Nordhaus-Gaddum Theorem for the Distinguishing Chromatic Number

Karen L. Collins
Department of Mathematics and Computer Science
Wesleyan University
Middletown, CT, U.S.A.
kcollins@wesleyan.edu

Ann Trenk
Department of Mathematics
Wellesley College
Wellesley, MA, U.S.A.
atrenk@wellesley.edu

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Abstract

Nordhaus and Gaddum proved, for any graph G, that $\chi(G) + \chi(\overline{G}) \leqslant n + 1$, where χ is the chromatic number and $n = |V(G)|$. Finck characterized the class of graphs, which we call NG-graphs, that satisfy equality in this bound. In this paper, we provide a new characterization of NG-graphs, based on vertex degrees, which yields a new polynomial-time recognition algorithm and efficient computation of the chromatic number of NG-graphs. Our motivation comes from our theorem that generalizes the Nordhaus-Gaddum theorem to the distinguishing chromatic number. For any graph G, $\chi_D(G) + \chi_D(\overline{G}) \leqslant n + D(G)$. We call the set of graphs that satisfy equality in this bound NGD-graphs, and characterize the set of graphs that are simultaneously NG-graphs and NGD-graphs.

Keywords: distinguishing number, distinguishing chromatic number, chromatic number, Nordhaus-Gaddum theorem

1 Introduction

We provide a generalization of the classic Nordhaus and Gaddum Theorem for the chromatic number to the distinguishing chromatic number. First, we recall their theorem, which gives bounds on the sum and the product of the chromatic number of a graph with that of its complement. We write $\chi(G)$ for the chromatic number of graph G, and for the complement of graph G we write $\overline{G}$. The upper bound in (2) below was proved by Zykov [23] and the remaining three inequalities were proved by Nordhaus and Gaddum [17].

Theorem 1. *If G is a graph with $|V(G)| = n$ and $\chi(G)$ is the chromatic number of G, then*

$$2\sqrt{n} \leqslant \chi(G) + \chi(\overline{G}) \leqslant n + 1. \tag{1}$$

$$n \leqslant \chi(G)\chi(\overline{G}) \leqslant \left(\frac{n+1}{2}\right)^2. \tag{2}$$

Finck [11] characterized the graphs that achieve equality for each of the four bounds in Equations (1.1) and (1.2).

A labeling (or coloring) of the vertices of a graph G, $h : V(G) \to \{1, \ldots, r\}$, is said to be *$r$-distinguishing* (or just *distinguishing*) if the only automorphism of the graph that preserves all of the vertex labels is the identity. The *distinguishing number of G,* denoted by $D(G)$, is defined as the minimum number r so that G has an r-distinguishing labeling. Albertson and Collins study the distinguishing number in [2] and subsequently other authors have studied the distinguishing number of graphs and of other structures, see for example [1, 3, 4, 10, 13, 14, 15, 20, 22], and many others.

The automorphism group of a graph is the same as the automorphism group of its complement, hence we get the following remark.

Remark 2. For any graph G, we have $D(G) = D(\overline{G})$.

In [9] we introduce the distinguishing chromatic number of a graph G, denoted by $\chi_D(G)$, that requires the coloring to be proper as well as distinguishing. Together with Hovey we explored the distinguishing chromatic number from the perspective of group theory in [8]. The subject has received considerable attention from others, who considered the distinguishing chromatic number in [7], and others both the distinguishing number and the distinguishing chromatic number [6, 16, 19, 21]. In this paper we ask whether there is a version of Theorem 1 for the distinguishing chromatic number.

Definition 3. A labeling (or coloring) of the vertices of a graph G, $h : V(G) \to \{1, \ldots, r\}$, is said to be *proper r-distinguishing* (or just *proper distinguishing*) if it is a proper labeling (i.e., coloring) of the graph and the only automorphism of the graph that preserves all of the vertex labels is the identity. The *distinguishing chromatic number* of a graph G, denoted by $\chi_D(G)$, is the minimum r such that G has a proper r-distinguishing labeling.

We note that the two lower bounds from Theorem 1 are still valid for the distinguishing chromatic number since $\chi(G) \leqslant \chi_D(G)$ for all graphs G. Thus for any graph G with $n = |V(G)|$ we have:

$$2\sqrt{n} \leqslant \chi_D(G) + \chi_D(\overline{G}) \text{ and} \tag{3}$$

$$n \leqslant \chi_D(G) \cdot \chi_D(\overline{G}) \tag{4}$$

For any graph G with $D(G) = 1$, we have $\chi(G) = \chi_D(G)$, and so any graph G with $D(G) = 1$ that satisfies equality in one of the lower bounds of Equations (1.1) and (1.2) will be an example of a graph for which the corresponding bound in Equations (1.3) and

(1.4) is tight. Finck's constructions [11] of such graphs include examples with $D(G) = 1$. Cavers and Seyffarth provide further examples in [5].

Before concluding this section, we present some background definitions and Brooks' Theorem. We use $|S|$ to denote the size of set S and $\Delta(G)$ to denote the largest vertex degree in graph G. The independent set with s vertices is denoted by I_s. For a vertex $u \in V(G)$, we let $N(u)$ be the set of neighbors of u in G. We will routinely use $\overline{G} - v$ in place of $\overline{G - v}$ and it is easy to see that these are equivalent. If S is a set of vertices in G, we write $G[S]$ to denote the subgraph induced in G by S. We write $Aut(G)$ for the group of all automorphisms of the graph G. We say that graph H is *color-critical* if $\chi(H - x) < \chi(H)$ for every vertex $x \in V(H)$. We will also need Brooks' Theorem:

Theorem 4. *(Brooks [1941]) If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leqslant \Delta(G)$.*

In this paper we revisit the Nordhaus-Gaddum inequalities (Theorem 1) and the classes of graphs for which the upper bound in Equation (1.1) is tight. In Section 2 we give analogues of the upper bounds in Equations (1.1) and (1.2) for the distinguishing chromatic number. In Section 3 we give a new characterization of those graphs that achieve equality for the upper bound in Equation (1.1), based on vertex degrees. Our characterization leads to a new polynomial-time recognition algorithm for this class and efficient computation of the chromatic number of graphs in this class. In Section 4 we characterize those graphs that achieve equality in the upper bound of Equation (1.1) and our distinguishing chromatic number analog of this Nordhaus-Gaddum inequality.

