A Note on Total and Paired Domination of Cartesian Product Graphs

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Abstract

A dominating set D for a graph G is a subset of $V(G)$ such that any vertex not in D has at least one neighbor in D. The domination number $\gamma(G)$ is the size of a minimum dominating set in G. Vizing's conjecture from 1968 states that for the Cartesian product of graphs G and H, $\gamma(G)\gamma(H) \leqslant \gamma(G \square H)$, and Clark and Suen (2000) proved that $\gamma(G)\gamma(H) \leqslant 2\gamma(G \square H)$. In this paper, we modify the approach of Clark and Suen to prove similar bounds for total and paired domination in the general case of the n-Cartesian product graph $A_1 \square \cdots \square A_n$. As a by-product of these results, improvements to known total and paired domination inequalities follow as natural corollaries for the standard $G \square H$.

1 Introduction

We consider simple undirected graphs $G = (V, E)$ with vertex set V and edge set E. The open neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and closed neighborhood by $N_G[v]$. A dominating set D of a graph G is a subset of $V(G)$ such that for all v, $N_G[v] \cap D \neq \emptyset$. A γ-set of G is a minimum dominating set for G, and its size is denoted $\gamma(G)$. A *total* dominating set D of a graph G is a subset of $V(G)$ such that for all v, $N_G(v) \cap D \neq \emptyset$. A γ_t-set of G is a minimum total dominating set for G, and its size is denoted $\gamma_t(G)$. A *paired* dominating set D for a graph G is a dominating set such that the subgraph of G induced by D (denoted $G[D]$) has a perfect matching. A γ_{pr}-set of G is a minimum paired dominating set for G, and its size is denoted $\gamma_{pr}(G)$. In general, for a graph containing no isolated vertices, $\gamma(G) \leqslant \gamma_t(G) \leqslant \gamma_{pr}(G)$.

The Cartesian product graph, denoted $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, where vertices gh and $g'h'$ are adjacent whenever $g = g'$ and $(h, h') \in E(H)$, or $h = h'$ and $(g, g') \in E(G)$. Just as the Cartesian product of graphs G and H is denoted $G\Box H$, the n-product of graphs $A_1, A_2, \ldots, A_n$ is denoted as $A_1\Box A_2\Box\cdots\Box A_n$, and has vertex set $V(A_1)\times V(A_2)\times\cdots\times V(A_n)$, where vertices $u_1\cdots u_n$ and $v_1\cdots v_n$ are adjacent if and only if for some i, $(u_i, v_i) \in E(A_i)$, and $u_j = v_j$ for all other indices $j \neq i$.

Vizing's conjecture from 1968 states that $\gamma(G)\gamma(H) \leqslant \gamma(G\Box H)$. For a thorough review of the activity on this famous open problem, see [1] and references therein. In 2000, Clark and Suen [3] proved that $\gamma(G)\gamma(H) \leqslant 2\gamma(G\Box H)$ by a sophisticated double-counting argument which involved projecting a γ-set of the product graph $G\Box H$ down onto the graph H. In this paper, we slightly modify the Clark and Suen double-counting approach and instead project subsets of $G\Box H$ down onto both graphs G and H, which allows us to prove several theorems/corollaries relating to total and paired domination. In this section, we state the results, and in Section 2, we prove the results.

Theorem 1. *Given graphs $A_1, A_2, \ldots, A_n$ containing no isolated vertices,*

$$\gamma(A_1)\prod_{i=2}^{n}\gamma_t(A_i) \leqslant n\gamma(A_1\Box A_2\Box\cdots\Box A_n)\ .$$

In 2008, Ho [4] proved $\gamma_t(G)\gamma_t(H) \leqslant 2\gamma_t(G\Box H)$, an inequality for total domination precisely analogous to the Clark and Suen inequality for domination. In this paper, we extend this result to the n-product case, and then Ho's inequality becomes a special case of a more general result.

