{i+1 \bmod q+1} + Q_{i-1}) \cap l_i$$

which forces $l_{i+1 \bmod q+1} + P_{i-1} = l_{i+1 \bmod q+1} + Q_{i-1}$, and thus $P_{i-1} = Q_{i-1}$, a contradiction.

Finally, it is easy to see that if $\mathcal{P}_{P_0}$ and $\mathcal{P}_{Q_0}$ shared a line, then they would also share a point. □

Lemma 5. *We can partition the points of $\alpha_q \setminus \{O\}$ into s cycles, $\mathcal{C}_1, \ldots, \mathcal{C}_s$, where the length of C_i is $t_i(q+1)$ for some integer t_i, $1 \leqslant i \leqslant s \leqslant q-1$, $1 \leqslant t_1 \leqslant \ldots \leqslant t_s$, and $t_1 + \cdots + t_s = q-1$.*

Proof. If we label the points on $l_0 \setminus \{O\}$ by $x_1, x_2, \cdots, x_{q-1}$, by Lemma 4, $\mathcal{P}_{x_1}$, $\mathcal{P}_{x_2}, \ldots$, $\mathcal{P}_{x_{q-1}}$ yields a partition of the points of $\alpha_q \setminus \{O\}$ into $q-1$ disjoint paths each having $q+1$ vertices. If $q \geqslant 3$, then we have at least two such paths, and we may want to connect them to create longer paths and/or cycles.

Note that in the paths $\mathcal{P}_{x_i}$, no line parallel to l_1 has been used. Now, if we consider a path $\mathcal{P}_{P_0}$, then the line $l_1 + P_q$ intersects l_0 at a point Q, which can never be equal to O, otherwise $l_1 + P_q = l_1$, and thus $P_q \in l_1 \cap l_q = \{O\}$, a contradiction. This point Q is uniquely determined by P_0 (and the way we do this construction, of course). If $Q = P_0$, then we get a $(q+1)$-cycle. If $Q \neq P_0$, then we re-label $Q = Q_0$ and consider the path $\mathcal{P}_{Q_0}$. This will give us a path with $2(q+1)$ vertices, namely

$$P_0 \to P_1 \to \cdots \to P_q \to Q_0 \to Q_1 \to \cdots Q_q.$$

We then proceed to find $R_0 := (l_1 + Q_q) \cap l_0$. If $R_0 = P_0$ we get a cycle of length $2(q+1)$. If $R_0 = Q_0$, then we get that Q_0 is on two lines that are parallel to l_1, namely $l_1 + Q_q$ and $l_1 + P_q$. This forces P_q and Q_q to coincide, but this is impossible because of Lemma 4. It follows that we either get a cycle of length $2(q+1)$ or we can keep extending this path using $\mathcal{P}_{R_0}$. Since l_0 contains finitely many points this process must end. Moreover, it is impossible to 'close' this cycle at any point that is not P_0, as this would yield the same contradiction we obtained above when we assumed $R_0 = Q_0$. Hence, by combining paths we can construct cycles of length $t(q+1)$, for some positive integer t, these are the cycles $\mathcal{C}_i$ we wanted to find. □

In order to prove Theorem 1 we will need to construct paths out of the cycles $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_s$. Firstly, we define terms and set notation that will be necessary for the rest of this article.

Definition 6. For every $i = 1, \ldots, s$, let $P_{i,i-1}$ be an arbitrary point on $l_{i-1} \cap \mathcal{C}_i$ (note that there are t_i such points), and let $P_{i,i}$ be its neighbor on l_i.

We construct two different types of paths in $\mathcal{C}_i$: all of them start at $P_{i,i-1}$ and

1. the next vertex is $P_{i,i}$. The other vertices in the path are easily determined from these first two, or

2. the next vertex is the neighbor of $P_{i,i-1}$ in $\mathcal{C}_i$ that is on the line $l_{i-2 \mod q+1}$. The other vertices in the path are easily determined from these first two.

We will say that the first path is a *positive* path, and that the second is a *negative* path.

Lemma 7. *k-cycles can be embedded in α_q, for all $3 \leqslant k \leqslant t_1(q+1)$.*

Proof. If $q = 2, 3$ the result is immediate. We assume $q \geqslant 4$ for the rest of this proof.