2 The Nordhaus Gaddum inequalities for χ and χ_D

Nordhaus and Gaddum [17] describe three classes of graphs to illustrate that their bounds are tight. The first class is the complete graphs, which are tight for the upper bound in Equation (1.1) and the lower bound in Equation (1.2); next is the complete multipartite graphs with q parts, each of size q, which are tight for the lower bounds in Equation (1.1) and Equation (1.2); and third, the disjoint union of a complete graph and an independent set with one fewer vertex, $K_n + I_{n-1}$ which are tight for the upper bounds in Equation (1.1) and Equation (1.2). They note that it is not possible to satisfy the lower bound in Equation (1.1) and the upper bound in Equation (1.2) simultaneously. In Table 1, we record the values for the distinguishing number and the distinguishing chromatic number for these examples.

The examples in Table 1 make clear that we will need to increase the upper bounds in Equations (1.1) and (1.2) in order to prove analogues for the distinguishing chromatic number. Each of these examples in the table is a complete multipartite graph (K_n and $K_{q,q,\ldots,q}$) or the complement of a complete multipartite graph ($K_t + I_{t-1}$). Collins and Trenk [9] have shown that complete multipartite graphs are exactly the graphs G for which $\chi_D(G) = |V(G)|$, that is, the graphs with the largest possible distinguishing chromatic number. Note that the distinguishing number of each graph in the table is equal to

G	$\lvert V(G)\rvert$	$\chi_D(G)$	$\chi_D(\overline{G})$	$\chi_D(G)+\chi_D(\overline{G})$	$\chi_D(G)\cdot\chi_D(\overline{G})$	$D(G)$
K_n	n	n	n	$2n$	n^2	n
$K_{\underbrace{q,q,\ldots,q}_{q \text{ times}}}$	q^2	q^2	$q+1$	q^2+q+1	$q^2(q+1)$	$q+1$
$K_t + I_{t-1}$	$2t-1$	t	$2t-1$	$3t-1$	$(2t-1)t$	t

Table 1: Examples from Nordhaus and Gaddum [17], together with their distinguishing chromatic numbers.

either its distinguishing chromatic number or the distinguishing chromatic number of its complement. This leads us to our first step in finding an appropriate generalization of the Nordhaus-Gaddam theorems, which is to consider the distinguishing chromatic number of the complements of complete multipartite graphs.

Proposition 5. *Let G be a complete multipartite graph. Then $\chi_D(\overline{G}) = D(\overline{G})$.*

Proof. By the definition of χ_D and D, the inequality $\chi_D(\overline{G}) \geqslant D(\overline{G})$ holds for all graphs G. Since G is a complete multipartite graph, we know that $\overline{G}$ is a collection of disjoint complete graphs. Let $\phi : V(\overline{G}) \to \{1, 2, \ldots, D(\overline{G})\}$ be a distinguishing labeling of $\overline{G}$. Let $u, v \in V(\overline{G})$ be adjacent in $\overline{G}$. Then u, v are both in the same complete subgraph of $\overline{G}$. The automorphism of $\overline{G}$ that switches u and v and fixes all other vertices must not preserve labels, so $\phi(u) \neq \phi(v)$. Thus, ϕ is both proper and distinguishing, so $\chi_D(\overline{G}) \leqslant D(\overline{G})$ and $\chi_D(\overline{G}) = D(\overline{G})$. □

This suggests a natural generalization of the Nordhaus-Gaddum bound. Theorem 6 presents an upper bound generalizing Equation (1.1), and Corollary 7 gives the resulting upper bound generalizing Equation (1.2). Note that the analogous lower bounds were presented in Equations (1.3) and (1.4).

Theorem 6. *If G is a graph with $n = |V(G)|$ then*

$$\chi_D(G) + \chi_D(\overline{G}) \leqslant n + D(G)$$

Proof. Fix a distinguishing coloring of graph G using colors in the set

$$C = \{1, 2, 3, \ldots, D(G)\}.$$

This simultaneously provides a distinguishing coloring of $\overline{G}$. For each $i \in C$, we let V_i be the vertices of color i, and let $G_i = G[V_i]$, thus $\overline{G_i} = \overline{G}[V_i]$. By Theorem 1, we know

$\chi(G_i)+\chi(\overline{G_i}) \leqslant |V_i|+1$ for each $i \in C$. Thus we may recolor the graph G_i and separately recolor the graph $\overline{G_i}$ using $|V_i|+1$ new colors so that both new colorings are proper. We do this for each $i \in C$ using a new set of $|V_i|+1$ colors for each i. The result is a coloring of G and a coloring of $\overline{G}$ using a total of $\sum_{i=1}^{D(G)}(|V_i|+1) = |V(G)| + D(G) = n + D(G)$ colors. By construction, these colorings of G and $\overline{G}$ are proper. Moreover, we show they are distinguishing. Suppose there were a non-trivial automorphism σ of G that preserved colors. Since a new set of colors is used for each i, we know that σ must preserve membership in V_i for each i. However, the original coloring was distinguishing, so the only automorphism of G that preserves membership in V_i for each i must be the identity.

□

Corollary 7. *If G is a graph with $n = |V(G)|$ then $\chi_D(G)\chi_D(\overline{G}) \leqslant \left(\frac{n+D(G)}{2}\right)^2$.*

Proof. We follow the proof given in [17]. For all real numbers x, y we know $0 \leqslant (x-y)^2$ and thus $4xy \leqslant (x+y)^2$ and $xy \leqslant (\frac{x+y}{2})^2$. Substitute $x = \chi_D(G)$ and $y = \chi_D(\overline{G})$ into this last inequality and then apply Theorem 6 to finish the proof. □

Theorem 6 is robust, and in Proposition 10 we extend it to any group action on our graph G.