Theorem 2. *Given graphs $A_1, A_2, \ldots, A_n$ containing no isolated vertices,*

$$\prod_{i=1}^{n}\gamma_t(A_i) \leqslant n\gamma_t(A_1\Box A_2\Box\cdots\Box A_n)\ .$$

In 2007, Brešar, Henning and Rall [2] proved that $\gamma_{pr}(G)\gamma_{pr}(H) \leqslant 8\gamma_{pr}(G\Box H)$, and in 2010, Hou and Jiang [5] proved that $\gamma_{pr}(G)\gamma_{pr}(H) \leqslant 7\gamma_{pr}(G\Box H)$. We extend these results to the n-product case, and attain an improvement to these inequalities as a corollary:

Theorem 3. *Given graphs $A_1, \ldots, A_n$ containing no isolated vertices,*

$$\prod_{i=1}^{n}\gamma_{pr}(A_i) \leqslant 2^{n-1}(2n-1)\gamma_{pr}(A_1\Box\cdots\Box A_n)\ .$$

Corollary. *Given graphs G and H containing no isolated vertices,*

$$\gamma_{pr}(G)\gamma_{pr}(H) \leqslant 6\gamma_{pr}(G\Box H)\ .$$

2 Main Results

We begin by introducing some notation which will be utilized throughout the proofs in this section. Given $S \subseteq V(A_1 \Box \cdots \Box A_n)$, the projection of S onto graph A_i is defined as

$$\Phi_{A_i}(S) = \{a \in V(A_i) \mid \exists\, u_1 \cdots u_n \in S \text{ with } a = u_i\} .$$

We partition the set of edges $E(A_1 \Box \cdots \Box A_n)$ into n sets. Thus, we define E_i to be

$$E_i = \Big\{ \big(u_1 \cdots u_n, v_1 \cdots v_n\big) \mid (u_i, v_i) \in E(A_i), \text{ and } u_j = v_j, \text{ for all other indices } j \neq i \Big\} .$$

An edge $e \in E_i$ is said to be an E_i-edge. For $u \in V(A_1 \Box \cdots \Box A_n)$, the i-neighborhood of u is defined as follows:

$$N_{\Box A_i}(u) = \Big\{ v \in V(A_1 \Box \cdots \Box A_n) \mid v \text{ and } u \text{ are connected by } E_i\text{-edge} \Big\} .$$

Finally, we present a proposition utilized throughout our proofs. Although the more general n-dimensional case stated in Prop. 2 is the proposition referenced within the proofs, we begin by separately stating the 2-dimensional case to clarify the overall idea.

Proposition 1. *Let M be a matrix containing only 0/1 entries. Then at least one of the following two statements are true:*

(a) each column contains a 1,

(b) each row contains a 0 .

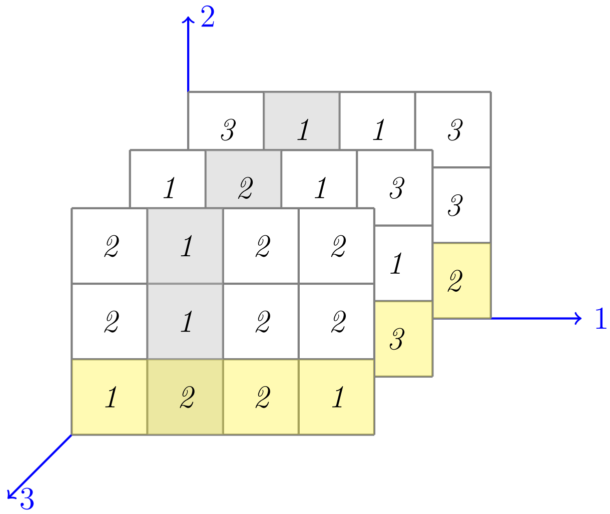
Prop. 1 refers only to binary matrices, or matrices containing only 0/1 entries. Prop. 2 refers to *n-ary* matrices, or (in this case) matrices containing only entries in $\{1, \ldots, n\}$. Furthermore, Prop. 1 refers only to two-dimensional matrices, or $d_1 \times d_2$ matrices M. Prop. 2 refers to n-dimensional matrices, or $d_1 \times d_2 \times \cdots \times d_n$ matrices M.