The cycle $\mathcal{C}_1$ is of length $t_1(q+1)$, and so we only need to construct k-cycles with $3 \leqslant k < t_1(q+1)$.
If $k \equiv 1 \pmod{q+1}$, then, since $k \geqslant 3$, we consider a positive k-path in $\mathcal{C}_1$ starting at $P_{1,0}$. As $k \equiv 1 \pmod{q+1}$, this path ends at some $Q_0 \in l_0$, and $Q_0 \neq P_0$. Connect P_0 to Q_0 using l_0 to get a k-cycle.
If $k \equiv 2 \pmod{q+1}$ and $2 < k < t_1(q+1)$, then $t_1 > 1$. Consider a positive $(k-2)$-path $\mathcal{P}$ in $\mathcal{C}_1$ starting at $P_{1,0}$. This path ends at a point $P_q \in l_q$. Since $k < t_1(q+1)$ then there is a 2-path in $\mathcal{C}_1$, disjoint from $\mathcal{P}$, of the form $Q_{q-1} \to Q_q$ with $Q_i \in l_i$, for $i = q-1, q$. Consider the following k cycle

$$O \xrightarrow{l_0} \underbrace{P_0 \to \cdots \to P_{q-1}}_{in\ \mathcal{P}} \xrightarrow{l_{q-1}} Q_{q-1} \to Q_q \xrightarrow{l_q} O,$$

where $P_{q-1} \in l_{q-1}$ was the neighbor of P_q in $\mathcal{P}$.
If $k \not\equiv 1, 2 \pmod{q+1}$, then, since $3 \leqslant k \leqslant t_1(q+1)$, take a positive $(k-1)$-path in $\mathcal{C}_1$ starting at $P_{1,0}$. This path will end on a point $P_{k-2} \in l_{k-2}$. Connect P_0 and P_{k-2} to O using l_0 and l_{k-2}, respectively, to get a k-cycle. □

We now focus on the construction of k-cycles for $k > t_1(q+1)$. In order to do that we will use the following construction.

Construction 1. Let $\lambda_m = t_1 + t_2 + \cdots + t_m$. We will construct a $\lambda_m(q+1)$-path $\mathcal{P}_m$ out of the cycles $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$, where $2 \leqslant m \leqslant s$ (recall that $s \leqslant q-1$).
For each $i = 1, \ldots, m-1$, we connect $\mathcal{C}_i$ with $\mathcal{C}_{i+1}$ by joining $P_{i,i}$ with $P_{i+1,i}$ using l_i. Then, for all $i = 1, \ldots, m$, we take the $P_{i,i-1}P_{i,i}$ path in $\mathcal{C}_i$ having $t_i(q+1)$ vertices, and construct the following path

$$\underbrace{P_{1,0} \to \cdots \to P_{1,1}}_{in\ \mathcal{C}_1} \xrightarrow{l_1} \underbrace{P_{2,1} \to \cdots \to P_{2,2}}_{in\ \mathcal{C}_2} \xrightarrow{l_2} \cdots \xrightarrow{l_{m-1}} \underbrace{P_{m,m-1} \to \cdots \to P_{m,m}}_{in\ \mathcal{C}_m}$$

Since no vertices were eliminated or added, and all new lines are distinct and through O (none used in the construction of the $\mathcal{C}_i$'s), this construction yields a $P_{1,0}P_{m,m}$-path with $\lambda_m(q+1)$ vertices.
Note that O has not been used in the construction of $\mathcal{P}_m$, and that neither have the lines $l_m, \ldots, l_q$, and l_0.
Finally, we will denote the neighbor of $P_{1,0}$ in $\mathcal{P}_m$, which is a point on l_q, by $P_{1,q}$.
Figure 2, at the top of the next page, depicts this construction.

We now prove Theorem 1.

Proof of Theorem 1. In this proof we follow the notation introduced in Construction 1.

If $q = 2$, the existence of 3- and 4-cycles is obvious. If $q = 3$, pancyclicity can be easily verified. In what follows we assume $q \geqslant 4$, though most arguments hold for $q \geqslant 3$.