Definition 8. Let G be a graph and let Γ be a subgroup of $Aut(G)$. The *distinguishing number of G with respect to Γ*, denoted by $D^\Gamma(G)$, is the minimum number of colors needed to color the vertices of G so that no non-identity element of Γ preserves the colors.

Definition 9. Let G be a graph and let Γ be a subgroup of $Aut(G)$. The *distinguishing chromatic number of G with respect to Γ*, denoted by $\chi_D^\Gamma(G)$, is the minimum number of colors needed to color the vertices of G so that the coloring is proper, and no non-identity element of Γ preserves the colors.

Proposition 10. *If G be a graph with $n = |V(G)|$ and Γ is a subgroup of $Aut(G)$, then*

$$\chi_D^\Gamma(G) + \chi_D^\Gamma(\overline{G}) \leqslant n + D^\Gamma(G).$$

Proof. The proof is similar to the proof of Theorem 6. □

We now turn to the question of characterizing those graphs that achieve equality in the upper bounds of identity of Theorem 1 and the related question of characterizing the analogous graphs for Theorem 6.

Definition 11. A graph G with $|V(G)| = n$ is an *NG-graph* if it satisfies $\chi(G) + \chi(\overline{G}) = n + 1$, and is an *NGD-graph* if it satisfies $\chi_D(G) + \chi_D(\overline{G}) = n + D(G)$.

Proposition 5 shows that all complete multipartite graphs, including K_n and $\overline{K_t + I_{t-1}}$, are NGD-graphs. However, they are not all NG-graphs, see Table 2 at the beginning of Section 4.

Corollary 12. *If G is an NGD-graph with a fixed distinguishing coloring using $D(G)$ colors, then each color class induces an NG-graph.*

Proof. Let G be an NGD-graph and fix a distinguishing coloring of G using $D(G)$ colors: $1, 2, 3, \ldots, D(G)$. Let V_i be the vertices of color i and let $G_i = G[V_i]$. If $\chi(G_i) + \chi(\overline{G_i}) < |V_i| + 1$ for any i, then following the proof of Theorem 6, we would have a distinguishing coloring of G and $\overline{G}$ using fewer than $n + D(G)$ colors. This contradicts the assumption that G is an NGD-graph. □

3 Characterizing NG-graphs

In this section we focus on the ordinary chromatic number χ and the inequality $\chi(G) + \chi(\overline{G}) \leqslant n + 1$ of Theorem 1. Our main result of this section is a characterization of NG-graphs, that is, the graphs that satisfy this with equality. Our characterization leads to a polynomial-time recognition algorithm for NG-graphs and an efficient computation of the chromatic number of NG-graphs.

Finck [11] characterizes the graphs that achieve equality for each of the four inequalities in Theorem 1. His characterizations involve arrays and in the case of NG-graphs, he gives an induction proof based on $\chi(H)$ for certain induced subgraphs H of G. This proof is not constructive, nor does it lead to a polynomial-time algorithm for recognizing whether a given graph is an NG-graph. Starr and Turner [18] give a characterization of NG-graphs that is simpler to state but relies explicitly on $\chi(G)$ and thus also can not be used to recognize NG-graphs in polynomial-time. Our characterization depends on partitioning vertices according to their degree and leads to a polynomial-time algorithm to determine whether a graph is an NG-graph and if so to find its chromatic number.

Definition 13. If G is an NG-graph, then the *ABC-partition* of $V(G)$ is as follows:
$A_G = \{v \in V(G) : deg(v) = \chi(G) - 1\}$
$B_G = \{v \in V(G) : deg(v) > \chi(G) - 1\}$
$C_G = \{v \in V(G) : deg(v) < \chi(G) - 1\}$
When it is unambiguous, we write $A = A_G$, $B = B_G$, $C = C_G$.

The following theorem characterizes NG-graphs and Figure 1 illustrates the three possible forms.

Theorem 14. *A graph G is an NG-graph if and only if when $V(G) = A_G \cup B_G \cup C_G$ is an ABC-partition of $V(G)$, we have*
(i) $A_G \neq \emptyset$ and $G[A_G]$ is a clique, an independent set, or a 5-cycle
(ii) $G[B_G]$ is a clique.
(iii) $G[C_G]$ is an independent set
(iv) $uv \in E(G)$ for all $u \in A_G$, $v \in B_G$
(v) $uw \notin E(G)$ for all $u \in A_G$, $w \in C_G$.

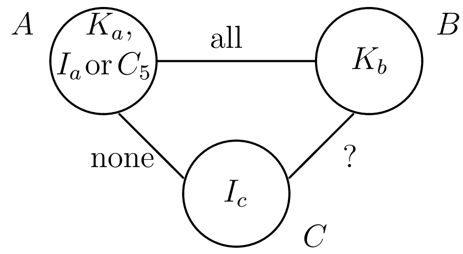

Figure 1: The forms of an NG-graph

Proof. ($\Longleftarrow$) Let $A = A_G, B = B_G$, and $C = C_G$. In the case that $G[A]$ is a clique, we can write $G[A] = K_a$, $G[B] = K_b$, and $G[C] = I_c$ for some integers a, b, c where $a \geqslant 1$. We observe that $\chi(G) = a + b$ since we need $a + b$ colors for $A \cup B$ and we may reuse a color from A for all vertices in C. In addition, $\chi(\overline{G}) = c + 1$ since we need c colors for C and one new color for $A \cup B$. We have $\chi(G) + \chi(\overline{G}) = a + b + c + 1 = n + 1$, so G is an NG-graph.

In the case that $G[A]$ is an independent set, let $G[A] = I_a$, $G[B] = K_b$, and $G[C] = I_c$ where a, b, c are integers with $a \geqslant 1$. Then $\chi(G) = b + 1$ (b colors for B, one for the rest of the vertices), $\chi(\overline{G}) = a + c$ ($a + c$ colors for $A \cup C$, then reuse a color from A for all of B). Thus $\chi(G) + \chi(\overline{G}) = b + 1 + a + c = n + 1$, so G is an NG-graph.

