Intersecting k-uniform families containing all the k-subsets of a given set

Wei-Tian Li*
Department of Applied Mathematics
National Chung Hsing University
Taichung 40227, Taiwan
weitianli@dragon.nchu.edu.tw

Bor-Liang Chen
Department of Business Administration
National Taichung University of Science and Technology
Taichung 40401, Taiwan
blchen@ntit.edu.tw

Kuo-Ching Huang †
Department of Financial and Computational Mathematics
Providence University
Taichung 43301, Taiwan
kchuang@gm.pu.edu.tw

Ko-Wei Lih ‡
Institute of Mathematics
Academia Sinica
Taipei 10617, Taiwan
makwlih@sinica.edu.tw

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Abstract

Let m, n, and k be integers satisfying $0 < k \leqslant n < 2k \leqslant m$. A family of sets $\mathcal{F}$ is called an (m, n, k)-*intersecting family* if $\binom{[n]}{k} \subseteq \mathcal{F} \subseteq \binom{[m]}{k}$ and any pair of members of $\mathcal{F}$ have nonempty intersection. Maximum (m, k, k)- and $(m, k+1, k)$-intersecting families are determined by the theorems of Erdős-Ko-Rado and Hilton-Milner, respectively. We determine the maximum families for the cases $n = 2k-1$, $2k-2, 2k-3$, or m sufficiently large.

Keywords: intersecting family; cross-intersecting family; Erdős-Ko-Rado; Milner-Hilton; Kneser graph

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1 Introduction

For positive integers $a \leqslant b$, define $[a,b] = \{a, a+1, \ldots, b\}$ and let $[a] = [1,a]$. The cardinality of a set X is denoted by $|X|$. A set of cardinality n is called an n-set. A family of subsets of X is said to be *intersecting* if no two members are disjoint. The family of all k-subsets of X is denoted by $\binom{X}{k}$. Note that $\binom{[m]}{k}$ is intersecting if $0 < k \leqslant m < 2k$. If all members of a family $\mathcal{F} \subseteq \binom{[m]}{k}$ contain a fixed element, then $\mathcal{F}$ is obviously an intersecting family and is said to be *trivial*. A trivial intersecting family can have at most $\binom{m-1}{k-1}$ members. One of the cornerstones of the extremal theory of finite sets is the following pioneering result of Erdős, Ko, and Rado [5].

Theorem 1. *Suppose $0 < 2k < m$. Let $\mathcal{F} \subseteq \binom{[m]}{k}$ be an intersecting family. Then $|\mathcal{F}| \leqslant \binom{m-1}{k-1}$. Moreover, the equality holds if and only if $\mathcal{F}$ consists of all k-subsets containing a fixed element.*

Let $A \in \binom{[m]}{k}$ and $t \notin A$. Define $\mathcal{M}_1(A;t) = \{A\} \cup \{B \in \binom{[m]}{k} \mid t \in B \text{ and } A \cap B \neq \emptyset\}$. Clearly $|\mathcal{M}_1(A;t)| = \binom{m-1}{k-1} - \binom{m-1-k}{k-1} + 1$. Let $X \in \binom{[m]}{3}$. Define $\mathcal{M}_2(X) = \{B \in \binom{[m]}{k} \mid |X \cap B| \geqslant 2\}$. Both $\mathcal{M}_1(A;t)$ and $\mathcal{M}_2(X)$ are intersecting families. The largest size of a non-trivial intersecting family was determined in the following result of Hilton and Milner [10].

Theorem 2. *Suppose $0 < 2k < m$. Let $\mathcal{F} \subseteq \binom{[m]}{k}$ be an intersecting family such that $\cap\{A \mid A \in \mathcal{F}\} = \emptyset$. Then $|\mathcal{F}| \leqslant \binom{m-1}{k-1} - \binom{m-1-k}{k-1} + 1$. Moreover, the equality holds if and only if $\mathcal{F}$ is of the form $\mathcal{M}_1(A;t)$ or the form $\mathcal{M}_2(X)$, the latter occurs only for $k = 3$.*

In a more general form, the Erdő-Ko-Rado theorem describes the size and structure of the largest collection of k-subsets of an n-set having the property that the intersection of any two subsets contains at least t elements. This theorem has motivated a great deal of development of finite extremal set theory since its first publication in 1961. The complete establishment of the general form was achieved through cumulative works of Frankl [6], Wilson [12], and Ahlswede and Khachatrian [2]. Ahlswede and Khachatrian [1] even extended the Hilton-Milner theorem in the general case. The reader is referred to Deza and Frankl [4], Frankl [7], and Borg [3] for surveys on relevant results.

Let $0 < k \leqslant n < 2k \leqslant m$. We call an intersecting family $\mathcal{F}$ an (m,n,k)-*intersecting family* if $\binom{[n]}{k} \subseteq \mathcal{F} \subseteq \binom{[m]}{k}$. Define $\alpha(m,n,k) = \max\{|\mathcal{F}| \mid \mathcal{F} \text{ is an } (m,n,k)\text{-intersecting family}\}$. An (m,n,k)-intersecting family with cardinality $\alpha(m,n,k)$ is called a *maximum* family. The focus for our study is the following.

Problem 3. For $0 < k \leqslant n < 2k \leqslant m$, determine $\alpha(m,n,k)$ and the corresponding maximum families.

