

Enumerating finite racks, quandles and kei

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Abstract

A rack of order n is a binary operation \triangleright on a set X of cardinality n , such that right multiplication is an automorphism. More precisely, (X, \triangleright) is a rack provided that the map $x \mapsto x \triangleright y$ is a bijection for all $y \in X$, and $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ for all $x, y, z \in X$.

The paper provides upper and lower bounds of the form 2^{cn^2} on the number of isomorphism classes of racks of order n . Similar results on the number of isomorphism classes of quandles and kei are obtained. The results of the paper are established by first showing how an arbitrary rack is related to its operator group (the permutation group on X generated by the maps $x \mapsto x \triangleright y$ for $y \in Y$), and then applying some of the theory of permutation groups. The relationship between a rack and its operator group extends results of Joyce and of Ryder; this relationship might be of independent interest.

Keywords: rack, quandle, kei, asymptotic enumeration

1 Introduction

We begin by defining the objects of interest to us.

Definition 1. A *rack* is a set X together with a binary operator $\triangleright: X \times X \rightarrow X$ such that the following two conditions hold.

- (i) For all $y \in X$, the map $f_y: X \rightarrow X$ is a bijection, where we define $xf_y := (x \triangleright y)$ for all $x \in X$.
- (ii) For all $x, y, z \in X$, $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

Definition 2. A *quandle* is a rack such that $xf_x = x$ for all $x \in X$. A *kei* is a quandle such that f_y has order dividing 2 for all $y \in X$.

Definition 3. An isomorphism θ from a rack (X, \triangleright) to a rack (X', \triangleright') is a bijection from X to X' such that $x\theta \triangleright' y\theta = (x \triangleright y)\theta$ for all $x, y \in X$.

Some illustrative examples of kei, quandles and racks are as follows. If X is a set and π is permutation in $\text{Sym}(X)$, the symmetric group on X , then defining $x \triangleright y := x\pi$ we have a rack (X, \triangleright) . If G is a group, then defining $X = G$ and $x \triangleright y := y^{-1}xy$, we find that (X, \triangleright) is a quandle (a *conjugation quandle*); if instead we take X to be the set of all elements of order 2 in G , then (X, \triangleright) is a kei.

A rack can be motivated purely combinatorially, as a binary operation where right multiplication is an automorphism. But another motivation comes from knot theory: kei, quandles and racks have recently led to the discovery of new invariants of classical knots, and new classes of generalised knots; see the recent inspiring article by Nelson [10] (which contains a nice exposition of how the kei, quandle and rack axioms relate to Reidemeister moves in knot diagrams, as well as mentioning connections with many other areas of mathematics, and giving more examples of racks). Kei were first studied by Takasaki [14] in 1943; racks originated in unpublished correspondence between J.H. Conway and G.C. Wraith in 1959 on conjugation in groups; the special case of a quandle was studied in detail from the perspective of knot theory by Joyce [6]. See Fenn and Rourke [3] for a brief history of these concepts.

The operator group of a rack is the subgroup of $\text{Sym}(X)$ generated by the permutations f_y for $y \in X$ (see Section 2). We will establish tight results that show how a rack can be built from its operator group. These results extend those of Joyce [6, Section 7] and of Ryder [13, Section 5]. As an application of these structural results we prove an enumeration theorem (Theorem 4) below, though we hope that the results are of more general interest.

If X is a finite set of order n , we say that a rack, quandle or kei with underlying set X has order n . We write $f_{\text{rack}}(n)$, $f_{\text{quandle}}(n)$ and $f_{\text{kei}}(n)$ for the number of isomorphism classes of racks, quandles and kei of order n respectively. We aim to prove the following theorem:

Theorem 4. *There exist constants c_1 and c_2 such that*

$$2^{c_1 n^2} \leq f_{\text{kei}}(n) \leq f_{\text{quandle}}(n) \leq f_{\text{rack}}(n) \leq 2^{c_2 n^2}$$

for all sufficiently large integers n .

Theorem 4 follows from Theorems 11 and 12 below. The proofs of these theorems show that we may take $c_1 = \frac{1}{4} - \epsilon$ for any positive ϵ , and we may take $c_2 = c + \epsilon$ for any positive ϵ , where $c = \frac{1}{6}(\log_2 24) + \frac{1}{2}(\log_2 3) \approx 1.5566$.