Finally, if $G[A]$ is a 5-cycle, let $G[A] = C_5$, $G[B] = K_b$, and $G[C] = I_c$ where b and c are integers. Then $\chi(G) = b + 3$ (reuse a color from A for all vertices of C), $\chi(\overline{G}) = c + 3$ (reuse a color from A for all vertices of B), and $n = 5 + b + c$, so G is an NG-graph.

We prove the converse of Theorem 14 after a series of lemmas.

Lemma 15. *If x is a vertex in an NG-graph H and $deg(x) > \chi(H) - 1$ then $deg_{\overline{H}}(x) < \chi(\overline{H}) - 1$. Moreover, x is color-critical in H but not in $\overline{H}$.*

Proof. Using the assumption that H is an NG-graph and the given degree condition we have $\chi(\overline{H}) - 1 = |V(H)| - \chi(H) > |V(H)| - (deg(x) + 1) = deg_{\overline{H}}(x)$, thus $deg_{\overline{H}}(x) < \chi(\overline{H}) - 1$. If x were color-critical in $\overline{H}$, then $\chi(\overline{H} - x) = \chi(\overline{H}) - 1$, and we could color x with one of the $\chi(\overline{H}) - 1$ colors not appearing on its neighbor set. This would give a proper $(\chi(\overline{H})-1)$-coloring of $\overline{H}$, a contradiction. Thus x is not color-critical in $\overline{H}$, and so $\chi(\overline{H} - x) = \chi(\overline{H})$. Applying Theorem 1 to the graph $H - x$ yields $\chi(H - x) + \chi(\overline{H} - x) \leqslant (n - 1) + 1 = n$. If $\chi(H - x) = \chi(H)$ then $\chi(H) + \chi(\overline{H}) \leqslant n$, a contradiction because H is an NG-graph. Thus $\chi(H - x) < \chi(H)$ and x is color-critical in H. This completes the proof of all the assertions of Lemma 15. □

Lemma 16. *Suppose G is an NG-graph and define A_G, B_G and C_G as in Definition 13. For any $y \in B_G$ and any $x \in A_G \cup B_G$ we have $xy \in E(G)$.*

Proof. For a contradiction, assume $xy \notin E(G)$. By the definition of B_G and Lemma 15 we know y is color-critical in G and not color-critical in $\overline{G}$. Thus $\chi(G - y) = \chi(G) - 1$

and $\chi(\overline{G}-y)=\chi(\overline{G})$, and so $\overline{G}-y$ is an NG-graph. Then

$$deg_{G-y}(x)=deg_G(x)\geqslant\chi(G)-1=\chi(G-y)>\chi(G-y)-1.$$

By Lemma 15 applied to $\overline{G}-y$, x is color-critical in $G-y$, so $\chi(G-y-x)=\chi(G-y)-1=\chi(G)-2$. Now we can properly color G using $\chi(G)-1$ colors by taking a proper coloring of $G-x-y$ using $\chi(G)-2$ colors and using one additional color for x and y, a contradiction. □

Lemma 17. *Suppose G is an NG-graph and $A=A_G$, $B=B_G$ and $C=C_G$ are defined as in Definition 13. Then $G[C]$ is an independent set and there are no edges in G between vertices of A and vertices of C.*

Proof. Given that G is an NG-graph, Definition 11 implies that $\overline{G}$ is also an NG-graph. Let $A'=A_{\overline{G}}=\{v:deg_{\overline{G}}(v)=\chi(\overline{G})-1\}$, $B'=B_{\overline{G}}=\{v:deg_{\overline{G}}(v)>\chi(\overline{G})-1\}$, and $C'=C_{\overline{G}}=\{v:deg_{\overline{G}}(v)<\chi(\overline{G})-1\}$. Combining Definition 11 with $deg_G(v)+deg_{\overline{G}}(v)=n-1$ yields $(deg_G(v)-\chi(G)+1)+(deg_{\overline{G}}(v)-\chi(\overline{G})+1)=0$. Thus $A'=A$, $B'=C$ and $C'=B$.

Now Lemma 16 applied to $\overline{G}$ implies that $\overline{G}[B']$ is complete, hence $G[C]$ is an independent set. For $x\in A'$ and $y\in B'$ the same lemma tells us that $xy\in E(\overline{G})$ and thus for $x\in A$ and $y\in C$ we know $xy\notin E(G)$. □

Lemma 18. *If G is an NG-graph and $A=A_G$ and $B=B_G$ are defined as in Definition 13, and $A\neq\emptyset$ then $H=G[A]$ is an NG-graph and $\chi(G)=\chi(H)+|B|$.*

Proof. From the structure of graph G deduced in Lemmas 16 and 17, we know $\chi(G)=\chi(H)+|B|$ and $\chi(\overline{G})=\chi(\overline{H})+|C|$. Thus $1+|A|+|B|+|C|=1+n=\chi(G)+\chi(\overline{G})=\chi(H)+\chi(\overline{H})+|B|+|C|$, so $1+|A|=\chi(H)+\chi(\overline{H})$ and H is an NG-graph as desired. □

Lemma 19. *If G is an NG-graph and with $A=A_G,B=B_G$ and $C=C_G$ are defined as in Definition 13, and $A\neq\emptyset$ then $H=G[A]$ is either a 5-cycle, a complete graph, or an independent set.*

Proof. Let $v\in A$. By our definition of the set A we know $deg_G(v)=\chi(G)-1$. To compute the degree of v in H, we subtract all neighbors in B (and C), and apply Lemmas 16, 17 and 18 to obtain $deg_H(v)=\chi(G)-1-|B|=\chi(H)-1$. Since this holds for all $v\in A$ we have $\Delta(H)=\chi(H)-1$. Now apply Brooks' Theorem (Theorem 4).

If H is connected we conclude that H is an odd cycle or a complete graph. First consider the case $H=C_{2k+1}$. If $k\geqslant 3$ we have $\chi(H)=3$ and $\chi(\overline{H})=k+1$ so $\chi(H)+\chi(\overline{H})=k+4\leqslant k+(k+1)=2k+1=n$ which contradicts Lemma 18. If $k=1$ then H is the complete graph C_3, hence we conclude that either H is a 5-cycle ($k=2$) or H is a complete graph.