Suppose that $\mathcal{F}$ is an (m,n,k)-intersecting family. If any $A \in \mathcal{F}$ satisfies $|A \cap [n]| \leqslant n-k$, then $|[n] \setminus A| \geqslant n-(n-k) = k$. Hence, there exists a k-subset $B \subseteq [n] \setminus A$. It is clear that $B \in \mathcal{F}$ and $B \cap A = \emptyset$, violating the intersecting condition on $\mathcal{F}$. Hence, we have a size constraint on any $A \in \mathcal{F}$: $|A \cap [n]| \geqslant n-k+1$, or equivalently, $|A \setminus [n]| \leqslant 2k-n-1$.

For any fixed $t \in [n]$, define $\mathcal{H}_t^{m,n,k}$ to be the family consisting of all k-subsets of $[n]$ and those k-subsets which contain t and at least $n-k$ other elements from $[n]$, i.e.

$$\mathcal{H}_t^{m,n,k} = \binom{[n]}{k} \cup \bigcup_{i=1}^{2k-n-1} \left\{ A \cup B \cup \{t\} \,\middle|\, A \in \binom{[n]\setminus\{t\}}{k-i-1}, B \in \binom{[n+1,m]}{i} \right\}.$$

We often write $\mathcal{H}_t$ for $\mathcal{H}_t^{m,n,k}$ if the context is clear. It is easy to see that $\mathcal{H}_t$ is an (m,n,k)-intersecting family and its cardinality is equal to

$$h(m,n,k) = \binom{n}{k} + \sum_{i=1}^{2k-n-1} \binom{n-1}{k-i-1}\binom{m-n}{i}.$$

Hence, $\alpha(m,n,k) \geqslant h(m,n,k)$.

For the case $n = k$, Theorem 1 shows that $\alpha(m,k,k) = \binom{m-1}{k-1} = h(m,n,k)$ and all maximum families are of the form $\mathcal{H}_t$ for some $t \in [k]$. For the case $n = k+1$, a maximum family is non-trivial since $\binom{[k+1]}{k} = \{[k+1]\setminus\{i\} \mid 1 \leqslant i \leqslant k+1\}$ and $\cap\{A \mid A \in \binom{[k+1]}{k}\} = \emptyset$. Theorem 2 shows that $\alpha(m,k+1,k) = \binom{m-1}{k-1} - \binom{m-1-k}{k-1} + 1 = h(m,k+1,k)$ and all maximum families are of the form $\mathcal{M}_1(A;t) = \mathcal{H}_t$, where $t \in [k+1]$ and $A = [k+1]\setminus\{t\}$, or the form $\mathcal{M}_2(X)$, where $X \in \binom{[4]}{3}$, the latter occurs only for $k = 3$.

In view of the above paragraph, the theorems of Erdős-Ko-Rado and Hilton-Milner can be regarded as special solutions to Problem 3. For these two particular cases, the obvious lower bound $h(m,n,k)$ coincides with the maximum value and, except the case for $k = 3$ and $n = 4$, all maximum families are of the form $\mathcal{H}_t$. This phenomenon leads us to pose the following.

Problem 4. When does $\alpha(m,n,k) = h(m,n,k)$ hold? When it does, are $\mathcal{H}_t$'s the only maximum families?

In this paper, we give an affirmative answer $\alpha(m,n,k) = h(m,n,k)$ for the above questions when $n = 2k-1, 2k-2, 2k-3$, or m sufficiently large.

2 Main Tools

Frequently, extremal problems concerning sub-families of $\binom{[m]}{k}$ can be translated into the context of Kneser graphs so that graph-theoretical tools may be employed to solve them. For $0 < 2k \leqslant n$, a *Kneser graph* $\mathsf{KG}(n,k)$ has vertex set $\binom{[n]}{k}$ such that two vertices A and B are adjacent if and only if they are disjoint as subsets. By stipulation, we use $\mathsf{KG}(n,k)$ to denote the graph consisting of $\binom{n}{k}$ isolated vertices when $0 < k \leqslant n < 2k$. An *independent* set in a graph is a set of vertices no two of which are adjacent. The maximum cardinality of an independent set in a graph G is called the *independence number* of G and is denoted by $\alpha(G)$. The Erdős-Ko-Rado theorem just gives the independence number of a Kneser graph and characterizes all maximum independent sets.

The *direct product* $G \times H$ of two graphs G and H is defined on the vertex set $\{(u,v) \mid u \in G \text{ and } v \in H\}$ such that two vertices (u_1,v_1) and (u_2,v_2) are adjacent if and only if

u_1 is adjacent to u_2 in G and v_1 is adjacent to v_2 in H. The cardinality of the vertex set of a graph G is denoted by $|G|$. The following result is due to Zhang [13].

Theorem 5. *Let G and H be vertex-transitive graphs. Then $\alpha(G \times H) = \max\{\alpha(G)|H|, |G|\alpha(H)\}$. Furthermore, every maximum independent set of $G \times H$ is the pre-image of an independent set of G or H under projection.*

Since Kneser graphs are vertex-transitive, we are going to use the above theorem for $G = \mathsf{KG}(n_1, k_1)$ and $H = \mathsf{KG}(n_2, k_2)$. The version of Theorem 5 for Kneser graphs was established in an earlier paper [8] of Frankl.

