

Delay colourings of cubic graphs

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Abstract

In this note we prove the conjecture of Wilfong, Haxell and Winkler (2001) that every bipartite multigraph with integer edge delays admits an edge colouring with $d + 1$ colours in the special case where $d = 3$. A connection to the Brualdi-Ryser-Stein conjecture is discussed.

1 Introduction

Motivated by scheduling issues in optical networks, Wilfong Haxell and Winkler [6] made the following elegant combinatorial conjecture:

Conjecture 1.1. *Let G be a bipartite multigraph with partition classes A, B and maximum degree d . Suppose that each edge e is associated with an integer ‘delay’ $r(e)$. Then G admits an edge-colouring $f : E(G) \rightarrow \{0, \dots, d\}$ such that f is proper on A and $f + r \pmod{d + 1}$ is proper on B .*

See also [2]. When the graph consists of just two vertices joined by d parallel edges, this is implied by a theorem of Hall [5] as noted in [6]. This fact has been dubbed ‘the fundamental theorem of juggling’ (think of the two vertices as the two hands of a juggler juggling d balls).

More generally, one can consider the case where the edges impose an arbitrary distortion of the colours, given by a permutation r of the set of colours, rather than ‘delaying’ the colour by a constant:

Problem 1.2. *Let G be a bipartite multigraph with partition classes A, B and maximum degree d . Suppose that each edge e is associated with a permutation r_e of the set $\{0, 1, \dots, d\}$. Then G admits a proper distortion-colouring with colours $\{0, 1, \dots, d\}$ (definitions in the next section).*

It turns out that in this generality the problem becomes much harder than its special case in Conjecture 1.1: as noticed by N. Alon (private communication), the special case of Problem 1.2 where each class A, B consists of just one vertex is equivalent to the following conjecture of [1], which is a strengthening of a well-known conjecture of Brualdi and Stein on transversals in Latin squares.

Conjecture 1.3 ([1, Conjecture 2.4¹]). *Let H be a 3-partite 3-uniform hypergraph, with partition classes V_1, V_2, V_3 such that $|V_2| = |V_3| = |V_1| + 1$. Suppose that for every $x \in V_1$, the set of hyperedges containing x induces a perfect matching of $V_2 \cup V_3$. Then V_1 is matchable.*

Here, V_1 being matchable means that there is a matching in H containing all vertices in V_1 .

Indeed, to see the equivalence, represent each edge in Problem 1.2 by a vertex in V_1 , and let V_2, V_3 be sets of size $d + 1 = |V_1| + 1$, to be thought of as the colour on the left endvertex and the colour on the right endvertex respectively.

Conjecture 1.3 strengthens the following well-known conjecture, made independently by Brualdi [3] and Stein [8]

Conjecture 1.4 (Brualdi-Stein). *In every $n \times n$ Latin square there exists a transversal of size $n - 1$.*

An $n \times n$ matrix with entries in $\{1, \dots, n\}$ is a *Latin square*, if no two entries in the same row or in the same column are equal. A *transversal* is a set of entries, each in a different row and in a different column, and each containing a different symbol. (For odd n , Ryser [7] conjectured that there is even a transversal of size n .)

To see that Conjecture 1.3 implies Conjecture 1.4, construct a 3-partite hypergraph H with V_1 being the set of rows of that Latin square L , V_2 the set of columns, and $V_3 = \{1, \dots, n\}$, and for each entry of L introduce an edge containing the three corresponding vertices of H . Then delete an arbitrary vertex in V_1 with all edges containing it to obtain the setup of Conjecture 1.3.

The aim of this note is to prove that Problem 1.2 has a positive answer for $d = 3$.

2 Definitions

Let $G = (V, E)$ be a bipartite multigraph of maximum degree d with bipartition $\{A, B\}$. Let $Col := \{0, 1, \dots, d\}$ be the set of *colours*, and suppose that every edge $e \in E$ is associated with a bijection r_e on Col , called the *distortion* of e ; intuitively, we are going to colour e at its A end and the colour will be distorted by r_e when seen from B (in [6] r_e was addition, mod $d + 1$, with the ‘delay’ of e). If a, b is the endvertex of e in A, B respectively, then we use the notation $ba(\cdot)$ to denote $r_e(\cdot)$ and $ab(\cdot)$ to denote $r_e^{-1}(\cdot)$.

¹The conjecture of [1] is more general than Conjecture 1.3: it allows for larger sets V_2, V_3 with an appropriate modification of the perfect matching condition.

A k -colouring of a set E (of edges) is a function $f : E \rightarrow \{0, 1, \dots, k-1\}$. A k -colouring f of the edges of a multigraph G as above is called a *proper distortion-colouring* (with respect to the permutations r_e), if for every $a \in A$ we have $f(e) \neq f(g)$ for every two edges e, g incident with a and for every $b \in B$ we have $r_e(f(e)) \neq r_g(f(g))$ for every two edges e, g incident with b .