Next we consider the case that H is not connected. Let H_1 be a component with maximum chromatic number and let H_2 be the rest of H, so H_2 is not empty. Then

$\chi(H) = \chi(H_1)$ and every vertex in H_1 has degree $\Delta(H_1) = \Delta(H) = \chi(H)-1 = \chi(H_1)-1$, so by Brooks' Theorem, H_1 is an odd cycle or a complete graph.

Case 1: $H_1 = K_r$ for some $r \geqslant 1$. Then $\chi(H) = \chi(H_1) = r$, so $\chi(\overline{H}) = 1 + \chi(\overline{H_2})$. By Lemma 18, we know H is an NG-graph, so $r+|V(H_2)|+1 = \chi(H)+\chi(\overline{H}) = r+1+\chi(\overline{H_2})$. Thus $|V(H_2)| = \chi(\overline{H_2})$ and H_2 is an independent set.

By definition of A, every vertex in H has degree $\chi(H) - 1$. Since $H_2 \neq \emptyset$, each vertex in H_2 has degree 0 in H, thus $\chi(H) - 1 = 0$ and H is an independent set.

Case 2: $H_1 = C_{2k+1}$ for some $k \geqslant 2$. In this case, $\chi(H) = \chi(H_1) = 3$ and $\chi(\overline{H}) = \chi(\overline{C_{2k+1}}) + \chi(\overline{H_2}) = k+1+\chi(\overline{H_2})$. Again, using Lemma 18, we know H is an NG-graph, so $2k+1+|V(H_2)|+1 = \chi(H)+\chi(\overline{H}) = 3+k+1+\chi(\overline{H_2})$. Thus $2-k = |V(H_2)|-\chi(\overline{H_2}) \geqslant 0$ and $k \leqslant 2$. By the assumptions of this case, $k \geqslant 2$ so $k = 2$ and the cycle C_{2k+1} is a 5-cycle. Substituting $k = 2$ into the equation above yields $|V(H_2)| = \chi(\overline{H_2})$, so H_2 is an independent set. Now vertices in H_1 have degree 2 in H and vertices in H_2 have degree 0 in H, contradicting the definition of A. □

Now we are ready to prove the other direction of Theorem 14.

($\Longrightarrow$) Suppose we are given an NG-graph G and let $A = A_G$, $B = B_G$, $C = C_G$ as in Definition 13. We show that the partition $V(G) = A \cup B \cup C$ satisfies $(i) - (v)$ of Theorem 14. Conditions (ii) and (iv) follow from Lemma 16 and conditions (iii) and (v) follow from Lemma 17. If $A \neq \emptyset$, condition (i) follows from Lemma 19.

Finally, we consider the case $A = \emptyset$, and show it leads to a contradiction. Suppose $A = \emptyset$, so $V(G) = B \cup C$ where $G[B] = K_b$, and $G[C] = I_c$. First observe that $\chi(G) \geqslant b$ because G contains $G[B] = K_b$. We know that $\chi(G) \leqslant b+1$ since C is an independent set. Moreover, for any $x \in C$, $deg(x) < \chi(G) - 1 \leqslant b+1-1 = b$. So $deg(x) < b$, and thus each vertex in C can be colored using one of the b colors not appearing among its neighbors.

Similarly, we show $\chi(\overline{G}) = c$. We know $\overline{G}[B] = I_b$ and $\overline{G}[C] = K_c$, so $\chi(\overline{G}) \geqslant c$ and each vertex of C requires its own color. By Lemma 15, we know each vertex $x \in B$ has $deg_{\overline{G}}(x) \leqslant \chi(\overline{G}) - 1$, so no additional colors are needed for vertices in B and $\chi(\overline{G}) = c$. Thus $\chi(G) + \chi(\overline{G}) = b + c = |V(G)|$, contradicting the assumption that G is an NG-graph. □

We next present an algorithm for determining whether a graph is an NG-graph and in the affirmative case, computing its chromatic number. The proof of correctness and an analysis of the complexity are given in Theorem 20.

Algorithm: NG

Input: A graph G with $n = |V(G)|$.

Output: A determination of whether G is an NG-graph and if so, its chromatic number.

Initialize $k = 1$.

Loop: Partition $V(G)$ according to vertex degrees as follows:

$A = \{v \in V(G) : deg(v) = k-1\}$

$B = \{v \in V(G) : deg(v) > k-1\}$

$C = \{v \in V(G) : deg(v) < k-1\}$.

Consider questions (i) – (v). If the answer to any of the questions is 'no', continue to step $(*)$. Otherwise (if all answers are yes), go to step (vi).

(i) Is $G[A]$ a complete graph, an independent set or a 5-cycle?

(ii) Is $G[B]$ complete?

(iii) Is $G[C]$ an independent set?

(iv) Is $ab \in E(G)$ for all $a \in A$ and all $b \in B$?

(v) Is $ac \notin E(G)$ for all $a \in A$ and all $c \in C$?

If the answers to (i) – (v) are 'yes', then check if $\chi(G) = k$ as follows:

(vi) For $G[A]$ a complete graph, check that $|A| + |B| = k$. For $G[A]$ an independent set, check that $|B| + 1 = k$. For $G[A]$ a 5-cycle, check that $|B| + 3 = k$.

If (vi) is affirmative, then graph G is an NG-graph and $\chi(G) = k$ and the algorithm ends. If not, continue to $(*)$.

$(*)$ If $k < n$, increment $k := k + 1$ and return to the beginning of the loop. If $k = n + 1$, then graph G is not an NG-graph.

Theorem 20. *Algorithm NG determines whether graph G is an NG-graph in polynomial time.*

Proof. We first establish correctness. We know $\chi(G)$ is between 1 and n, so start with $k = 1$, thinking of k as a potential value of χ.

If the answers to (i) – (vi) are all 'yes', then Theorem 14 ensures that G is a an NG-graph, where (vi) verifies that $\chi(G) = k$. If any of the answers to (i) – (vi) are no, we try the next possible value of k. If we reach $k = n + 1$, then we have tried all possible values of $\chi(G)$, and $V(G)$ can not be partitioned so that G has the necessary form. Thus the algorithm correctly determines whether G is an NG-graph, and if so, computes the chromatic number.