We can derive the following by Theorem 1, Theorem 5, and direct computation.

Lemma 6. *When $2(k-i) \leqslant n$ and $2i \leqslant m-n$,*

$$\alpha(\mathsf{KG}(n, k-i) \times \mathsf{KG}(m-n, i)) = \begin{cases} \binom{n-1}{k-i-1}\binom{m-n}{i} & \text{if } m \geqslant nk/(k-i), \\ \binom{n}{k-i}\binom{m-n-1}{i-1} & \text{otherwise.} \end{cases}$$

When $2(k-i) > n$ or $2i > m-n$, $\alpha(\mathsf{KG}(n, k-i) \times \mathsf{KG}(m-n, i)) = \binom{n}{k-i}\binom{m-n}{i}$.

Two families of sets $\mathcal{A}$ and $\mathcal{B}$ are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ for any pair $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Frankl and Tokushige [9] proved the following.

Theorem 7. *Let $\mathcal{A} \subseteq \binom{X}{a}$ and $\mathcal{B} \subseteq \binom{X}{b}$ be nonempty cross-intersecting families of subsets of X. Suppose that $|X| \geqslant a+b$ and $a \leqslant b$. Then*

$$|\mathcal{A}| + |\mathcal{B}| \leqslant \binom{|X|}{b} - \binom{|X|-a}{b} + 1.$$

The above inequality provides a useful tool for handling our problems.

3 The cases for $m = 2k$, $n = 2k-1$, and $n = 2k-2$

Proposition 8. *We have $\alpha(2k, n, k) = \frac{1}{2}\binom{2k}{k} = h(2k, n, k)$ for all n $(k \leqslant n < 2k)$.*

This is true because any $(2k, n, k)$-intersecting family cannot contain a k-subset and its complement in $[2k]$ simultaneously. Any maximum family $\mathcal{F}$ can be obtained in the following manner. Pick a pair of a k-subset A and its complement $A' = [2k] \setminus A$. If A or A' is a subset of $[n]$, then we put it in $\mathcal{F}$. Otherwise, we put any one of them in $\mathcal{F}$.

A special case of the above construction for a maximum family is to choose the one that contains a prescribed element t when neither A nor A' is a subset of $[n]$. If $t \in [n]$, then the family so constructed is precisely $\mathcal{H}_t$.

Convention. From now on, we always assume that $0 < k \leqslant n < 2k < m$ for any (m, n, k)-intersecting family.

Proposition 9. *For $n = 2k-1$ and all $m > 2k$, we have $\alpha(m,n,k) = \binom{n}{k} = h(m,n,k)$ and $\binom{[n]}{k}$ is the unique maximum (m,n,k)-intersecting family.*

Proof. Let $\mathcal{F}$ be a maximum (m,n,k)-intersecting family. For any $A \in \mathcal{F}$, we know $k \geqslant |A \cap [n]| \geqslant n-k+1 = k$. Thus, $A \in \binom{[n]}{k}$, and hence $\mathcal{F} \subseteq \binom{[n]}{k}$. Therefore, $\mathcal{F} = \binom{[n]}{k}$ and $\alpha(m,n,k) = |\mathcal{F}| = \binom{n}{k} = h(m,n,k)$. Note that all $\mathcal{H}_t$'s are equal to $\binom{[n]}{k}$. □

Suppose that $\mathcal{F}$ is an (m,n,k)-intersecting family. Define its *canonical partition* as follows.

$$\mathcal{F} = \binom{[n]}{k} \cup \left(\bigcup_{i=1}^{2k-n-1} \mathcal{F}_i \right),$$

where $\mathcal{F}_i = \{F \in \mathcal{F} \mid |F \cap [n]| = k-i \text{ and } |F \cap [n+1,m]| = i\}$. For each i, we define an injection f_i from $\mathcal{F}_i$ to the vertex set of $\mathsf{KG}(n,k-i) \times \mathsf{KG}(m-n,i)$ such that $f_i(F) = (A, B^*)$, where $A = F \cap [n]$ and $B^* = \{b-n \mid b \in F \text{ and } b \geqslant n+1\}$. Since $\mathcal{F}_i$ is intersecting, it is easy to verify that the image of f_i is an independent set of $\mathsf{KG}(n,k-i) \times \mathsf{KG}(m-n,i)$. Thus, $|\mathcal{F}_i| \leqslant \alpha(\mathsf{KG}(n,k-i) \times \mathsf{KG}(m-n,i))$. We immediately obtain the following upper bound.

$$|\mathcal{F}| \leqslant \binom{n}{k} + \sum_{i=1}^{2k-n-1} \alpha(\mathsf{KG}(n,k-i) \times \mathsf{KG}(m-n,i)).$$

Theorem 10. *For $n = 2k-2$, we have $\alpha(m,n,k) = h(m,n,k)$. All the maximum families are of the form $\binom{[2k-2]}{k} \cup \{F \cup \{b\} \mid F \in \mathcal{F}^*, b \in [2k-1,m]\}$, where $\mathcal{F}^*$ is any maximum intersecting family of $(k-1)$-subsets of $[2k-2]$.*