Definition 1. *Let M be a $d_1 \times d_2 \times \cdots \times d_n$, n-ary matrix. Then M is a $\underline{j\text{-matrix}}$ if there exists a $j \in \{1, \ldots, n\}$ (not necessarily unique), such that* ***each*** *of the $d_1 \times \cdots \times d_{j-1} \times 1 \times d_{j+1} \times \cdots \times d_n$ submatrices of M contains an entry with value j.*

Proposition 2. *Every $d_1 \times d_2 \times \cdots \times d_n$, n-ary matrix M is a j-matrix for some $j \in \{1, \ldots, n\}$ (not necessarily unique).*

Note that, given any $d_1 \times d_2 \times \cdots \times d_n$ matrix, there are d_j submatrices of the form $d_1 \times \cdots \times d_{j-1} \times 1 \times d_{j+1} \times \cdots \times d_n$. Following standard Matlab notation, we will denote such a submatrix as $M[:, \ldots, :, i_j, :, \ldots, :]$ with $1 \leqslant i_j \leqslant d_j$.

Example 1. *Here we see a $4 \times 3 \times 3$, 3-ary (the entries are contained in the set $\{1, 2, 3\}$) matrix M. Using the notation specified above, $M[2, :, :]$ is denoted by the gray shaded region and $M[:, 1, :]$ is denoted by yellow shaded region. M is both 1-matrix and 2-matrix, but not a 3-matrix.*



Proof. For a pigeon-hole principle style proof, let M be a $d_1 \times d_2 \times \cdots \times d_n$ n-ary matrix which is **not** a j-matrix for $1 \leqslant j \leqslant n-1$. We will show that M is an n-matrix.

Consider $j = 1$. Since M is not a 1-matrix, there exists at least one $1 \times d_2 \times d_3 \times \cdots \times d_n$ submatrix that does *not* contain a 1. Without loss of generality, let $M[i_1, :, \ldots, :]$ with $1 \leqslant i_1 \leqslant d_1$ be such a matrix. Next, consider $j = 2$. Since M is also not a 2-matrix, let $M[:, i_2, :, \ldots, :]$ with $1 \leqslant i_2 \leqslant d_2$ be a $d_1 \times 1 \times d_3 \times \cdots \times d_n$ submatrix that does *not* contain a 2. Therefore, $M[i_1, i_2, :, \ldots, :]$ is a $1 \times 1 \times d_3 \times \cdots \times d_n$ submatrix that contains neither a 1 nor a 2. We continue this pattern for $1 \leqslant j \leqslant n-1$. Since M is *not* a j-matrix for $1 \leqslant j \leqslant n-1$, let $M[i_1, \ldots, i_{n-1}, :]$ be the $1 \times 1 \cdots 1 \times d_n$ submatrix containing no elements in the set $\{1, \cdots, n-1\}$. Therefore, for all $1 \leqslant i \leqslant d_n$, $M[i_1, \ldots, i_{n-1}, i] = n$, *all* of the $d_1 \times \cdots \times d_{n-1} \times 1$ submatrices of M contain an entry with value n. Thus, M is an n-matrix. □

We now present the proofs of Theorems 1 through 3.

2.1 Proof of Theorem 1

Proof. Let $\{u_1^1, \ldots, u_{\gamma(A_1)}^1\}$ be a γ-set of A_1. Partition $V(A_1)$ into sets $D_1^1, \ldots, D_{\gamma(A_1)}^1$ such that $u_j^1 \in D_j^1 \subseteq N_{A_1}[u_j^1]$ for $1 \leqslant j \leqslant \gamma(A_1)$. Having partitioned $V(A_1)$ based on a minimum dominating set, we will now partition each of $V(A_2), \ldots, V(A_n)$ based on a minimum *total* dominating set. For $i = 2, \ldots, n$, let $\{u_1^i, \ldots, u_{\gamma_t(A_i)}^i\}$ be a γ_t-set of A_i, and $D_1^i, \ldots, D_{\gamma_t(A_i)}^i$ be the corresponding partitions of $V(A_i)$ such that $D_j^i \subseteq N_{A_i}(u_j^i)$. Note that this implies $u_j^i \notin D_j^i$.