Each of the questions can be answered in time $O(n^2)$, and we potentially have to increment k from 1 to n so the running time of the algorithm is $O(n^3)$. □

4 Characterizing those NG-graphs that are NGD-graphs

In Section 3, we characterized NG-graphs and according to Theorem 14, there are three possibilities for A_G. We name them for convenience in the next definition.

Definition 21. An NG-graph G is of Type 1 if $G[A_G]$ is a clique, Type 2 if $G[A_G]$ is an independent set and Type 3 if $G[A_G]$ is a 5-cycle.

Note that an NG-graph with $|A_G| = 1$ is both Type 1 and Type 2.

Analogously we would like to characterize NGD-graphs. The set of NG-graphs and the set of NGD-graphs intersect, but neither is contained in the other, as demonstrated in Table 2.

G	NG-graph	NGD-graph
$K_{3,1,1}$	yes	yes
$K_{3,2}$	no	yes
C_5	yes	no
C_7	no	no

Table 2: Examples of graphs that separate the classes of NG-graphs and NGD-graphs

In this section we make progress toward this goal by characterizing the NG-graphs that are also NGD-graphs. We show in Theorem 24 that none of the Type 3 NG-graphs are NGD-graphs and in Theorem 32 and Corollary 33, we characterize those Type 1 and Type 2 NG-graphs that are NGD-graphs. The next result shows that the complement of a Type 1 NG-graph is a Type 2 NG-graph.

Proposition 22. *If G is an NG-graph, then $\overline{G}$ is an NG-graph. Moreover, an NG-graph G is of Type 1 iff $\overline{G}$ is of Type 2, and G is of Type 3 iff $\overline{G}$ is of Type 3.*

Proof. The first sentence follows immediately from Definition 11, so we focus on the statements in the second sentence. Since G is an NG-graph, $\chi(G)+\chi(\overline{G}) = n+1$ and A_G is the set of vertices whose degree is $\chi(G)-1$. Therefore A_G is the set of vertices whose degree in $\overline{G}$ is $(n-1)-(\chi(G)-1) = n-\chi(G) = \chi(\overline{G})-1$. Hence $A_G = A_{\overline{G}}$, and $G[A_G]$ is a clique if and only if $G[A_{\overline{G}}]$ is an independent set. So G is a Type 1 NG-graph if and only if $\overline{G}$ is a Type 2 NG-graph. The complement of a 5-cycle is a 5-cycle, so G is a Type 3 NG-graph if and only if $\overline{G}$ is a Type 3 NG-graph. □

In proving that a Type 3 NG-graph G is not an NGD-graph, it will be helpful to have an optimal coloring of G in which one color appears on only one vertex. This is possible by our next lemma.

Lemma 23. *(i) If G is a Type 1 NG-graph and $x \in A_G \cup B_G$ then there exists a proper coloring of G using $\chi(G)$ colors in which vertex x is uniquely colored. (ii) If G is a Type 2 NG-graph and $x \in B_G$ then there exists a proper coloring of G using $\chi(G)$ colors in which vertex x is uniquely colored.*

Proof. We start by proving (i). Let G be a Type 1 NG-graph and fix a proper coloring of G using $\chi(G)$ colors. Since the vertices in $A_G \cup B_G$ induce a clique in G, they are all colored distinctly. If there are other vertices with x's color, they must be in C_G. First consider the case in which $x \in B_G$. Since $A_G \neq \emptyset$, there exists $y \in A_G$, and all vertices of C_G may be recolored to have y's color. This leaves x as the only vertex in its color class.

Next consider the case in which $x \in A_G$ and $|A_G| = 1$. By the definition of Type 1 NG-graphs, we know $N(x) = B_G$ and for each $c \in C_G$ we know $deg(c) \neq deg(x) = |B_G|$. Thus each $c \in C_G$ has a non-neighbor in B_G. We can recolor each $c \in C_G$ to be the color of any of its non-neighbors in B_G. This leaves x as the only vertex in its color class.

Finally, consider the case in which $x \in A_G$ and $|A_G| > 1$. Let $a \in A_G$ where $a \neq x$. Since $ax \in E(G)$ we know a's color is different from x's color. Then each vertex in C_G can be colored with a's color and this leaves x as the only vertex in its color class.

The proof of (ii) is similar to the first paragraph of the proof of (i).

□

Theorem 24. *Type 3 NG-graphs are not NGD-graphs.*

Proof. Let G be a Type 3 NG-graph, and for a contradiction, assume G is also an NGD-graph. By definition of a Type 3 NG-graph, the vertices in A_G induce a 5-cycle in G, which we represent by $A_G = \{v_1, v_2, v_3, v_4, v_5\}$ with adjacencies $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$.

By the structure of NG-graphs, we know that any particular $v \in B_G \cup C_G$ has the same relationship to each vertex in A_G. Furthermore, any automorphism of G preserves the sets A_G, B_G, C_G because these sets are defined in terms of vertex degrees. Thus a coloring of G is distinguishing if and only if it is distinguishing on both $G[A_G]$ and $G[B_G \cup C_G]$. Fix a distinguishing coloring of G using colors $1, 2, 3, \ldots, D(G)$. Since $D(C_5) = 3$, we may recolor the vertices in A_G so that they use three colors and without loss of generality, vertices v_1, v_3 get color 1, vertices v_2, v_4 get color 2, and vertex v_5 gets color 3. Let V_i be the vertices of color i and let $G_i = G[V_i]$ be the graph induced by the vertices of color i.