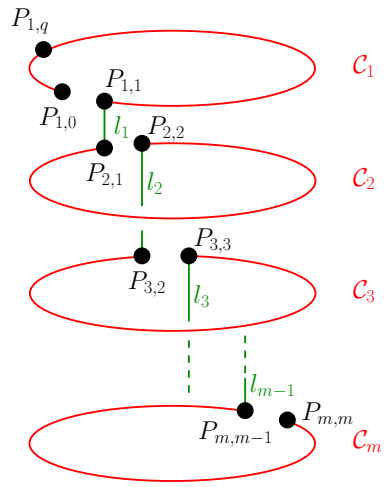

Figure 2: Construction of $\mathcal{P}_m$

We want to embed all possible k-cycles in α_q that have not been already discussed in Lemma 7. For any given k, we write it as either $k = \lambda_s(q+1)$, $k = \lambda_s(q+1)+1 = q^2$, or $k = \lambda_m(q+1)+r$, for some $m = 1, \ldots, s-1$ and $0 \leqslant r < t_{m+1}(q+1)$. Note that the case $m = 0$ was taken care of in Lemma 7.

Firstly, we can join $P_{1,0}$ and $P_{s,s}$ with O, using the lines l_0 and l_s respectively, to obtain a cycle of length $\lambda_s(q+1)+1$. Note that this grants hamiltonicity. Moreover, if we cut $\mathcal{P}_s$ short one vertex, and thus we ask it to end at $P_{1,q}$, then joining the endpoints of this new path to O yields a $\lambda_s(q+1)$-cycle.

From now on, let $k = \lambda_m(q+1)+r$, for some $m = 1, \ldots, s-1$ and some $0 \leqslant r < t_{m+1}(q+1)$. Our strategy for constructing a k-cycle in α_q will be to connect a path on $\mathcal{C}_{m+1}$ (note that $m < s$) to O and $\mathcal{P}_m$. The paths on $\mathcal{C}_{m+1}$ we will consider always starts at $P_{m+1,m}$, which will be connected to $P_{m,m} \in \mathcal{P}_m$ by using l_m.

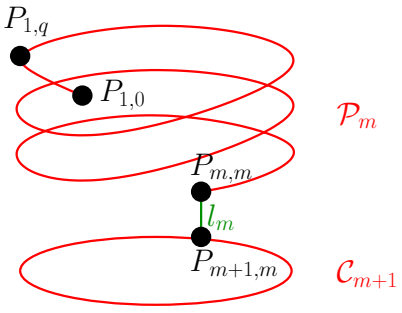

Figure 3: Connecting $\mathcal{P}_m$ and $\mathcal{C}_{m+1}$

We consider several cases.

(a) If $r \equiv 3 \pmod{q+1}$ we first get a positive path on $r-1$ vertices on $\mathcal{C}_{m+1}$ that ends

on a point $Q_{m+1} \in l_{m+1}$. We then join $P_{1,0}$ with O using l_0, Q_{m+1} with O using l_{m+1}. The result is a cycle of the desired length. **(b)** If $r \equiv 1 \pmod{q+1}$ we consider a negative $(r-2)$-path on $\mathcal{C}_{m+1}$ that ends on a point $Q_{m+1} \in l_{m+1}$. We finish the construction as in case **(a)**. **(c)** If $r \equiv 2 \pmod{q+1}$ or $r \equiv 0 \pmod{q+1}$, then we cut $\mathcal{P}_m$ short one vertex, so it ends at $P_{1,q}$. We get the path in $\mathcal{C}_{m+1}$ as in part **(a)** (for $r \equiv 2 \pmod{q+1}$) or **(b)** (for $r \equiv 0 \pmod{q+1}$). We close the cycle by joining $P_{1,q}$ with O using l_q, Q_{m+1} with O using l_{m+1}.
(d) If $r \equiv i \pmod{q+1}$, where $4 \leqslant i \leqslant q$. We want to get a positive $(r-2)$-path on $\mathcal{C}_{m+1}$ starting at $P_{m+1,m}$. This path would end at a point on $l_{m+i-2 \mod q+1}$.
(i) If $i \leqslant q+2-m$, then $m+i-2 \leqslant q$, and thus this path on $r-1$ vertices ends at a point $Q_{m+i-2} \in l_{m+i-2}$. We then get a cycle of the desired length by joining $P_{1,0}$ with O using l_0, Q_{m+i-2} with O using l_{m+i-2}.
(ii) If $i \geqslant q+3-m$, then $m+i-2 \geqslant q+1$, and thus this path on $r-1$ vertices ends at a point $Q_{t-2} \in l_{t-2}$, where $0 \leqslant t-2 \leqslant m-3$. We next 'shift' this path to make it start at $P_{m+1,m+1}$ instead of $P_{m+1,m}$ and add a vertex to make it a path on r vertices. Now this path ends at $Q_t \in l_t$, where $2 \leqslant t \leqslant m-1$. Since the line l_t is needed to construct $\mathcal{P}_m$ we will need to modify the construction of $\mathcal{P}_m$ by connecting the cycles $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{m+1}$ in the following way