Now let $Q = \{D_1^1, \ldots, D_{\gamma(A_1)}^1\} \times \{D_1^2, \ldots, D_{\gamma_t(A_2)}^2\} \times \cdots \times \{D_1^n, \ldots, D_{\gamma_t(A_n)}^n\}$. Then Q forms a partition of $V(A_1 \Box \cdots \Box A_n)$ with $|Q| = \gamma(A_1) \prod_{i=2}^{n} \gamma_t(A_i)$.

Let D be a γ-set of $A_1 \Box \cdots \Box A_n$. Then, for each $u \in V(A_1 \Box \cdots \Box A_n)$ not in D, there exists an i such that $N_{\Box A_i}(u) \cap D$ is non-empty. Based on this observation, we define an n-ary $|V(A_1)| \times \cdots \times |V(A_n)|$ matrix F such that:

$$F(u_1,\ldots,u_n)=\begin{cases}1 & \text{if } (u_1\cdots u_n)\in D \text{ , else}\\ i_{\min} & \text{where } i_{\min}=\min\{i \mid N_{\Box A_i}(u_1\cdots u_n)\cap D\neq\emptyset\} \text{ .}\end{cases}$$

Observe that $F(u_1,\ldots,u_n)=1$ has two meanings: either $(u_1\cdots u_n)\in D$ or $(u_1\cdots u_n)$ is dominated an an A_1-edge.

For $j=1,\ldots,n$, let $d_j\subseteq Q$ be the set of the elements in Q such that the corresponding submatrices of F are j-matrices. By Prop. 2, each element of Q belongs to at least one d_j-set. Then, $\gamma(A_1)\prod_{i=2}^{n}\gamma_t(A_i)\leqslant\sum_{j=1}^{n}|d_j|$.

Claim 1. $|d_1|\leqslant|D|$.

Proof. Similar to Q, let $B=\{D_1^2,\ldots,D_{\gamma_t(A_2)}^2\}\times\cdots\times\{D_1^n,\ldots,D_{\gamma_t(A_n)}^n\}$. For convenience, denote B as $\{B_1,\ldots,B_{|B|}\}$, where $|B|=\prod_{i=2}^{n}\gamma_t(A_i)$.

For $p=1,\ldots,|B|$, let $Z_p=D\cap(A_1\times B_p)$, and

$$S_p=\big\{D_x^1 \mid \text{the submatrix of } F \text{ determined by } D_x^1\times B_p \text{ is a 1-matrix, with } x\in\{1,\ldots,\gamma(A_1)\}\big\} \text{ .}$$

Note that if $D_x^1\times B_p$ is a 1-matrix for some $x\in\{1,\ldots,\gamma(A_1)\}$ and $p\in\{1,\ldots,|B|\}$, then for each $w\in D_x^1$, the submatrix of F corresponding to $\{w\}\times B_p$ contains a 1. Therefore, $\{w\}\times B_p$ contains a vertex in D or a vertex that is dominated by an A_1-edge. In either case, $\{w\}\times B_p$ contains a vertex that is dominated by Z_p. Thus, the projection of $D_x^1\times B_p$ on A_1 (i.e. D_x^1) is dominated by the projection of Z_p on A_1.