For simplicity of notation in what follows, let $H = G_1$ and $F = G_2$. By Corollary 12, graphs H and F are NG-graphs. Furthermore, H (and likewise F) is not of Type 3, since there is no C_5 induced in H (or in F). Thus H and F are Type 1 or Type 2 NG-graphs and by Proposition 22, so are their complements $\overline{H}$ and $\overline{F}$. Since v_1 and v_3 have the same degree in $\overline{H}$, they are in the same part of the $A_{\overline{H}} \cup B_{\overline{H}} \cup C_{\overline{H}}$ partition of $V(\overline{H})$. If $\overline{H}$ is Type 1, since v_1 is adjacent to v_3 in $\overline{H}$, we know $v_1, v_3 \in A_{\overline{H}} \cup B_{\overline{H}}$. If $\overline{H}$ is Type 2, since v_1 is adjacent to v_3 in $\overline{H}$, we know $v_1, v_3 \in B_{\overline{H}}$. Now applying Lemma 23 to $\overline{H}$ (and $\overline{F}$), we conclude that there exists a proper coloring of $\overline{H}$ using $\chi(\overline{H})$ colors in which v_1 is uniquely colored. Similarly, there exists a proper coloring of $\overline{F}$ using $\chi(\overline{F})$ colors in which v_2 is uniquely colored.

Following the proof of Theorem 6, for each $i = 1, 2, 3, \ldots, D(G)$, we create a new coloring of G_i using $\chi(G_i)$ colors and of $\overline{G_i}$ using $\chi(\overline{G_i})$ colors, so that $\chi(G_i) + \chi(\overline{G_i}) = |V_i| + 1$. Note that in this coloring we use a different palette of colors for each G_i and for each $\overline{G_i}$. Thus, we have used a total of $n + D(G)$ colors. Furthermore, we choose colorings of $\overline{H} = \overline{G_1}$ and $\overline{F} = \overline{G_2}$ so that v_1 is uniquely colored (yellow) in $\overline{H}$ and v_2 is uniquely colored (purple) in $\overline{F}$. Finally, we switch v_2's color to yellow, which will result in a total of $n + D(G) - 1$ colors used. Since v_1 and v_2 are not adjacent in $\overline{G}$, the new coloring is proper. It is also distinguishing as follows:

Recall that in G or $\overline{G}$, any automorphism preserves the set of vertices $\{v_1, v_2, v_3, v_4, v_5\}$. Note that v_1, v_2, v_3, v_4, v_5 were all given different colors in $\overline{G}$ before this final switch of v_2's color, since they come from three different palettes, and v_1, v_2 were uniquely colored. After the switching v_2's color, each of v_3, v_4, v_5 is fixed by every automorphism that preserves the colors. There is no automorphism that switches v_1 and v_2 and preserves the colors after the final switch since v_1 is adjacent to v_3 and v_2 is not adjacent to v_3 in $\overline{G}$. There are no vertices outside of $A_{\overline{G}}$ that needed v_1 and v_2 to distinguish them, since all vertices outside of A have the same relationship to each vertex inside of A.

Now we have given proper distinguishing colorings of G and $\overline{G}$ using a total of $n + D(G) - 1$ colors, contradicting G being an NG-graph. $\square$

We will need a refinement on the vertex partition of a Type 1 NG-graph, where we further partition the set C_G as $L_G \cup M_G$.

Definition 25. Let G be a Type 1 NG-graph with ABC-partition $A_G \cup B_G \cup C_G$. We define the $ABLM$-partition of G to be $V(G) = A_G \cup B_G \cup L_G \cup M_G$ where $L_G = \{v \in C_G : v \text{ is adjacent to every vertex in } B_G\}$ and $M_G = C_G - L_G$. When it is unambiguous we write $A = A_G, B = B_G, L = L_G$ and $M = M_G$.

From Definition 25, we know $deg(v) = |B_G|$ for each $v \in L_G$, and $deg(v) \leqslant |B_G| - 1$ for each $v \in M_G$. If $|A_G| = 1$, then $L_G = \emptyset$ because any $v \in L_G$ would have the same degree as the vertex in A_G, contradicting Definition 13.

Let $\oplus$ be the external direct product of groups, and recall that S_n is the group of permutations of the set $\{1, 2, 3, \ldots, n\}$.

Proposition 26. *Let G be a Type 1 NG-graph and define A, B, L, M as in Definition 25. Then*

$$Aut(G) \cong S_{|A|} \oplus S_{|L|} \oplus \Gamma$$

where Γ is the subgroup of automorphisms that act on $G[B \cup M]$ and fix A and L.

Proof. We note that any automorphism of G preserves the sets A, B, L, M, because of the different vertex degrees in each set. Thus, in a distinguishing coloring, we may use the same set of colors for each set of vertices. Further, since each vertex in A has the same set of neighbors outside of A, and each vertex in L has the same neighborhood, then the action of $Aut(G)$ is independent on the three subgraphs, $G[A], G[L], G[B \cup M]$. Then the automorphism group of G is isomorphic to $Aut(G[A]) \oplus Aut(G[L]) \oplus \Gamma$ where $\Gamma \subseteq Aut(G)$ is the subgroup of automorphisms that act on $G[B \cup M]$ and fix A and L. Since A is complete and L is an independent set, $Aut(A) = S_{|A|}$ and $Aut(L) = S_{|L|}$. Hence $Aut(G) \cong S_{|A|} \oplus S_{|L|} \oplus \Gamma$. $\square$

Corollary 27. *Let G be a Type 1 NG-graph and define A, B, L, M as in Definition 25. Let $a = |A|$ and $\ell = |L|$. Then*

$$D(G) = max\{a, \ell, D^{\Gamma}(G[B \cup M])\}$$

where Γ is the subgroup of automorphisms that act on $G[B \cup M]$ and fix A and L.

Proof. Recall the definition of the distinguishing number with respect to Γ from Definition 8. In any distinguishing coloring of G, the colors can be reused for each set in the $ABLM$-partition of G, and the number of colors needed for A is a and for L is ℓ, thus $D(G) = max\{a, \ell, D^{\Gamma}(G[B \cup M])\}$. $\square$

We now define some necessary parameters, x_G and y_G.

Definition 28. Let G be a Type 1 NG-graph and define A, B, L, M as in Definition 25. Let $b = |B|$, $m = |M|$ and let Γ be defined as in Proposition 26. We define $x = x_G = \chi_D^\Gamma(G[B \cup M]) - b$, that is, the minimum number of colors, above the b colors used on the vertices in B, needed to color the vertices in M, to get a proper distinguishing coloring of $G[B \cup M]$ under the action of Γ. Similarly, we define $y = y_G = \chi_D^\Gamma(\overline{G}[B \cup M]) - m$, that is, the minimum number of colors, above the m colors used on the vertices in M, needed to color the vertices in B to get a proper distinguishing coloring of $\overline{G}[B \cup M]$ under the action of Γ.