$$\mathcal{C}_1 \xrightarrow{l_1} \mathcal{C}_2 \xrightarrow{l_2} \cdots \xrightarrow{l_{t-2}} \mathcal{C}_{t-1} \xrightarrow{l_{t-1}} \mathcal{C}_t \xrightarrow{l_{t+1}} \mathcal{C}_{t+1} \xrightarrow{l_{t+2}} \cdots \xrightarrow{l_{m+1}} \mathcal{C}_{m+1}$$

Note that this can be done for all $2 \leqslant t \leqslant m-1$, and that doing this means that $P_{t,t}$ is a 'loose' vertex, not used in $\mathcal{P}_m$ anymore.
Now we connect this path to the path on $\mathcal{C}_{m+1}$ that ends on Q_t. The line l_t is now free, and thus it can be used to close the cycle at O. We get the cycle

$$O \xrightarrow{l_0} \mathcal{C}_1 \xrightarrow{l_1} \cdots \xrightarrow{l_m} \mathcal{C}_m \xrightarrow{l_{m+1}} P_{m+1,m+1} \to \cdots \to Q_t \xrightarrow{l_t} O$$

This cycle has length:

$$\lambda_m(q+1) - 1 + r + 1 = \lambda_m(q+1) + r = k$$

The 'minus one' is because of the loose vertex, the 'plus one' is because of O. □

3 Cycles in Projective Planes

In this section we will study embeddings of cycles in finite projective planes. Let π_q denote a projective plane of order q. We think about π_q as obtained from an affine plane α_q by adding a line, denoted ℓ_∞, consisting of points (i), for $i = 0, \cdots, q$. Using the notations from the previous section, each of these points (i) is incident with only the following lines: ℓ_∞, line l_i of α_q, and the $q-1$ lines of α_q parallel to l_i. The next statement follows immediately from our work in Section 2.

Corollary 8. *Let π_q be a projective plane of order q. Then, a k-cycle can be embedded in π_q, for all $k = 3, \ldots, q^2$.*

Therefore in order to prove the pancyclicity of π_q, we need to show that k-cycles can be embedded into π_q for all k, $q^2+1 \leqslant k \leqslant q^2+q+1$. At this point one would expect to use heavily the pancyclicity of α_q for the construction of 'long' cycles in π_q, but we could not make use of this idea. Instead, we base our construction methods on using the cycles $\mathcal{C}_i$ in similar ways to that in the proof of Theorem 1.

Let W_1 be any of the vertices of $\mathcal{C}_1$ that are on l_q, and let $V_1 = (l_1+W_1)\cap l_0$. It follows that V_1 is a vertex of $\mathcal{C}_1$, and that l_1+W_1 is an edge of $\mathcal{C}_1$. Similarly, for $2 \leqslant i \leqslant s$, let $W_i \in l_{i-2}$ be a vertex of $\mathcal{C}_i$, and $V_i = (l_i+W_i)\cap l_{i-1}$. Hence, V_i is a vertex of $\mathcal{C}_i$, and l_i+W_i is an edge of $\mathcal{C}_i$. For each $i = 1, \cdots, s$, let $[V_i, W_i]$ denote the V_iW_i-path in $\mathcal{C}_i$, different from the edge V_iW_i. Next we define U_i to be the vertex of $\mathcal{C}_i$ that is on l_q and that is the closest to V_i, when we move from V_i towards W_i along $\mathcal{C}_i$. By $[V_iU_i]$ we denote the subpath of $[V_i, W_i]$ having $q-i+2$ vertices and endpoints V_i and U_i.