We remark that Theorem 4 shows that the number of isomorphism classes of kei, quandles and racks grows much faster than the number of isomorphism classes of groups of order n (which Pyber [11] proved is at most $2^{O((\log n)^3)}$; see [1]). In particular, most quandles are not isomorphic to conjugation quandles.

The number of kei, quandles and racks grows significantly more slowly than n^2 , the number of binary operations on a set of cardinality n . This contrasts with the situation for semigroups, for example: Kleitman, Rothschild and Spencer [7] have shown that the number of semigroups of order n is $n^{(1-o(1))n^2}$.

We are not aware of any previous asymptotic enumeration results for racks, quandles and kei, but there has been interest in enumerating the quandles of small order. In particular, Ho and Nelson [5], and Henderson, Macedo and Nelson [4] have enumerated the isomorphism classes of quandles of order 8 or less; Vendramin [15], extending computations of Clauwens [2], has enumerated the isomorphism classes of quandles of order 35 or less whose operator group is transitive.

The structure of the remainder of the paper is as follows. In Section 2 we establish the structural results that relate the structure of a rack with its operator group. We prove a lower bound (Theorem 11) on $f_{\text{kei}}(n)$ in Section 3, and an upper bound (Theorem 12) on $f_{\text{rack}}(n)$ in Section 4.

2 The structure of a rack

In this section, we recap some terminology we need from the theory of racks, and prove two structural theorems. These theorems are used to prove Theorem 11 in Section 3, and Theorem 12 in Section 4. Racks in this section need not be finite.

Definition 5. Let (X, \triangleright) be a rack. The *augmentation map* of (X, \triangleright) is the map $f : X \rightarrow \text{Sym}(X)$ defined by $(y)f = f_y$. The *operator group* (or *inner automorphism group*) of X is the subgroup $G \leq \text{Sym}(X)$ generated by the image of f . So

$$G = \langle (y)f : y \in X \rangle.$$

Note that, in contrast to some of the literature, we define the operator group as a permutation group on X , rather than as an abstract group. The following lemma is well-known; see the second form of the rack identity in [3].

Lemma 6. *Let (X, \triangleright) be a rack with operator group G , and let $f : X \rightarrow G$ be the augmentation map of X . Then*

$$(\alpha g)f = g^{-1}(\alpha f)g$$

for all $g \in G$ and $\alpha \in X$.

Proof. The identity (ii) in the definition of a rack may be rewritten as

$$x(yf)(zf) = x(zf)((y(zf))f)$$

for all $x, y, z \in X$. Set $g = zf$ and set $x' = xg$. Then the equation above becomes

$$x'g^{-1}(yf)g = x'((yg)f)$$

for all $x', y \in X$ and all $g \in \text{im} f \subseteq G$, and so

$$g^{-1}(yf)g = (yg)f \tag{1}$$

for all $x', y \in X$ and all $g \in \text{im} f$. Setting $y = \alpha$, we see that the lemma holds whenever $g \in \text{im} f$. Setting $y = \alpha g^{-1}$ and multiplying both sides of (1) on the left by g and on the right by g^{-1} , we see that the lemma holds when $g^{-1} \in \text{im} f$. Since the image of f generates G , any element of G may be written as a product as elements from the image of f and their inverses; the lemma now holds for any $g \in G$, by induction on the length of such a product. \square

Theorems 7 and 8 below show how to build a rack from its operator group. The theorems strengthen those of Joyce [6, Section 7] (who worked with quandles rather than racks) and Ryder [13, Section 5].

For a group G and an element $\pi \in G$, we write $C_G(\pi)$ for the centraliser of π in G . If $G \leq \text{Sym}(X)$ and $\alpha \in X$, we write G_α for the point stabiliser of α in G .

Theorem 7. *Let X be a set, and let G be a subgroup of $\text{Sym}(X)$. Let I be an index set for the set of orbits of G , and let $\{\alpha_i : i \in I\} \subseteq X$ be a complete set of representatives for the orbits of G . For each $i \in I$ let $\pi_i \in G$, and suppose that*

$$C_G(\pi_i) \supseteq G_{\alpha_i} \text{ for all } i \in I. \tag{2}$$

Let $f : X \rightarrow \text{Sym}(X)$ be defined by

$$(\alpha_i g)f = g^{-1}\pi_i g \text{ for } g \in G \text{ and } i \in I.$$

Define $x \triangleright y = x(yf)$ for all $x, y \in X$. Then (X, \triangleright) is a rack with operator group contained in G . If

$$G = \langle g^{-1}\pi_i g : g \in G, i \in I \rangle. \tag{3}$$

then the operator group of (X, \triangleright) is equal to G .