Proof. Let $\mathcal{F}$ be a largest $(m, 2k-2, k)$-intersecting family with canonical partition $\binom{[2k-2]}{k} \cup \mathcal{F}_1$. Now, all the conditions $2(k-1) \leqslant n$, $2 \leqslant m-n$, and $m \geqslant nk/(k-1)$ hold. It follows from Lemma 6 that $|\mathcal{F}_1| \leqslant \binom{2k-3}{k-2}\binom{m-2k+2}{1}$. Then $|\mathcal{F}| = \binom{2k-2}{k} + |\mathcal{F}_1| \leqslant h(m,2k-2,k)$. As a consequence, $|\mathcal{F}| = h(m,2k-2,k)$ and $|\mathcal{F}_1| = \binom{2k-3}{k-2}\binom{m-2k+2}{1}$. By Theorem 5, $f_1(\mathcal{F}_1)$ is a maximum independent set in $\mathsf{KG}(2k-2,k-1) \times \mathsf{KG}(m-2k+2,1)$ and the collection $\mathcal{F}^*$ of all the first components of $f_1(\mathcal{F}_1)$ is an independent set of $\mathsf{KG}(2k-2,k-1)$. Clearly, $\mathcal{F}^*$ is maximum because of its cardinality. □

Remark. When $k = 3$, an $(m, 2k-2, k)$-family is also an $(m, k+1, k)$ family. There are other maximum families besides the collection of all $\mathcal{H}_t$'s. This phenomenon is consistent with the Hilton-Milner theorem for the case $k = 3$.

4 The case for $n = 2k-3$

Theorem 11. *For $n = 2k-3$, we have $\alpha(m,n,k) = h(m,n,k)$. All the maximum families are of the form $\mathcal{H}_t$ for some $t \in [n]$.*

Proof. Let $\mathcal{F}$ be a largest $(m, 2k-3, k)$-intersecting family with canonical partition $\binom{[2k-3]}{k} \cup \mathcal{F}_1 \cup \mathcal{F}_2$. We further partition $\mathcal{F}_1$ and $\mathcal{F}_2$ into subfamilies. Let $N = \binom{2k-3}{k-1}$. Partition $\binom{[2k-3]}{k-1}$ into $A_1, \ldots, A_N$ and $\binom{[2k-3]}{k-2}$ into $A'_1, \ldots, A'_N$ such that $A_j \cup A'_j = [2k-3]$ for all j. Define $\mathcal{F}(A_j) = \{F \in \mathcal{F} \mid F \cap [2k-3] = A_j\}$ and $\mathcal{F}(A'_j) = \{F \in \mathcal{F} \mid F \cap [2k-3] = A'_j\}$. Then

$$\mathcal{F} = \binom{[2k-3]}{k} \cup \left(\bigcup_{j=1}^{N} \Big(\mathcal{F}(A_j) \cup \mathcal{F}(A'_j) \Big) \right).$$

Observation. If $\mathcal{F}(A_j) \neq \emptyset$, then $|\mathcal{F}(A_j)| + |\mathcal{F}(A'_j)| \leqslant m - 2k + 3$.

If $\mathcal{F}(A'_j) = \emptyset$, then $|\mathcal{F}(A_j)| + |\mathcal{F}(A'_j)| = |\mathcal{F}(A_j)| \leqslant |\{A_j \cup \{b\} \mid b \in [2k-2, m]\}| = m - 2k + 3$. If $\mathcal{F}(A'_j) \neq \emptyset$, then $\{\{b\} \mid A_j \cup \{b\} \in \mathcal{F}(A_j)\} \subseteq \binom{[2k-2,m]}{1}$ and $\{\{b_1, b_2\} \mid A'_j \cup \{b_1, b_2\} \in \mathcal{F}(A'_j)\} \subseteq \binom{[2k-2,m]}{2}$ are cross-intersecting. By Theorem 7, $|\mathcal{F}(A_j)| + |\mathcal{F}(A'_j)| \leqslant \binom{m-2k+3}{2} - \binom{m-2k+2}{2} + 1 = m - 2k + 3$. Hence, the observation holds.

Now suppose that all of $\mathcal{F}(A_1), \ldots, \mathcal{F}(A_s)$ are nonempty, yet $\mathcal{F}(A_{s+1}) = \cdots = \mathcal{F}(A_N) = \emptyset$. Then we have

$$|\mathcal{F}| \leqslant \binom{2k-3}{k} + s(m-2k+3) + (N-s)\binom{m-2k+3}{2}. \tag{1}$$

Case 1. $m \geqslant 2k+2$.