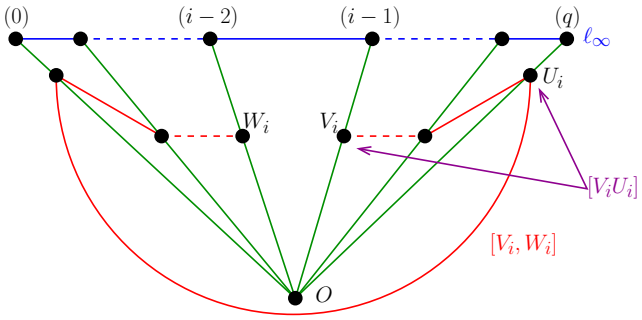

Figure 4: Paths $[V_i, W_i]$ and $[V_iU_i]$

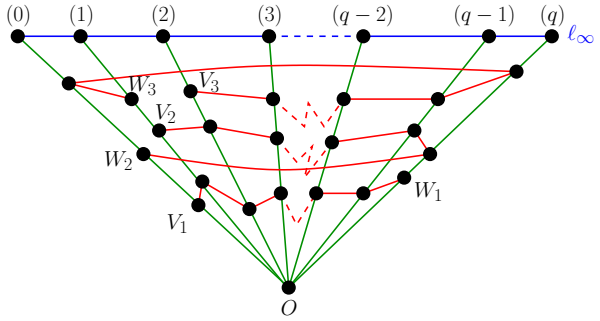

Figure 5: Paths $[V_1, W_1]$, $[V_2, W_2]$, and $[V_3, W_3]$

Recall that $(i) = l_i \cap \ell_\infty$. We now construct a path $\mathcal{P}$ (for $s \geqslant 2$) by connecting W_i with (i) using l_i+W_i (which is not an edge of $[V_i, W_i]$), and connecting (i) with V_{i+1} using l_i. Thus $\mathcal{P}$ is the path:

$$[V_1, W_1] \xrightarrow{l_1+W_1} (1) \xrightarrow{l_1} [V_2, W_2] \xrightarrow{l_2+W_2} (2) \to \cdots \to (s-1) \xrightarrow{l_{s-1}} [V_s, W_s].$$

For $s = 1$, $\mathcal{P}$ is obtained from the cycle $\mathcal{C}_1$ by removing the edge V_1W_1.

Note that for all s, $\mathcal{P}$ has $(q^2-1)+(s-1) = q^2+s-2$ vertices. The lines $l_s, \cdots, l_q, l_0$, $l_s + W_s$, and ℓ_∞ have not been used in the construction of $\mathcal{P}$, and neither have the points $(s), \cdots, (q), (0)$, and O.

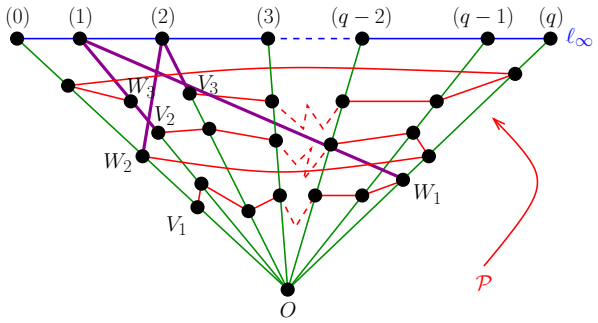

Figure 6: Paths $[V_1, W_1]$, $[V_2, W_2]$, $[V_3, W_3]$ joined into a path $\mathcal{P}$. Its endpoints are V_1 and W_3

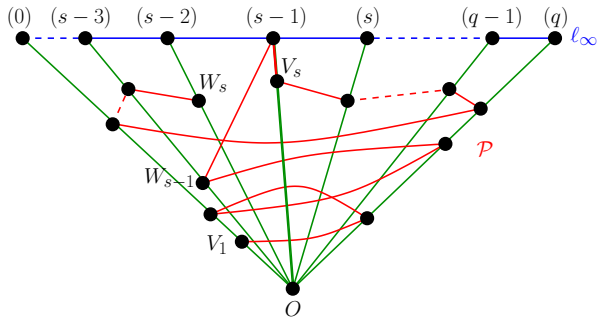

Figure 7: A simple diagram of the path $\mathcal{P}$

Now we begin our construction of k-cycles in π_q of lengths from q^2+1 to q^2+q+1 by using the path $\mathcal{P}$ and/or modifications of it. Recall that s denotes the number of all cycles $\mathcal{C}_i$, or of all paths $[V_i, W_i]$, and that $1 \leqslant s \leqslant q-1$. We will first construct cycles of length between q^2+1 to q^2+s+2, and then the ones that are longer than q^2+s+2.