Proof. Suppose that (X, \triangleright) is constructed in this way. Note that the map f is well defined, since (2) is satisfied. Let $x, y, z \in X$. Let $i, j \in I$ and $g, h \in G$ be such that $y = \alpha_i g$ and $z = \alpha_j h$. Then

$$\begin{aligned} (x \triangleright y) \triangleright z &= xg^{-1}\pi_i gh^{-1}\pi_j h \\ &= x(h^{-1}\pi_j h)(h^{-1}\pi_j h)^{-1}g^{-1}\pi_i gh^{-1}\pi_j h \\ &= (xh^{-1}\pi_j h)(gh^{-1}\pi_j h)^{-1}\pi_i(gh^{-1}\pi_j h) \\ &= (x \triangleright z) \triangleright (\alpha_i gh^{-1}\pi_j h) \\ &= (x \triangleright z) \triangleright (y \triangleright z). \end{aligned}$$

The maps $x \mapsto x \triangleright y$ are all permutations, since $(X)f \subseteq G$. Hence (X, \triangleright) is a rack. Clearly f is the augmentation map for (X, \triangleright) . Moreover, we may write the operator

group of (X, \triangleright) as:

$$\begin{aligned} \langle (\alpha)f : \alpha \in X \rangle &= \langle (\alpha_i g)f : i \in I, g \in G \rangle \\ &= \langle g^{-1} \pi_i g : i \in I, g \in G \rangle \\ &\leq G, \end{aligned}$$

with equality in the last line if (3) holds. \square

Theorem 8. *Every rack on X with operator group G arises in the manner of Theorem 7. More precisely, let (X, \triangleright) be a rack with operator group G , and let f be the augmentation map for (X, \triangleright) . Let $\{\alpha_i : i \in I\}$ be a complete set of orbit representatives for G , and define $\pi_i = \alpha_i f$ for $i \in I$. Then (2) and (3) hold. Moreover, $(\alpha_i g)f = g^{-1} \pi_i g$ for all $g \in G$ and $i \in I$, and $x \triangleright y = x(yf)$ for all $x, y \in X$.*

Proof. The last sentence of the theorem follows from Lemma 6, and the definition of the augmentation map. If $g \in G_{\alpha_i}$ we have that $(\alpha_i g)f = (\alpha_i)f = \pi_i$, and so $g \in C_G(\pi_i)$ by Lemma 6. Thus (2) holds. Finally, as G is the operator group of (X, \triangleright) ,

$$\begin{aligned} G &= \langle (\alpha)f : \alpha \in X \rangle \\ &= \langle (\alpha_i g)f : i \in I, g \in G \rangle \\ &= \langle g^{-1}(\alpha_i f)g : i \in I, g \in G \rangle, \end{aligned}$$

by Lemma 6 and so (3) holds. \square

We remark that ‘rack’ may be replaced by ‘quandle’ in the theorems above, provided that we also add the condition that $\pi_i \in G_{\alpha_i}$ for $i \in I$. ‘Quandle’ may in turn be replaced by ‘kei’ if we insist in addition that π_i has order dividing 2 for $i \in I$.

As an aside, we end this section by giving two simple consequences of the Theorems 7 and 8. The first is a strengthening of a result of Ryder [13, Theorem 3.2], which asserts that every abstract group is the operator group of some rack.

Corollary 9. *Every abstract group is the operator group of some quandle. An abstract group is the operator group of some kei if and only if it is generated by its involutions.*

Proof. Let G be an abstract group, and let $\{\pi_i \in G : i \in I\}$ be a set of elements whose normal closure in G is equal to G . Without loss of generality, we may assume that there exists $i_0 \in I$ such that $\pi_{i_0} = 1$. For $i \in I \setminus \{i_0\}$, let X_i be the set of right cosets of $C_G(\pi_i)$ in G , and let $\alpha_i = C_G(\pi_i) \in X_i$. Define $X_{i_0} = G$, and $\alpha_{i_0} = 1 \in X_{i_0}$. Let X be the disjoint union of the sets X_i (for $i \in I$), and let G act on X by right multiplication. Note that G acts faithfully on X_{i_0} , so we have realised G as a subgroup of $\text{Sym}(X)$. Theorem 7 shows that G is the operator group of a rack. Moreover, using the fact that $\alpha_i \pi_i = \alpha_i$ for $i \in I$, it is easy to check that this rack is in fact a quandle. This establishes the first statement of the corollary.