We now claim that for $p=1,\ldots,|B|$, $|S_p|\leqslant|Z_p|$. Let $S_p=\{D_{i_1}^1,D_{i_2}^1,\ldots,D_{i_t}^1\}$ and let $\Phi_{A_1}(Z_p)$ be the projection of Z_p on A_1. Then, $\Phi_{A_1}(Z_p)$ dominates $\cup_{x=1}^{t}D_{i_x}^1$, and for $i\notin\{i_1,i_2,\ldots,i_t\}$, u_i^1 dominates D_i^1. Thus, $\Phi_{A_1}(Z_p)\cup\big\{u_i^1 \mid i\notin\{i_1,i_2,\ldots,i_t\}\big\}$ is a dominating set for A_1. Now, $\big|\Phi_{A_1}(Z_p)\cup\big\{u_i^1 \mid i\notin\{i_1,i_2,\ldots,i_t\}\big\}\big|\geqslant\gamma(A_1)$, and $\big|\big\{u_i^1 \mid i\notin\{i_1,i_2,\ldots,i_t\}\big\}\big|=\gamma(A_1)-t$. Thus, $|\Phi_{A_1}(Z_p)|\geqslant t$. Hence, $|S_p|=t\leqslant|\Phi_{A_1}(Z_p)|\leqslant|Z_p|$. Finally, we see that $|d_1|=\sum_{p=1}^{|B|}|S_p|\leqslant\sum_{p=1}^{|B|}|Z_p|=|D|$. □

Claim 2. *For $j=2,\ldots,n$, $|d_j|\leqslant|D|$.*

Proof. We prove here that $|d_n|\leqslant|D|$, but a similar proof can be performed for any other $j\geqslant 2$. Let $B=\{D_1^1,\ldots,D_{\gamma(A_1)}^1\}\times\{D_1^2,\ldots,D_{\gamma_t(A_2)}^2\}\times\cdots\times\{D_1^{n-1},\ldots,D_{\gamma_t(A_{n-1})}^{n-1}\}$. For convenience, denote B as $\{B_1,\ldots,B_{|B|}\}$, where $|B|=\gamma(A_1)\prod_{i=2}^{(n-1)}\gamma_t(A_i)$.

For $p=1,\ldots,|B|$, let $Z_p=D\cap(B_p\times A_n)$, and

$$S_p=\big\{D_x^n \mid \text{the submatrix of } F \text{ determined by } B_p\times D_x^n \text{ is an } n\text{-matrix, with } x\in\{1,\ldots,\gamma_t(A_n)\}\big\} \text{ .}$$

Note that if $B_p \times D_x^n$ is a n-matrix for some $p \in \{1, \ldots, |B|\}$ and $x \in \{1, \ldots, \gamma_t(A_n)\}$, then, for each $w \in D_x^n$, the submatrix of F corresponding to $B_p \times \{w\}$ contains an n. Therefore, each $B_p \times \{w\}$ contains a vertex that is dominated by an A_n-edge. Therefore, each vertex in the projection of $B_p \times D_x^n$ on A_n (i.e., each vertex $w \in D_x^n \subseteq V(A_n)$) is dominated by a vertex from $\Phi_{A_n}(Z_p)$, *other than itself*. In other words, the projection of $B_p \times D_x^n$ on A_n (i.e. D_x^n) is *non-self-dominated* by the projection of Z_p on A_n.

We now claim that for $p = 1, \ldots, |B|$, $|S_p| \leqslant |Z_p|$. Let $S_p = \{D_{i_1}^n, D_{i_2}^n, \ldots, D_{i_t}^n\}$ and let $\Phi_{A_n}(Z_p)$ be the projection of Z_p on A_n. Then, $\cup_{x=1}^t D_{i_x}^n$ is *non-self-dominated* by $\Phi_{A_n}(Z_p)$, and $\Phi_{A_n}(Z_p) \cup \left\{u_i^n \mid i \notin \{i_1, i_2, \ldots, i_t\}\right\}$ is a total dominating set of A_n. Now, $\left|\Phi_{A_n}(Z_p) \cup \left\{u_i^n \mid i \notin \{i_1, i_2, \ldots, i_t\}\right\}\right| \geqslant \gamma_t(A_n)$, and $\left|\left\{u_i^n \mid i \notin \{i_1, i_2, \ldots, i_t\}\right\}\right| = \gamma_t(A_n) - t$. Thus, $|\Phi_{A_n}(Z_p)| \geqslant t$. Hence, $|S_p| = t \leqslant |\Phi_{A_n}(Z_p)| \leqslant |Z_p|$. Finally, we see that $|d_n| = \displaystyle\sum_{p=1}^{|B|} |S_p| \leqslant \sum_{p=1}^{|B|} |Z_p| = |D|$. □