Since $h(m, 2k-3, k) \leqslant |\mathcal{F}|$ and $N = \binom{2k-4}{k-2} + \binom{2k-4}{k-3}$, it follows $s \leqslant \binom{2k-4}{k-2}$. We may assume $k \geqslant 5$ because $\alpha(m, 3, 3)$ and $\alpha(m, 5, 4)$ are known by the theorems of Erdős-Ko-Rado and Hilton-Milner. It follows that $m \geqslant (2k-3)k/(k-2)$. Together with $2(k-2) < 2k-3$ and $4 < m-2k+3$, we have $\alpha(\mathsf{KG}(2k-3, k-2) \times \mathsf{KG}(m-2k+3, 2)) = \binom{2k-4}{k-3}\binom{m-2k+3}{2}$ by Lemma 6. Recall that $f_2(\mathcal{F}_2)$ is an independent set of $\mathsf{KG}(2k-3, k-2) \times \mathsf{KG}(m-2k+3, 2)$. Hence, $|\mathcal{F}_2| \leqslant \binom{2k-4}{k-3}\binom{m-2k+3}{2}$. If $s < \binom{2k-4}{k-2}$, then $|\mathcal{F}_1| = \sum_{j=1}^{s} |\mathcal{F}(A_j)| < \binom{2k-4}{k-2}(m-2k+3)$. This leads to $|\mathcal{F}| = \binom{2k-3}{k} + |\mathcal{F}_1| + |\mathcal{F}_2| < h(m, 2k-3, k)$, a contradiction. Thus, $s = \binom{2k-4}{k-2}$ and $\alpha(m, 2k-3, k) = h(m, 2k-3, k)$ for $m \geqslant 2k+2$.

Case 2. $m = 2k+1$.

Suppose that $\binom{2k-3}{k} + \binom{2k-4}{k-2}\binom{4}{1} + \binom{2k-4}{k-3}\binom{4}{2} = h(2k+1, 2k-3, k) < |\mathcal{F}|$. Since $N = \binom{2k-4}{k-2} + \binom{2k-4}{k-3}$, it follows from inequality (1) that $|\{j \mid |\mathcal{F}(A_j)| + |\mathcal{F}(A'_j)| \geqslant 5\}| > \binom{2k-4}{k-3}$. By our Observation, $|\mathcal{F}(A_j)| + |\mathcal{F}(A'_j)| \geqslant 5$ implies $\mathcal{F}(A_j) = \emptyset$ for any j. Thus $|\{A'_j \mid |\mathcal{F}(A'_j)| \geqslant 5\}| > \binom{2k-4}{k-3}$. By Theorem 1, there exist disjoint sets A'_{j_1} and A'_{j_2} in $\{A'_j \mid |\mathcal{F}(A'_j)| \geqslant 5\} \subseteq \binom{[2k-3]}{k-2}$. Then it is easy to find two disjoint sets, one in $\mathcal{F}(A'_{j_1})$ and the other in $\mathcal{F}(A'_{j_2})$. This contradicts the assumption that $\mathcal{F}$ is intersecting. Therefore $|\mathcal{F}| = h(2k+1, 2k-3, k)$.

Let us examine the maximum families. Note that $\alpha(m, 2k-3, k) = h(m, 2k-3, k)$ implies that inequality (1) becomes equality, $s = \binom{2k-4}{k-2}$, and $N - s = \binom{2k-4}{k-3}$. It follows

that $\mathcal{F}(A'_j) = \{A'_j \cup B \mid B \in \binom{[2k-2,m]}{2}\}$ for $s < j \leqslant N$. Since there exists a non-intersecting pair B_1 and B_2 in $\binom{[2k-2,m]}{2}$, $\{A'_j \mid s+1 \leqslant j \leqslant N\}$ must be a maximum intersecting family in view of its cardinality. By Theorem 1, there exists $t \in \cap_{j=s+1}^{N} A'_j$. For $1 \leqslant j \leqslant s$, if there exists $\mathcal{F}(A'_{j_1}) \neq \emptyset$ for some $1 \leqslant j_1 \leqslant s$, then there exists some A'_{j_2}, $s+1 \leqslant j_2 \leqslant N$ such that $A'_{j_1} \cap A'_{j_2} = \emptyset$. We can find two disjoint sets, one in $\mathcal{F}(A'_{j_1})$ and the other in $\mathcal{F}(A'_{j_2})$, a contradiction. Therefore we have $\mathcal{F}(A'_j) = \emptyset$ and $\mathcal{F}(A_j) = \{A_j \cup B \mid B \in \binom{[2k-2,m]}{1}\}$ for $1 \leqslant j \leqslant s$. Suppose that $t \notin A_{j_0}$ for some $1 \leqslant j_0 \leqslant s$. Then $t \in A'_{j_0}$. For any A'_j, $s+1 \leqslant j \leqslant N$, we have $A'_{j_0} \neq A'_j$ since $\mathcal{F}(A_j) = \emptyset$, yet $\mathcal{F}(A_{j_0}) \neq \emptyset$. Then $\{A'_{j_0}, A_{s+1}, \ldots, A_N\}$ is an intersecting family in $\binom{[2k-3]}{k-2}$ having more than $\binom{2k-4}{k-3}$ members, a contradiction. Hence $\mathcal{F}$ has the form $\mathcal{H}_t$ for $t \in [2k-3]$. $\square$

5 The case for m sufficiently large

We have solved Problem 3 for $n = 2k-1, 2k-2$, and $2k-3$. In this section, we are going to assume that $k \leqslant n < 2k-3$ and solve the problem when m is sufficiently large.

Let r, l, n be positive integers satisfying $r < l \leqslant n/2$, and let X_1 and X_2 be disjoint n-sets. Wang and Zhang [11] characterized the maximum intersecting families $\mathcal{F} \subseteq \{F \in \binom{X_1 \cup X_2}{r+l} \mid |F \cap X_1| = r \text{ or } l\}$ of maximum cardinality. We consider a similar extremal problem.

