Coloring 2-intersecting hypergraphs

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Abstract

A hypergraph is 2-intersecting if any two edges intersect in at least two vertices. Blais, Weinstein and Yoshida asked (as a first step to a more general problem) whether every 2-intersecting hypergraph has a vertex coloring with a constant number of colors so that each hyperedge has at least $\min\{|e|, 3\}$ colors. We show that there is such a coloring with at most 5 colors (which is best possible).

A *proper coloring* of a hypergraph is a coloring of its vertices so that no edge is monochromatic, i.e. contains at least two vertices with distinct colors. It is well-known that intersecting hypergraphs without singleton edges have proper colorings with at most three colors. This statement is from the seminal paper of Erdős and Lovász [3]. Recently Blais, Weinstein and Yoshida suggested a generalization in [1]. They consider t-intersecting hypergraphs, in which any two edges intersect in at least t vertices and they call a coloring of the vertices *c-strong* if every edge e is colored with at least $\min\{|e|, c\}$ distinct colors. One of the problems they consider is the following.

Problem 1. *([1]) Suppose that $\mathcal{H}$ is a t-intersecting hypergraph. Is there a $(t+1)$-strong vertex coloring of $\mathcal{H}$ where the number of colors is bounded by a function of t? In particular, is there a $t+1$-strong vertex coloring with at most $2t+1$ colors? If true, it would be best possible, as the $2t$-element sets of a $3t$ element set demonstrate.*

Notice that for $t = 1$ the answer to Problem 1 is affirmative (for both parts) according to the starting remark but open for $t \geqslant 2$ [1]. Our aim is to give an affirmative answer to both parts of the problem in case of $t = 2$. Notice that intersecting hypergraphs do

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not always have 3-strong colorings with any fixed number of colors: if every edge of a $(k+1)$-chromatic graph is extended by the same new vertex, the resulting intersecting hypergraph has no 3-strong coloring with k colors. Thus the 2-intersecting condition is important in the following theorem.

Theorem 2. *Every 2-intersecting hypergraph G has a 3-strong coloring with at most five colors.*

We learned from a referee that a weaker form of Theorem 2 (with 21 colors instead of 5) is proved recently in [2]. We also prove a lemma that will be used in the proof of Theorem 2 but has independent interest. A hypergraph has property P_t for some integer $t \geqslant 2$ if any i edges intersect in at least $t+1-i$ vertices, for all $i, 2 \leqslant i \leqslant t$.

Lemma 3. *Suppose that $\mathcal{H}$ is a hypergraph with property P_t. Then $\mathcal{H}$ has a t-strong coloring with at most $t+1$ colors.*

Proof. Let $\mathcal{H}$ be a hypergraph with property P_t for $t \geqslant 2$. Select an edge e of $\mathcal{H}$ which is minimal for containment. Let $\mathcal{F}$ be the hypergraph defined on the vertex set of e with edge set $\{h \cap e : h \in E(\mathcal{H})\}$. Color each vertex not in e with color $t+1$. If $t = 2$, color the vertices of e arbitrarily using colors 1,2 (or just color 1 if e has just one vertex). If $|e| = t-1$, color vertices of e by $1, 2, \ldots, t-1$. Otherwise, since $\mathcal{F}$ has property P_{t-1}, we can find by induction a $(t-1)$-strong coloring C on $\mathcal{F}$ with colors $1, 2, \ldots, t$. We may suppose that C uses all colors $1, 2, \ldots, t$ on e, otherwise we may change some repeated colors to the missing colors maintaining the $(t-1)$-strong coloring. Thus C colors e with at least t colors and, since for any other edge $h \in \mathcal{H}$, $|h \cap e| \geqslant t-1$, C uses at least $t-1$ colors on $h \cap e$ and h also has at least one vertex of color $t+1$. Therefore we have a t-strong coloring of $\mathcal{H}$ with $t+1$ colors. $\square$

It is worth noting that Lemma 3 does not hold if we require a t-strong coloring with at most t colors. Indeed, all t-sets of $t+1$ elements have property P_t but a t-strong coloring must use $t+1$ colors. $\square$

Proof of Theorem 2. By the condition, there are no singleton edges. Also, if some edge e has just two vertices, coloring them with colors $1, 2$ and all other vertices by 3, we obviously have a 3-strong coloring. Thus we may assume that every edge has at least three vertices, therefore a 3-strong coloring on the minimal edges of G is also a 3-strong coloring on G. Thus we may assume that G is an antichain.