To conclude the proof, we observe that

$$\gamma(A_1) \prod_{i=2}^{n} \gamma_t(A_i) \leqslant \sum_{j=1}^{n} |d_j| \leqslant n|D| = n\gamma(A_1 \Box \cdots \Box A_n) \ .$$

□

We conclude this section with the following corollary.

Corollary. *Given graphs G and H containing no isolated vertices,*

$$\max\{\gamma(G)\gamma_t(H), \gamma_t(G)\gamma(H)\} \leqslant 2\gamma(G \Box H) \ .$$

2.2 Proof of Theorem 2

Proof. For $i = 1, \ldots, n$, let $\{u_1^i, ..., u_{\gamma_t(A_i)}^i\}$ be a γ_t-set of A_i, and $D_1^i, \ldots, D_{\gamma_t(A_i)}^i$ be the corresponding partitions of $V(A_i)$ such that $D_j^i \subseteq N_{A_i}(u_j^i)$.

Let $Q = \{D_1^1, \ldots, D_{\gamma_t(A_1)}^1\} \times \cdots \times \{D_1^n, \ldots, D_{\gamma_t(A_n)}^n\}$. Then Q forms a partition of $V(A_1 \Box \cdots \Box A_n)$ with $|Q| = \displaystyle\prod_{i=1}^{n} \gamma_t(A_i)$.

Let D be a γ_t-set of $A_1 \Box \cdots \Box A_n$. Then, for each $u \in V(A_1 \Box \cdots \Box A_n)$, there exists an i such that $N_{\Box A_i}(u) \cap D$ is non-empty. Based on this observation we define an n-ary $|V(A_1)| \times \cdots \times |V(A_n)|$ matrix F such that:

$$F(u_1, \ldots, u_n) = \min\{i \mid N_{\Box A_i}(u_1 \cdots u_n) \cap D \neq \emptyset\} \ .$$

For $j = 1, \ldots, n$, let $d_j \subseteq Q$ be the set of the elements in Q which are j-matrices. By Prop. 2, each element of Q belongs to at least one d_j-set. Then, $\displaystyle\prod_{i=1}^{n} \gamma_t(A_i) \leqslant \sum_{j=1}^{n} |d_j|$.

Claim 3. *For $j = 1, \ldots, n$, $|d_j| \leqslant |D|$.*

Proof. The proof is copy of proof of Claim 2, except that here $B = \{D_1^1, \ldots, D_{\gamma_t(A_1)}^1\} \times \cdots \times \{D_1^{n-1}, \ldots, D_{\gamma_t(A_{n-1})}^{n-1}\}$, and $|B| = \prod_{i=1}^{(n-1)} \gamma_t(A_i)$. □

Thus we have, $\prod_{i=1}^{n} \gamma_t(A_i) \leqslant \sum_{j=1}^{n} |d_j| \leqslant n|D| = n\gamma_t(A_1 \square \cdots \square A_n)$. □

We conclude by observing that Ho's inequality follows as a corollary to Thm. 2.

Corollary (Ho [4])**.** *Given graphs G and H containing no isolated vertices,*

$$\gamma_t(G)\gamma_t(H) \leqslant 2\gamma_t(G \square H) \ .$$

2.3 Proof of Theorem 3

Proof. For $i = 1, \ldots, n$, let $k_i = \gamma_{pr}(A_i)/2$, and $\{x_1^i, y_1^i, \ldots, x_{k_i}^i, y_{k_i}^i\}$ be a γ_{pr}-set of A_i, where for each A_i and $1 \leqslant j \leqslant k_i$, $(x_j^i, y_j^i) \in E(A_i)$. Partition each $V(A_i)$ into sets $D_1^i, \ldots, D_{k_i}^i$, such that $\{x_j^i, y_j^i\} \subseteq D_j^i \subseteq N_{A_i}[x_j^i] \cup N_{A_i}[y_j^i]$ for $1 \leqslant j \leqslant k_i$.