Problem 12. Given integers m, n, k, c, d satisfying $n < m$, $k \leqslant n < 2k-3$, $d < c < k$, and $c + d = n$, characterize the intersecting families $\mathcal{F} \subseteq \{F \in \binom{[m]}{k} \mid |F \cap [n]| = c \text{ or } d\}$ of maximum cardinality.

We can derive an asymptotic solution of the above problem as follows.

Lemma 13. *For given n, k, c, d satisfying conditions in the above problem, if m is sufficiently large, then a maximum intersecting family $\mathcal{F}$ has the form $\{A \cup B \cup \{t\} \mid A \in \binom{[n]\setminus\{t\}}{c-1}, B \in \binom{[n+1,m]}{k-c}\} \cup \{A \cup B \cup \{t\} \mid A \in \binom{[n]\setminus\{t\}}{d-1}, B \in \binom{[n+1,m]}{k-d}\}$ for some $t \in [n]$, and hence $|\mathcal{F}| = \binom{n-1}{c-1}\binom{m-n}{k-c} + \binom{n-1}{d-1}\binom{m-n}{k-d}$.*

Proof. Let $\mathcal{F}$ be a maximum intersecting family satisfying the conditions of Problem 12. Any special form stated in the lemma is an intersecting family, hence its cardinality $\binom{n-1}{c-1}\binom{m-n}{k-c} + \binom{n-1}{d-1}\binom{m-n}{k-d}$ supplies a lower bound for $|\mathcal{F}|$.

Let us consider upper bounds for $|\mathcal{F}|$. First partition $\mathcal{F}$ into two subfamilies $\mathcal{F}_{k-c}$ and $\mathcal{F}_{k-d}$ such that $\mathcal{F}_{k-c} = \{F \in \mathcal{F} \mid |F \cap [n]| = c\}$ and $\mathcal{F}_{k-d} = \{F \in \mathcal{F} \mid |F \cap [n]| = d\}$. For $\mathcal{F}_{k-d}$, we consider the injection from $\mathcal{F}_{k-d}$ to the vertex set of $\mathsf{KG}(n,d) \times \mathsf{KG}(m-n, k-d)$ defined prior to Lemma 6. We may choose m sufficiently large so that $2(k-d) < m-n$ and $m > nk/d$ hold. By Lemma 6, we have $|\mathcal{F}_{k-d}| \leqslant \alpha(\mathsf{KG}(n,d))|\mathsf{KG}(m-n, k-d)| = \binom{n-1}{d-1}\binom{m-n}{k-d}$. Consider a further partition on $\mathcal{F}_{k-c}$ and $\mathcal{F}_{k-d}$. Denote $N = \binom{n}{c}$. For $A_j \in \binom{[n]}{c}$ and $A'_j = [n] \setminus A_j$, $1 \leqslant j \leqslant N$, let $\mathcal{F}(A_j) = \{F \in \mathcal{F}_{k-c} \mid F \cap [n] = A_j\}$ and $\mathcal{F}(A'_j) = \{F \in \mathcal{F}_{k-d} \mid F \cap [n] = A'_j\}$. Since $A_j \cap A'_j = \emptyset$, the two families $\{B \in \binom{[n+1,m]}{k-c} \mid A_j \cup B \in \mathcal{F}\}$ and $\{B \in \binom{[n+1,m]}{k-d} \mid A'_j \cup B \in \mathcal{F}\}$ are cross-intersecting of size $|\mathcal{F}(A_j)|$ and

$|\mathcal{F}(A'_j)|$, respectively. Let $r \leqslant s$ be integers such that $\mathcal{F}(A_j) = \emptyset$ for $1 \leqslant j \leqslant r$, $\mathcal{F}(A_j)$ and $\mathcal{F}(A'_j)$ are nonempty for $r+1 \leqslant j \leqslant s$ and $\mathcal{F}(A'_j) = \emptyset$ for $s+1 \leqslant j \leqslant N$. Then by Theorem 7,

$$\begin{aligned}
|\mathcal{F}| &= \sum_{j=1}^{r} |\mathcal{F}(A'_j)| + \sum_{j=r+1}^{s} (|\mathcal{F}(A_j)| + |\mathcal{F}(A'_j)|) + \sum_{j=s+1}^{N} |\mathcal{F}(A_j)| \\
&\leqslant r\binom{m-n}{k-d} + (s-r)\left(\binom{m-n}{k-d} - \binom{m-k-d}{k-d} + 1\right) \\
&\quad +(N-s)\binom{m-n}{k-c}.
\end{aligned}$$

We first show that $r = \binom{n-1}{d-1}$. If $r > \binom{n-1}{d-1}$, then

$$\begin{aligned}
|\mathcal{F}| &= \sum_{j=r+1}^{N} |\mathcal{F}(A_j)| + \sum_{j=1}^{s} |\mathcal{F}(A'_j)| \\
&\leqslant (N-r)\binom{m-n}{k-c} + |\mathcal{F}_{k-d}| \\
&< \binom{n-1}{c-1}\binom{m-n}{k-c} + \binom{n-1}{d-1}\binom{m-n}{k-d},
\end{aligned}$$

which cannot be true.