Lemma 9. *Cycles of length ranging from q^2+1 to q^2+s+2 can be embedded in π_q.*

Proof. Using the path $\mathcal{P}$ we can construct

1. a cycle of length q^2+s+2:

$$\underbrace{V_1 \to \cdots \to W_s}_{in\ \mathcal{P}} \xrightarrow{l_s+W_s} (s) \xrightarrow{l_s} O \xrightarrow{l_q} (q) \xrightarrow{\ell_\infty} (0) \xrightarrow{l_0} V_1,$$

2. a cycle of length q^2+s+1:

$$\underbrace{V_1 \to \cdots \to W_s}_{in\ \mathcal{P}} \xrightarrow{l_s+W_s} (s) \xrightarrow{\ell_\infty} (q) \xrightarrow{l_q} O \xrightarrow{l_0} V_1, \text{ and}$$

3. a cycle of length q^2+s:

$$\underbrace{V_1 \to \cdots \to W_s}_{in\ \mathcal{P}} \xrightarrow{l_s+W_s} (s) \xrightarrow{\ell_\infty} (0) \xrightarrow{l_0} V_1.$$

Note that the lines $l_s, \cdots, l_q$ have not been used in the construction of this last cycle, and neither have the points $(s+1), \cdots, (q)$, and O. We will denote this cycle by $\mathcal{C}$.

If $q+1 \leqslant 2s$, then, for every $q-s+1 \leqslant i \leqslant s$, let us modify $\mathcal{C}$ in the following way:

- Delete $q-i+1$ vertices of the path $[V_iU_i]$, all except U_i.
- Connect $(i-1)$ with O (recall that $(i-1)$ was connected to V_i in $\mathcal{C}$ via l_{i-1}).
- Connect O with U_i using l_q.

This yields the cycle

$$\underbrace{U_i \to \cdots \to (i-1)}_{in\ \mathcal{C}} \xrightarrow{l_{i-1}} O \xrightarrow{l_q} U_i$$

which has length $(q^2+s)-(q-i+1)+1 = q^2-q+s+i$.

Since $q-s+1 \leqslant i \leqslant s$, the length of this cycle ranges between q^2+1 and $(q^2+s)-(q-s)$. Note that if $q+1 > 2s$, then $(q^2+s)-(q-s) < q^2+1$. So, for all relevant values of s we have been able to construct cycles with lengths ranging from q^2+1 to $(q^2+s)-(q-s)$.

Next we want to construct k-cycles for $(q^2+s)-(q-s) < k < q^2+s$. In order to do that we need to set more notation.

Let us relabel the vertices in the path $[V_sU_s]$ by $V_s = P_{s-1}, P_s, \cdots, P_{q-1}, P_q$, where $P_i \in l_i$, for all $i = s, \ldots, q$. Note that $P_q = U_s$. For $s-1 \leqslant i < q$, let $[P_iP_j]$ denote the subpath of $[V_sU_s]$ joining P_i and P_j.

Note that a cycle of length $(q^2+s)-(q-s)$ vertices may be obtained using $i=s$ in the previous construction. We want to use a similar construction to get a cycle of length $(q^2+s)-(q-s)+1$. We modify $\mathcal{C}$ by replacing its subpath $[P_{s-1}P_{q-1}]$ by a path

$$(s-1) \xrightarrow{l_{s-1}} O \xrightarrow{l_{q-1}} P_{q-1} \to P_q$$

leading to the following cycle of length $(q^2+s)-(q-s)+1$.

$$\underbrace{U_s \to \cdots \to (s-1)}_{in\ \mathcal{C}} \xrightarrow{l_{s-1}} O \xrightarrow{l_{q-1}} P_{q-1} \to P_q$$

Note that we are using here that $s \leqslant q-1$.

Now, to create cycles of length larger than $(q^2+s)-(q-s)+1$ we use the following strategy.

For every $i = s, \cdots, q-1$, we modify $\mathcal{C}$, by connecting P_i with O, and O with P_q to get the cycle

$$\underbrace{P_q \to \cdots \to P_i}_{in\ \mathcal{C}} \xrightarrow{l_i} O \xrightarrow{l_q} P_q,$$

which has length $(q^2+s)-(q-i-1)+1 = (q^2+s)-(q-i-2)$. Since $i = s, \cdots, q-1$, then the length of this cycle ranges from $(q^2+s)-(q-s)+2$ to $(q^2+s)+1$. □

Corollary 10. *With the same notation used in Lemma 9, if $s = q-1$, then π_q is pancyclic.*

Now we want to construct cycles longer than q^2+s+2 for when $1 \leqslant s < q-1$.