The operator group of a kei is generated by its involutions, since the definitions of kei and operator group provide a generating set consisting of elements of order dividing 2. Let G be an abstract group generated by its involutions. If we define $\{\pi_i \in G : i \in I\}$ (for some suitable index set I) to be the set of all elements of order dividing 2 in G , then the construction above realises G as the operator group of a kei. \square

We remark that there are many groups that are not generated by their involutions, and so do not occur as the operator group of any kei. The most obvious examples of such groups are the non-trivial groups of odd order; more generally, any group whose Sylow 2-subgroup is normal and proper is not generated by its involutions.

The operator group of a rack (X, \triangleright) was defined to be a permutation group on its underlying set X . (In other words the operator group comes equipped with a permutation action.) One might wonder whether something stronger than Corollary 9 holds: is every permutation group isomorphic as a permutation group (not just as an abstract group) to the operator group of some rack? However, this is too much to hope for, as the following corollary shows.

Corollary 10. *Suppose $G \leq \text{Sym}(X)$ is transitive. If G is the operator group of a rack (X, \triangleright) , then there exists $\pi \in G$ whose normal closure is equal to G . Thus not all permutation groups occur as operator groups.*

Proof. Suppose (X, \triangleright) is a rack with operator group G . Then Theorem 8 implies that there exists $\pi \in G$ whose normal closure is equal to G and so the first statement of the corollary follows.

Let G be a non-cyclic abelian group acting transitively on X . Suppose, for a contradiction, that G is the operator group of a rack (X, \triangleright) . Let $\pi \in G$ be an element whose normal closure in G is equal to G . Then, since G is abelian, $G = \langle \pi \rangle$ and so G is cyclic. This contradiction establishes the final assertion of the corollary. \square

3 A lower bound

Theorem 11. *The number $f_{\text{kei}}(n)$ of isomorphism classes of kei of order n is at least $2^{\frac{1}{4}n^2 - O(n \log n)}$.*

We remark that Theorem 11 establishes the lower bound in Theorem 4.

Proof of Theorem 11. Let $X = \{1, 2, \dots, n\}$, and define $k = \lfloor n/2 \rfloor$. Set $T = \{1, 2, \dots, k\}$.

Let $E = (e_{ij})$ be a $k \times k$ matrix such that $e_{ij} \in \{0, 1\}$ for all $i, j \in T$ with $i \neq j$, and such that $e_{ii} = 0$. To prove the theorem, it suffices to construct kei X_E on X , in such a way that $X_E \neq X_{E'}$ whenever $E \neq E'$. (It will happen that $X_E \cong X_{E'}$ in some cases.) To see that the theorem follows from this, first note that there are $2^{(k-1)k}$ matrices E . Moreover, an isomorphism class of kei sharing the same underlying set X can contain at most $n!$ distinct elements, since there are at most $n!$ choices for an isomorphism $\theta : X \rightarrow X$. So we will have constructed at least $2^{(k-1)k}/n!$ distinct isomorphism classes of kei. Since $n! \leq n^n = 2^{n \log n}$ and $k \geq (n-1)/2$ the theorem will therefore follow.

For $i \in T$, let $X_i = \{2i-1, 2i\}$ and define $X_{k+1} = \{n\}$. Define $I = T$ when n is even, and $I = T \cup \{k+1\}$ when n is odd. So $\bigcup_{i \in I} X_i$ is a partition of X containing exactly k subsets of size 2, and possibly a single set of size 1.

Let

$$G = \{\pi \in \text{Sym}(X) : X_i \pi = X_i \text{ for all } i \in I\} \leq \text{Sym}(X).$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{array}{c|cccccccc} \triangleright & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 \\ 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 \\ 4 & 3 & 3 & 4 & 4 & 3 & 3 & 4 \\ 5 & 6 & 6 & 5 & 5 & 5 & 5 & 5 \\ 6 & 5 & 5 & 6 & 6 & 6 & 6 & 6 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \end{array}$$

Figure 1: An example of a kei X_E when $n = 7$

Then G is an elementary abelian 2-group of order 2^k , generated by the transpositions $\tau_i = (2i - 1, 2i)$ for $i \in T$. The orbits of G are the sets X_i where $i \in I$. Define $\alpha_i = 2i - 1$ for $i \in I$. Then $\{\alpha_i : i \in I\}$ is a complete set of representatives for the orbits of G .