3 Main

Theorem 3.1. *For every bipartite multigraph G of maximum degree 3, and any edge distortions, there is a proper distortion-colouring of $E(G)$ with 4 colours 0,1,2,3.*

Proof. We may assume without loss of generality that every vertex of G has degree precisely 3, for otherwise we can add dummy vertices and edges to make G cubic. It is well known that the edges of a regular bipartite multigraph can be decomposed into disjoint perfect matchings [4, Corollary 2.1.3]. So let M, M', M'' be perfect matchings of G with $M \cup M' \cup M'' = E(G)$. Then $M' \cup M''$ is a 2-factor, and it can be decomposed into a collection $\mathcal{C}$ of edge disjoint cycles.

Let $\{A, B\}$ be the bipartition of $V(G)$. We are going to let each element of $\mathcal{C}$ choose the colours of the edges of M incident with its A side. More precisely, given a $C \in \mathcal{C}$, let $M_{C \cap A}$ denote the set of edges in M incident with $C \cap A$, and let $M_{C \cap B}$ denote the set of edges in $M \setminus M_{C \cap A}$ incident with B . We are going to prove that

For every $C \in \mathcal{C}$, there is a 4-colouring f_A of $M_{C \cap A}$ such that for every 4-colouring f_B of $M_{C \cap B}$, there is a 4-colouring f_C of $E(C)$ such that $f_A \cup f_B \cup f_C$ is a proper distortion-4-colouring. (1)

Note that (1) easily implies a proper distortion-colouring of $E(G)$ with 4 colours: the sets $M_{C \cap A} \mid C \in \mathcal{C}$ are pairwise edge disjoint, and their union is M . Thus, we can begin by colouring each of them by a colouring f_A as in (1), and then we can extend the colouring to each $C \in \mathcal{C}$ keeping it proper.

So let us prove (1). Given a $C \in \mathcal{C}$, pick a 2-edge subarc uvy of C with $u, y \in A$. Distinguish two cases:

If the distortions of the edges uv, vy are identical, then give the edges m_u, m_y of M incident with u, y colours that are different (when seen from A). If C happens to be a 2-cycle, in which case $u = y$, give $m_u = m_y$ any colour.

If those distortions are not identical, then colour (the A side of) both m_u, m_y with a colour α such that $vu(\alpha) \neq vy(\alpha)$.

In both cases, colour the rest of $M_{C \cap A}$ arbitrarily; those colours will not matter.

We claim that this colouring f_A has the desired property. To prove this, let f_B be any colouring of $M_{C \cap B}$, and note that for every edge $e \in E(C)$ the set L_e of still available colours for e , that is, the colours that would not conflict with $f_A \cup f_B$ if given to e on its B side, say, has at least 2 elements; indeed, only 2 edges adjacent with e have been coloured so far and we had 4 colours to begin with.

Let us first deal with the case where C is not a 2-cycle, and consider again the two edges vu, vy as above. We claim that $L_{vu} \neq L_{vy}$; indeed, the colours we gave to m_u, m_y were chosen in such a way so as to forbid a different candidate colour at the v side of each of vu, vy , and so $L_{vu} \neq L_{vy}$ holds. On the other hand, the colour of the edge in M incident with v forbids the same colour for each of vu, vy , which implies that $L_{vu} \cap L_{vy} \neq \emptyset$.

Thus, since L_{vu}, L_{vy} are neither equal nor disjoint, and each contains at least two colours, we can find a common colour $\beta \in L_{vu} \cap L_{vy}$ and another 2 colours $\gamma \in L_{vu}, \delta \in L_{vy}$ so that β, γ, δ are all distinct. Now colour vu with $uv(\gamma)$ (so that its colour seems to be γ on its B side), and note that our colouring is still proper, since this colour came from the allowed list. Consider the next edge ux of C incident with u . This edge still has at least 1 available colour after we coloured uv (recall that $|L_e| \geq 2$), so give it that colour. Continue like this along C , to properly colour all its edges except the last edge vy . Now note that when we coloured vu we still left 2 colours available for the B side of vy , namely $\beta, \delta \neq \gamma$. At least one of them is still available now, and we assign it to the B side of vy completing the proper distortion-colouring of C .

If C is a 2-cycle then the situation is much simpler, and it is straightforward to check that (1) holds by distinguishing two cases according to whether its 2 edges bear the same distortions.

This completes the proof. Note that we proved something stronger than (1): for each $C \in \mathcal{C}$, all but one of the edges in $M_{C \cap A}$ can be precoloured arbitrarily. $\square$

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