If any three edges of G have non-empty intersection, we can apply Lemma 3 and get a 3-strong coloring with at most 4 colors. Thus, we may suppose that G contains three edges with empty intersection, select them with the smallest possible union, let these edges be e_1, e_2, e_3 and set $X = e_1 \cup e_2 \cup e_3$. A vertex $v \in X$ is called a private part of e_i $(i = 1, 2, 3)$ if $v \in e_i$ but v is not covered by any of the other two e_j-s.

We color the vertices in X as follows. The private parts of e_1, e_2, e_3 (if they exist) are colored with $1, 2, 3$ respectively. Notice that each intersection has at least two vertices, color $e_1 \cap e_3$ with colors $1, 3$ so that color 1 is used only once, color $e_1 \cap e_2$ with colors $2, 4$ so that color 2 is used only once. Vertices in $e_2 \cap e_3$ are all colored with color 5.

The coloring outside X varies according to the number of private parts of e_i-s.

Case 1. Each e_i has private parts, $i = 1, 2, 3$.

Here we color vertices not covered by X one by one with 1 or 2 by the following greedy type algorithm: if an uncolored vertex $w \notin X$ completes an edge f such that all vertices of $f - \{w\}$ are colored with colors $2, 3$ only (both present otherwise $|f \cap e_1| \leqslant 1$ or $|f \cap e_2| \leqslant 1$) then color w with color 1, otherwise color it with color 2. We claim that a 3-strong coloring is obtained.

Suppose there is an edge f_{ij} with colors i, j only, $1 \leqslant i < j \leqslant 5$. Edges f_{12}, f_{14}, f_{24} would intersect e_3 in at most one vertex, edge f_{25} would intersect e_1 in at most one vertex and f_{13} would not intersect e_2 at all. Edges f_{35}, f_{45} would form a proper subset of e_3, e_2, respectively, contradicting the antichain property.

Edge f_{34} cannot exist because the triple f_{34}, e_2, e_3 has no intersection and $Y = f_{34} \cup e_2 \cup e_3$ is a proper subset of X because e_1 has a private vertex. Thus we get a contradiction with the definition of e_1, e_2, e_3. The same argument can be applied to exclude $f_{15}, f_{23} \subset X$ (with $Y = f_{15} \cup e_1 \cup e_2, Y = f_{23} \cup e_2 \cup e_3$ and using that e_3, e_1 have private vertices).

Thus the only possibility is that there is an edge f_{15} or f_{23} with some vertex $w \notin X$. However, no such f_{15} exists since $w \notin X$ is colored with 1 only if there exists edge f of G such that $f - \{w\}$ is colored with colors $2, 3$ only thus $|f \cap f_{15}| = 1$ contradiction. Moreover, no such f_{23} can exist either, because its vertex in $V - X$ colored last got color 1 according to the rule governing Case 1.

Case 2. Two of e_1, e_2, e_3 have private parts, by suitable relabeling we may suppose that the private part of e_2 is empty.

In this case vertices not covered by X are colored with color 2 and claim that we have a 3-strong coloring. The nonexistence of $f_{12}, f_{13}, f_{14}, f_{24}, f_{25}$ follow as in Case 1 and here f_{23} can be excluded the same way since $|f_{23} \cap e_2| \leqslant 1$. The exclusion of f_{34}, f_{35}, f_{45} and $f_{15} \subset X$ is also exactly the same as in Case 1. Thus here we have to exclude only the existence of an edge f_{15} containing some vertices $w \notin X$. However, this cannot happen since here every vertex outside X is colored with color 2.

Case 3. Exactly one of e_1, e_2, e_3 has a private part, by suitable relabeling we may suppose that it is e_2.

Here all vertices not covered by X are colored with 1. Edges $f_{12}, f_{13}, f_{14}, f_{15}, f_{24}, f_{25}$ are all excluded since there is some e_i intersecting them in at most one vertex. The edges f_{34}, f_{35}, f_{45} are excluded since they are proper subsets of some e_i. The only possible edge is f_{23} but in this case we can replace the triple e_1, e_2, e_3 by the non-intersecting triple f_{23}, e_2, e_3 which has the same union but they have two private parts: the vertices of color 4 in e_2 and the vertex of color 1 in e_3. This reduces Case 3 to Case 2.

Case 4. None of the edges e_1, e_2, e_3 have private parts.

Vertices uncovered by X are colored with 1. Here $f_{12}, f_{13}, f_{14}, f_{15}, f_{23}, f_{24}, f_{25}$ are all excluded since there is some e_i intersecting them in at most one vertex. The other three edges f_{34}, f_{35}, f_{45} are excluded since they are proper subsets of some e_i.

In all cases we found a 3-strong coloring with at most five colors. $\square$

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