For m sufficient large, say $m > 2n(n/2)^{k-d}\binom{n}{\lfloor n/2 \rfloor}$, we have

$$\begin{aligned}
& (s-r)\left(\binom{m-n}{k-d} - \binom{m-k-d}{k-d} + 1\right) + (N-s)\binom{m-n}{k-c} \\
< \;& (s-r)\left(\frac{m^{k-d}}{(k-d)!} - \frac{(m-2n)^{k-d}}{(k-d)!} + 1\right) + (N-s)m^{k-c} \\
< \;& (s-r)(2nm^{k-d-1} + 1) + (N-s)m^{k-c} \\
< \;& N(2n)m^{k-d-1} \\
\leqslant \;& \binom{n}{\lfloor n/2 \rfloor}(2n)(n/2)^{k-d}\frac{1}{m}\frac{m^{k-d}}{(n/2)^{k-d}} \\
< \;& \binom{m-n}{k-d}.
\end{aligned}$$

If $r < \binom{n-1}{d-1}$, then $|\mathcal{F}| < (1+r)\binom{m-n}{k-d} \leqslant \binom{n-1}{d-1}\binom{m-n}{k-d}$, which is impossible. Hence $r = \binom{n-1}{d-1}$. Now we show that $s = \binom{n-1}{d-1}$. Note that $s \geqslant r = \binom{n-1}{d-1}$. Suppose $s > \binom{n-1}{d-1}$. Then by Theorem 5, the image of the injection from $\mathcal{F}_{k-d}$ to $\mathsf{KG}(n,d) \times \mathsf{KG}(m-n, k-d)$ cannot be a maximal independent set and $|\mathcal{F}_{k-d}| < \binom{n-1}{d-1}\binom{m-n}{k-d}$. This leads to $|\mathcal{F}| \leqslant (N-r)\binom{m-n}{k-c} + |\mathcal{F}_{k-d}| < \binom{n-1}{c-1}\binom{m-n}{k-c} + \binom{n-1}{d-1}\binom{m-n}{k-d}$, contradicting the lower bound of $|\mathcal{F}|$ again. Since $r = s = \binom{n-1}{d-1}$, we have $|\mathcal{F}| \leqslant \binom{n-1}{d-1}\binom{m-n}{k-d} + \binom{n-1}{c-1}\binom{m-n}{k-c}$. The equality must hold as the right hand side is the known lower bound of $|\mathcal{F}|$.

When $\mathcal{F}$ has maximum cardinality, $\mathcal{F}(A_j) = \{A_j \cup B \mid B \in \binom{[n+1,m]}{k-c}\}$ for $j > \binom{n-1}{d-1}$ and $\mathcal{F}(A'_j) = \{A'_j \cup B \mid B \in \binom{[n+1,m]}{k-d}\}$ for $j \leqslant \binom{n-1}{d-1}$. Now $\{A'_j \mid 1 \leqslant j \leqslant \binom{n-1}{d-1}\} \subseteq \binom{[n]}{k}$ is a maximum intersecting family. Thus, there is a common element $t \in A'_j$ for $1 \leqslant j \leqslant \binom{n-1}{d-1}$. On the other hand, no A'_j contains t for $j > \binom{n-1}{d-1}$. That implies $t \in A_j$. So t belongs to every member of $\mathcal{F}$. □

Theorem 14. *If integers n and k satisfy $k \leqslant n < 2k-3$, then $\alpha(m,n,k) = h(m,n,k)$ holds for sufficiently large m. For such a large m, a maximum (m,n,k)-intersecting family is of the form $\mathcal{H}_t$ for some $t \in [n]$.*

Proof. Let an (m,n,k)-intersecting family $\mathcal{F}$ have canonical partition $\binom{[n]}{k} \cup (\bigcup_{i=1}^{2k-n-1} \mathcal{F}_i)$ as before. When n is odd, we put $\mathcal{F}_i$ and $\mathcal{F}_{2k-n-i}$ into a pair for $1 \leqslant i \leqslant (2k-n-1)/2$. When n is even, we put $\mathcal{F}_i$ and $\mathcal{F}_{2k-n-i}$ into a pair for $1 \leqslant i \leqslant \lfloor (2k-n-1)/2 \rfloor - 1$, and leave $\mathcal{F}_{\lfloor (2k-n-1)/2 \rfloor}$ unpaired.

Let $c = k-i$ and $d = n-k+i$. The subfamily $\mathcal{F}_i \cup \mathcal{F}_{2k-n-i}$ is an intersecting family and satisfies the conditions in Lemma 13. Therefore $|\mathcal{F}_i| + |\mathcal{F}_{2k-n-i}| \leqslant \binom{n-1}{k-i-1}\binom{m-n}{i} + \binom{n-1}{n-k+i-1}\binom{m-n}{2k-n-i}$ for sufficiently large m. When n is odd, we immediately have the following.

$$\begin{aligned} |\mathcal{F}| &\leqslant \binom{n}{k} + \sum_{i=1}^{(2k-n-1)/2} \binom{n-1}{k-i-1}\binom{m-n}{i} + \binom{n-1}{k-i}\binom{m-n}{2k-n-i} \\ &= \binom{n}{k} + \sum_{i=1}^{2k-n-1} \binom{n-1}{k-i-1}\binom{m-n}{i}. \end{aligned}$$

When n is even, we have $|\mathcal{F}_i| \leqslant \binom{n-1}{k-i-1}\binom{m-n}{i}$ for $i = \lfloor (2k-n-1)/2 \rfloor$ by Theorem 5. Together with other upper bounds of $|\mathcal{F}_i \cup \mathcal{F}_{2k-n-i}|$, we have shown $|\mathcal{F}| \leqslant \binom{n}{k} + \sum_{i=1}^{2k-n-1} \binom{n-1}{k-i-1}\binom{m-n}{i}$.