Let $Q = \{D_1^1, \ldots, D_{k_1}^1\} \times \cdots \times \{D_1^n, \ldots, D_{k_n}^n\}$. Then Q forms a partition of $V(A_1 \square \cdots \square A_n)$ with $|Q| = \prod_{i=1}^{n} \gamma_{pr}(A_i)/2 = \frac{1}{2^n} \prod_{i=1}^{n} \gamma_{pr}(A_i)$.

Let D be a γ_{pr}-set of $A_1 \square \cdots \square A_n$. Then, for each $u \in V(A_1 \square \cdots \square A_n)$, there exists an i such that $N_{\square A_i}(u) \cap D$ is non-empty. Based on this observation, we define n different matrices F^i with $i = 1, \ldots, n$, where each of the n matrices is an n-ary $|V(A_1)| \times \cdots \times |V(A_n)|$ matrix F^i such that:

$$F^i(u_1, \ldots, u_n) = \begin{cases} i & \text{if } u_1 \cdots u_n \in D \ , \text{ else} \\ j_{\min} & \text{where } j_{\min} = \min\{\ j \mid N_{\square A_j}(u_1 \cdots u_n) \cap D \neq \emptyset\} \ . \end{cases}$$

Thus, each of the n matrices F^i with $i = 1, \ldots, n$ differs only in the entries that correspond to vertices in the paired dominating set D.

For $j = 1, \ldots, n$ and $i = 1, \ldots, n$, let $d_j^i \subseteq Q$ be the set of the elements in Q which are j-matrices in the matrix F^i. By Prop. 2, each element of Q belongs to at least one d_j^i-set for each $i = 1, \ldots, n$. Now, if an element $q \in Q$ belongs to the d_j^i-set, then q also belongs to the d_j^j-set. To see this, if M_i and M_j are the submatrices determined by q with respect to the matrices F^i and F^j, respectively, then all the entries that do not match in M_i and M_j have value j in M_j. Thus, each $q \in Q$ belongs to at least one d_i^i-set for some $i \in \{1, \ldots, n\}$. Then, $\frac{1}{2^n} \prod_{i=1}^{n} \gamma_{pr}(A_i) \leqslant \sum_{i=1}^{n} |d_i^i|$.

Similar to Q, let $B = \{D_1^1, \ldots, D_{k_1}^1\} \times \cdots \times \{D_1^{n-1}, \ldots, D_{k_{n-1}}^{n-1}\}$. For convenience, we denote B as $\{B_1, \ldots, B_{|B|}\}$, where $|B| = \prod_{i=1}^{n-1} \gamma_{pr}(A_i)/2 = \frac{1}{2^{n-1}} \prod_{i=1}^{n-1} \gamma_{pr}(A_i)$. Since D is a γ_{pr}-set, the subgraph of $A_1 \square \cdots \square A_n$ induced by D has a perfect matching. Let

$$D_i = \{u \in D \mid \text{the matching edge incident to } u \text{ is in } E_i\} \ .$$

Then, D can be written as the disjoint union of the subsets D_i. For $p = 1, \ldots, |B|$ and $i = 1, \ldots, n$, let $Z_p^i = D_i \cap (B_p \times A_n)$, and

$$S_p = \big\{D_x^n \mid \text{the submatrix of } F^n \text{ determined by } B_p \times D_x^n \text{ is an } n\text{-matrix}, \\ \text{with } x \in \{1, \ldots, k_n\}\big\} \ .$$

Then, $|d_n^n| = \sum_{p=1}^{|B|} |S_p|$.