Lemma 11. *For every $1 \leqslant s < q-1$, cycles of length ranging from q^2+s+3 to q^2+q+1 can be embedded in π_q.*

Proof. Just as we did in the proof of Lemma 9, the idea is to modify the path $\mathcal{P}$ to get the desired cycles. Hence, we will use the same notation introduced earlier in this section, including that used in the proof of Lemma 9.

We first eliminate the edge $l_{s+1}+V_s$ from $\mathcal{P}$, and connect W_s with V_s using l_s+W_s. This gives us a path $\tilde{\mathcal{P}}$ that has the same length of $\mathcal{P}$ (q^2+s-2 vertices) with endpoints V_1 and P_s.

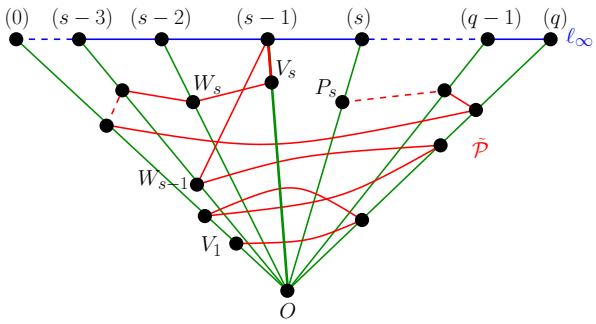

Figure 8: The path $\tilde{\mathcal{P}}$

Note that the lines $l_s, \cdots, l_q, l_0$, $l_{s+1}+V_s$, and ℓ_∞ have not been used, neither have the points $(s), \cdots, (q), (0)$, and O.

If we now eliminate the edge $l_{s+2}+P_s$ and, instead, connect P_s with P_{s+1} using

$$P_s \xrightarrow{l_{s+1}+V_s} (s+1) \xrightarrow{l_{s+1}} P_{s+1},$$

we get a path $\mathcal{G}_1$ in $(q^2+s-2)+1$ vertices (one more than $\tilde{P}$). We may close this path into a cycle by

$$V_1 \xrightarrow{l_0} (0) \xrightarrow{\ell_\infty} (s) \xrightarrow{l_s} P_s \xrightarrow{l_{s+1}+V_s} (s+1) \xrightarrow{l_{s+1}} \underbrace{P_{s+1} \to \cdots \to V_1}_{in\ \tilde{\mathcal{P}}}$$

which has length $(q^2+s-1)+2 = q^2+s+1$.

Now we eliminate the edge $l_{s+3 \mod q+1} + P_{s+1}$ from $\mathcal{G}_1$ and instead connect P_{s+1} with P_{s+2} using

$$P_{s+1} \xrightarrow{l_{s+2}+P_s} (s+2) \xrightarrow{l_{s+2}} P_{s+2}.$$

This yields a path $\mathcal{G}_2$ in $(q^2+s-2)+2$ vertices (two more than $\tilde{\mathcal{P}}$). We may close this path into a cycle by using $\tilde{\mathcal{P}}$ as above

$$V_1 \xrightarrow{l_0} (0) \xrightarrow{\ell_\infty} (s) \xrightarrow{l_s} P_s \xrightarrow{l_{s+1}+V_s} (s+1) \xrightarrow{l_{s+1}} P_{s+1} \xrightarrow{l_{s+2}+P_s} (s+2) \xrightarrow{l_{s+2}} \underbrace{P_{s+2} \to \cdots \to V_1}_{in\ \tilde{\mathcal{P}}}$$

which has length $(q^2+s)+2 = q^2+s+2$.

In general, for $1 \leqslant i < q-s$ (and thus $s+i+1 < q+1$), given a path $\mathcal{G}_i$ of length $(q^2+s-2)+i$ constructed as above we can eliminate the edge $l_{s+i+2 \mod q+1} + P_{s+i}$ from $\mathcal{G}_i$ and instead connect P_{s+i} with P_{s+i+1} using

$$P_{s+i} \xrightarrow{l_{s+i+1}+P_{s+i-1}} (s+i+1) \xrightarrow{l_{s+i+1}} P_{s+i+1}$$

this yields a path $\mathcal{G}_{i+1}$ in $(q^2+s-2)+i+1$ vertices ($i+1$ more than $\tilde{\mathcal{P}}$). We may close this path into a cycle as we did above