When $\mathcal{F}$ is a maximum (m,n,k)-intersecting family, for each pair $\mathcal{F}_i$ and $\mathcal{F}_{2k-n-i}$, there is an element t_i belonging to every member of $\mathcal{F}_i \cup \mathcal{F}_{2k-n-i}$. This also holds for $\mathcal{F}_i$, $i = \lfloor (2k-n-1)/2 \rfloor$ for even n. Suppose that there exist $\mathcal{F}_{i_1} \cup \mathcal{F}_{2k-n-i_1}$ and $\mathcal{F}_{i_2} \cup \mathcal{F}_{2k-n-i_2}$ for which $t_{i_1} \neq t_{i_2}$. (The case that one of them is $\mathcal{F}_i$, $i = \lfloor (2k-n-1)/2 \rfloor$ for even n, is the same.) Note that

$$\mathcal{F}_{2k-n-i_j} = \left\{ A \cup B \cup \{t_{i_j}\} \mid A \in \binom{[n] \setminus \{t_{i_j}\}}{n-k+i_j-1}, B \in \binom{[n+1,m]}{2k-n-i_j} \right\}$$

for $j = 1, 2$. Since $2(n-k+i_j-1) \leqslant n-1$ and $2(2k-n-i_j) < m-n$, we can find subsets $F_j \in \mathcal{F}_{2k-n-i_j}$ for $j = 1, 2$ such that $F_1 \cap F_2 = \emptyset$ if $t_{i_1} \neq t_{i_2}$. Therefore $t_{i_1} \neq t_{i_2}$ cannot happen. Consequently, $\mathcal{F} = \mathcal{H}_t$ for some $t \in [n]$. □

6 Conclusion

We have introduced the notion of an (m,n,k)-intersecting family and studied its maximum cardinality $\alpha(m,n,k)$. The well-known theorems of Erdős-Ko-Rado and Hilton-Milner in

finite extremal set theory are special cases for $n = k$ and $n = k + 1$. The common cardinality $h(m, n, k)$ of a particular collection of (m, n, k)-intersecting families $\mathcal{H}_t^{m,n,k}$ supplies a natural lower bound for $\alpha(m, n, k)$. A noticeable feature of $\mathcal{H}_t^{m,n,k}$ is that members of $\mathcal{H}_t^{m,n,k} \setminus \binom{[n]}{k}$ have a nonempty intersection. We have proved that the families $\mathcal{H}_t^{m,n,k}$ are precisely all the (m, n, k)-intersecting families of maximum cardinality for the cases $n = 2k-1, 2k-3$, or m sufficiently large. When $n = 2k-2$, there are other maximum families. Whether $\alpha(m, n, k) = h(m, n, k)$ is true in all cases and $\mathcal{H}_t^{m,n,k}$, $n \neq 2k-2$, always characterizes maximum families are interesting open problems. Analogue problems can be formulated with respect to intersecting families having intersection size greater than some prescribed positive integer.

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References

[1] R. Ahlswede and L. H. Khachatrian. The complete nontrivial-intersection theorem for systems of finite sets. *J. Combin. Theory Ser. A*, 76:121–138, 1996.

[2] R. Ahlswede and L. H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, 18:125–136, 1997.

[3] P. Borg. Intersecting families of sets and permutations: a survey, In A. R. Baswell (Ed.), *Advances in Mathematics Research*, volume 16, pages 283–299, Nova Science Publishers, 2011.

[4] M. Deza and P. Frankl. Erdős-Ko-Rado theorem – 22 years later. *SIAM J. Algebraic Discrete Methods*, 4:419–431, 1983.

[5] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. 2*, 12:313–320, 1961.

[6] P. Frankl. The Erdős-Ko-Rado Theorem is true for $n = ckt$. Proc. Fifth Hung. Comb. Coll., pages 365–375, North-Holland, Amsterdam, 1978.

[7] P. Frankl. The shifting technique in extremal set theory. In C. Whitehead (Ed.), *Combinatorial Surveys*, pages 81–110, Cambridge Univ. Press, 1987.

[8] P. Frankl. An Erdős-Ko-Rado theorem for direct products. *European J. Combin.*, 17:727–730, 1996.

[9] P. Frankl and N. Tokushige. Some best possible inequality concerning cross-intersecting families. *J. Combin. Theory Ser. A*, 61:87–97, 1992.

[10] A. J. W. Hilton and E. C. Milner. Some intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. 2*, 18:369–384, 1967.

[11] J. Wang and H. Zhang. Intersecting families in a subset of Boolean lattice. *Electron. J. Combin.*, 18:#P237, 2011.

[12] R. M. Wilson. The exact bound in the Erdős-Ko-Rado theorem. *Combinatorica*, 4:247–257, 1984.

[13] H. Zhang. Independent sets in direct products of vertex-transitive graphs. *J. Combin. Theory Ser. B*, 102:832–838, 2012.