We construct each kei X_E as follows (see Figure 1 for an example). Define permutations $\pi_i \in G$ for $i \in I$ as follows. For $i \in T$, define

$$\pi_i = \tau_1^{e_{1i}} \tau_2^{e_{2i}} \cdots \tau_k^{e_{ki}}.$$

For $i \in I \setminus T$ (so n is odd and $i = k + 1$), let π_i be the identity permutation. Note that distinct matrices E give rise to distinct lists of permutations $(\pi_i : i \in I)$. The condition (2) of Theorem 7 is satisfied since G is abelian and $\pi_i \in G$ for $i \in I$, so we may define $f : X \rightarrow G$ and (X, \triangleright) as in Theorem 7. Let $X_E = (X, \triangleright)$. By Theorem 7, X_E is a rack whose operator group is contained in G . It is not hard to check that $x \triangleright x = x$, using the fact that $e_{ii} = 0$ for all $i \in T$, and so X_E is a quandle. All the elements of the operator group of X_E have order dividing 2, since the operator group is contained in the elementary abelian 2-group G . Thus, X_E is a kei. Finally, since π_i is equal to the map $x \mapsto x \triangleright \alpha_i$, distinct matrices E give rise to distinct kei (X, \triangleright) . So the theorem follows. \square

4 An upper bound

This section aims to prove the following theorem.

Theorem 12. *The number $f_{\text{rack}}(n)$ of isomorphism classes of racks of order n is at most $2^{(c+o(1))n^2}$, where $c = \frac{1}{6}(\log_2 24) + \frac{1}{2}(\log_2 3) \approx 1.5566$.*

We remark that the proof of this theorem will complete the proof of Theorem 4. We require the following two results from the theory of permutation groups. The following theorem is due to Laci Pyber [11, Corollary 3.3].

Theorem 13. *The number of subgroups of $\text{Sym}(X)$ with $|X| = n$ is bounded above by $24^{(\frac{1}{6}+o(1))n^2}$.*

The next theorem is due to Attila Maróti [9], extending work of Kovács and Robinson [8], and of Riese and Schmid [12].

Theorem 14. *Let $n > 2$. Let G be a subgroup of $\text{Sym}(X)$, with $|X| = n$. Then the number of conjugacy classes of G is bounded above by $3^{(n-1)/2}$.*

Proof of Theorem 12. Let X be a set with $|X| = n$. By Theorem 13, there are at most $24^{(\frac{1}{6}+o(1))n^2}$ subgroups G of $\text{Sym}(X)$, and so the theorem will follow if we can provide a sufficiently good upper bound on the number of racks with operator group G , for any fixed G .

Let G be a fixed subgroup of $\text{Sym}(X)$. Suppose G has s orbits, of lengths n_1, n_2, \dots, n_s . Clearly $s \leq n$, and $\sum_{i=1}^s n_i = n$.

Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be a complete set of representatives for the orbits of G . By Theorem 8, a rack with operator group G is determined by a sequence of elements $\pi_1, \pi_2, \dots, \pi_s \in G$ such that $C_G(\pi_i) \geq G_{\alpha_i}$. Since G_{α_i} has index n_i in G , each π lies in a G -conjugacy class Π_i of order at most n_i .

By Theorem 14, the group G has at most $3^{\frac{1}{2}n}$ conjugacy classes. There are at most $3^{\frac{1}{2}ns}$ choices for the conjugacy classes $\Pi_1, \Pi_2, \dots, \Pi_s$, and $3^{\frac{1}{2}ns} \leq 3^{\frac{1}{2}n^2}$. Once these conjugacy classes are fixed, there are at most n_i choices for each element $\pi_i \in G$. So the number of choices for the elements π_i once the classes Π_i are chosen is at most $\prod_{i=1}^s n_i$. The product $\prod_{i=1}^t m_i$ of positive integers m_i such that $\sum_{i=1}^t m_i = n$ is maximised when $m_i \leq 3$ for all i , since $(m-2)m \geq m$ when $m \geq 4$. So $\prod_{i=1}^s n_i \leq 3^n = 2^{O(n)}$.

Thus there are at most $2^{(c+o(1))n^2}$ racks on X . Since every rack of order n is isomorphic to a rack with underlying set X , there are at most $2^{(c+o(1))n^2}$ isomorphism classes of racks of order n , as required. \square

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