Claim 4. *For $p = 1, \ldots, |B|$, $2|S_p| \leqslant 2|Z_p^1| + \cdots + 2|Z_p^{n-1}| + |Z_p^n|$.*

Proof. Let $S_p = \{D_{j_1}^n, D_{j_2}^n, \ldots, D_{j_t}^n\}$, and for $j = 1, \ldots, n$, let $V_j = \Phi_{A_n}(Z_p^j)$. Note that $|V_j| \leqslant |Z_p^j|$. Additionally, let $C = \big\{x_j^n \mid j \notin \{j_1, j_2, \ldots, j_t\}\big\} \cup \big\{y_j^n \mid j \notin \{j_1, j_2, \ldots, j_t\}\big\}$.

Let M be the matching on $V_n \cup C$ formed by taking all of the $\{x_j^n, y_j^n\}$ edges induced by the vertices in C, and then adding the edges from a maximal matching on the remaining unmatched vertices in V_n. Then, $E = V_1 \cup \cdots \cup V_n \cup C$ is a dominating set of A_n with M as a matching. Let $M_1 = V(M)$ and $M_2 = (V_n \cup C) \setminus M_1$. We note that M_1 consists of all the vertices in C plus the matched vertices from V_n, and M_2 contains only the unmatched vertices from V_n.

In order to obtain a perfect matching, we recursively modify E by choosing an unmatched vertex a in E, and then either matching it with an appropriate vertex, or removing it from E. Specifically, if $N_{A_n}(a) \setminus V(M)$ is non-empty, there exists a vertex $a' \in N_{A_n}(a) \setminus V(M)$ such that we can add a' to E and (a, a') to the matching M. Otherwise, a is incident on only matched vertices, and we can safely remove it from E without altering the fact that E is a dominating set.

Our recursively modified E (denoted by E_{rec}) is now a paired dominating set of A_n. Furthermore, in the worst case, we have doubled the unmatched vertices from V_n, and also doubled the vertices in $V_1, \ldots, V_{n-1}$. Thus,

$$2k_n \leqslant |E_{\text{rec}}| \leqslant 2|V_1| + \cdots + 2|V_{n-1}| + |M_1| + 2|M_2| \ .$$

This implies that $2k_n - |C| \leqslant 2|V_1| + \cdots + 2|V_{n-1}| + |Z_p^n|$. Since $2k_n - |C| = 2|S_p|$, therefore, $2|S_p| \leqslant 2|V_1| + \cdots + 2|V_{n-1}| + |Z_p^n| \leqslant 2|Z_p^1| + \cdots + 2|Z_p^{n-1}| + |Z_p^n|$. □

Using Claim 4, we now see

$$2\sum_{p=1}^{|B|} |S_p| \leqslant \sum_{p=1}^{|B|} \Big(2\sum_{j=1}^{n} |Z_p^j| - |Z_p^n|\Big) = 2|D| - \sum_{p=1}^{|B|} |Z_p^n| = 2|D| - |D_n| \ .$$

Therefore, $2|d_n^n| \leqslant 2|D| - |D_n|$. Similarly, we can show that $2|d_i^i| \leqslant 2|D| - |D_i|$ for $i = 1, \ldots, n$. To conclude the proof, we see

$$\frac{1}{2^{n-1}} \prod_{i=1}^{n} \gamma_{pr}(A_i) = 2(k_1 \cdots k_n) \leqslant 2 \sum_{i=1}^{n} |d_i^i| \leqslant 2n|D| - \sum_{i=1}^{n} |D_i| = (2n-1)|D| \ ,$$

$$\prod_{i=1}^{n} \gamma_{pr}(A_i) \leqslant 2^{n-1}(2n-1)\gamma_{pr}(A_1 \Box \cdots \Box A_n) \ .$$

□

We conclude this section by observing that an improvement to the Hou-Jiang [5] inequality $\gamma_{pr}(G)\gamma_{pr}(H) \leqslant 7\gamma_{pr}(G \Box H)$ follows as a corollary:

Corollary. *Given graphs G and H containing no isolated vertices,*

$$\gamma_{pr}(G)\gamma_{pr}(H) \leqslant 6\gamma_{pr}(G \Box H) \ .$$

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