$$V_1 \xrightarrow{l_0} (0) \xrightarrow{\ell_\infty} (s) \xrightarrow{l_s} P_s \xrightarrow{l_{s+1}+V_s} (s+1) \xrightarrow{l_{s+1}} P_{s+1} \xrightarrow{l_{s+2}+P_s} (s+2) \to \cdots$$

$$\cdots \to P_{s+i} \xrightarrow{l_{s+i+1}+P_{s+i-1}} (s+i+1) \xrightarrow{l_{s+i+1}} \underbrace{P_{s+i+1} \to \cdots \to V_1}_{in\ \tilde{\mathcal{P}}}$$

which has length $q^2+s+i+1$.

This will yield cycles of length up to q^2+q. The line not used in the (q^2+q)-cycle $\mathcal{Q}$ is $l_0 + P_{q-1}$, and the point not used is O. Figure 9 gives an idea of what $\mathcal{Q}$ looks like.

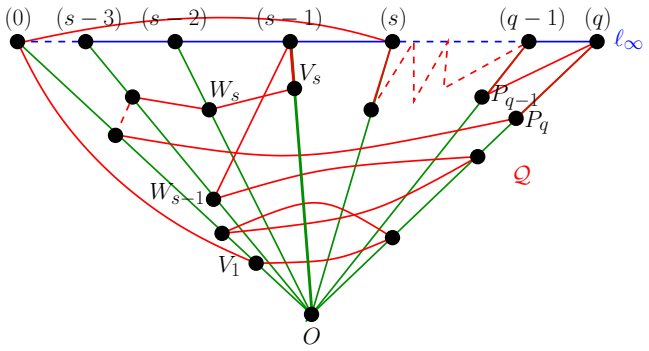

Figure 9: Cycle $\mathcal{Q}$

In order to construct a (q^2+q+1)-cycle we use $\mathcal{Q}$ and modified it as follows.

- eliminate ℓ_∞, which connected l_0 and l_s.

- eliminate l_{q-1}, which connected $(q-1)$ and P_{q-1}.
- eliminate l_q, which connected (q) and P_q.
- connect (s) and (q) using ℓ_∞.
- connect $(q-1)$ and O using l_{q-1}
- connect P_q and O using l_q
- connect $(q-1)$ and (0) using $l_0 + P_{q-1}$

We get the following hamiltonian cycle:

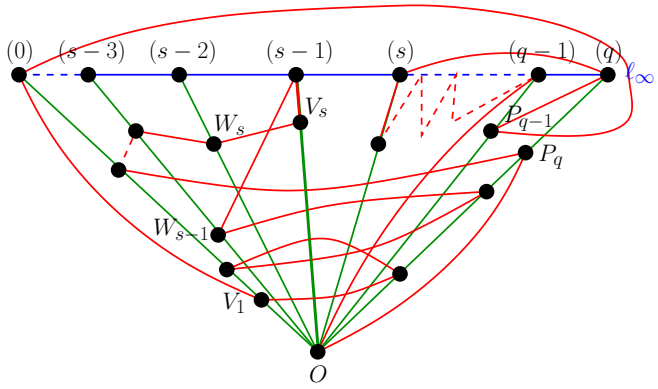

Figure 10: A hamiltonian cycle

□

Proof of Theorem 2. It follows from Lemmas 9 and 11. □

We wish to conclude this paper with a conjecture. Let $s \geqslant 1$ and $n \geqslant 2$. A finite partial plane $\mathcal{G} = (\mathcal{P}, \mathcal{L}; \mathcal{I})$ is called a *generalized n-gon of order s* if its Levi graph is $(s+1)$-regular, has diameter n, and has girth $2n$. It is known that a generalized n-gons of order s exists only for $n = 2, 3, 4, 6$, see Feit and Higman [6]. It is easy to argue that the number of points and the number of lines in the generalized n-gon is $p_s^{(n)} := s^{n-1} + s^{n-2} + \cdots + s + 1$. Note that a projective plane of order q is a generalized 3-gon (generalized triangle) of order q, and so $p_q^{(3)} = q^2 + q + 1 = n_q$ – the notation used in this paper earlier.

Conjecture 12. Let $s \geqslant 2$ and $n \geqslant 3$. Then $C_k \hookrightarrow \mathcal{G}$ for all k, $n \leqslant k \leqslant p_s^{(n)}$.

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