The next lemma gives a bound on $x + y$, and following it are more two technical lemmas.

Lemma 29. *Let G be a Type 1 NG-graph and define A, B, L, M as in Definition 25. Then*

$$x, y \leqslant x + y \leqslant D(G).$$

Proof. The first inequality follows immediately because x, y are both nonnegative. For the second inequality, let $b = |B|$ and $m = |M|$. By the definition of x, $\chi_D^\Gamma(G[B \cup M]) = b + x$, and by the definition of y, $\chi_D^\Gamma(\overline{G}[B \cup M]) = m + y$. Then, applying Proposition 10 to the graph $G[B \cup M]$, we get

$$(b + x) + (m + y) = \chi_D^\Gamma(G[B \cup M]) + \chi_D^\Gamma(\overline{G}[B \cup M]) \leqslant b + m + D^\Gamma(G[B \cup M]).$$

Furthermore, $D(G) \geqslant D^\Gamma(G[B \cup M])$ from Corollary 27. Thus, $x + y \leqslant D(G)$ as desired. □

Lemma 30. *Let G be a Type 1 NG-graph and define A, B, L, M as in Definition 25, and x as in Definition 28. Then $x < D(G)$.*

Proof. Let $b = |B|$. We define a proper distinguishing coloring of $G[B \cup M]$. First, color the vertices in B with the b colors $\{1, 2, 3, \ldots, b\}$. We know b colors are needed, since $G[B]$ is complete. Each vertex of B is then fixed by its unique label. The vertices in M form an independent set, so would need to be different colors only in order to be distinguished. If $u, v \in M$ have different neighborhoods in B, then the colors of the vertices in those neighborhoods are different sets, so an automorphism taking u to v will not preserve colors.

For each vertex $u \in M$, let $S_u \subseteq \{1, 2, 3, \ldots, b\}$ be the set of colors of the vertices in $N(u)$, and define $T_u = \{v \in M : N(v) = N(u)\}$. Note that for any two vertices in T_u, there is an automorphism of G that interchanges them and fixes the rest of G. Thus, $D(G) \geqslant |T_u|$ for every $u \in M$. In order to achieve a proper distinguishing coloring, for each $u \in M$, each set T_u must be colored distinctly and the colors used on the vertices in T_u must be disjoint from S_u. Conversely, if this is achieved, we have a proper and distinguishing coloring of $G[B \cup M]$. Let $u_1, u_2, \ldots, u_k$ be chosen so that $T_{u_1}, T_{u_2}, \ldots, T_{u_k}$ is a partition of M.

For $1 \leqslant i \leqslant k$, we color the vertices in T_{u_i} distinctly, using as many colors in $\{1, 2, 3, \ldots, b\} - S_{u_i}$ as possible. The smallest number of colors, in addition to our original b colors, that we need is $max_{1 \leqslant i \leqslant k}\{|T_{u_i}| - (b - |S_{u_i}|)\}$. Since x is the minimum number of colors, above the b colors used on the vertices in B, needed to color the vertices in M, to get a proper distinguishing coloring of $G[B \cup M]$ under the action of Γ, $x = max_{1 \leqslant i \leqslant k}\{|T_{u_i}| - (b - |S_{u_i}|)\}$. By the definition of M, each vertex in M is missing at least one edge to B, so for each i, we can use at least one color from $\{1, 2, 3 \ldots, b\}$, and thus $x < max_{1 \leqslant i \leqslant k}\{|T_{u_i}|\} \leqslant D(G)$.

□

Lemma 31. *Let G be a Type 1 NG-graph, and define A, B, L, M as in Definition 25, and y as in Definition 28. If $|L| = 0$, then $y < D(G)$.*

Proof. Given $|L| = 0$, we know $C = M$. Since each vertex in B has degree greater than each vertex in A in G, each vertex in B has an edge to some vertex in M in G. That means that in $\overline{G}$, every vertex in B is missing an edge to some vertex in M. For each $v \in B$, let $W_v = \{w \in B : N_{\overline{G}}(w) = N_{\overline{G}}(v)\}$. Following the argument of Lemma 30, $y < max_{v \in B}\{|W_v|\} \leqslant D(\overline{G}) = D(G)$. □

We are now ready to characterize those Type 1 NG-graphs that are also NGD-graphs. Note that the vertices in A_G in a Type 1 NG-graph G form a complete subgraph, and all have the same neighborhood in the rest of the graph. So $D(G) \geqslant |A_G|$. Similarly, an independent set L of vertices in C_G each of which is adjacent to every vertex in B_G would all need to be distinctly colored in any distinguishing coloring of G, so $D(G) \geqslant |L|$. In the theorem below, we show that for any Type 1 NG-graph G, $D(G)$ is the maximum of these quantities if and only if G is NGD-graph.

Theorem 32. *Let G be a Type 1 NG-graph G and $a = |A_G|$ and ℓ is the number of vertices in C_G that are adjacent to every vertex in B_G. Then G is an NGD-graph iff $D(G) = max\{a, \ell\}$.*

Proof. Recall that $b = |B_G|$, x, y are as in Definition 28, and $\ell = |L|$ and $m = |M|$ where L, M are as in Definition 25. In any proper distinguishing coloring of G, all vertices in B are colored distinctly since B is complete. There are no edges between the set A and the set $L \cup M$, so colors used on vertices in A can be reused in $L \cup M$. There are no edges between L and M, so colors used on L can be reused on M. Thus, $\chi_D(G) = b + max\{a, \ell, x\}$. In a proper distinguishing coloring of $\overline{G}$, every vertex in $A \cup L \cup M$ must be colored distinctly since $L \cup M$ is a complete graph, and all vertices in A are adjacent to all vertices in $L \cup M$, and all vertices in A must be colored distinctly to eliminate the symmetries. Colors used on A and L can be re-used on B, since there are no edges in $\overline{G}$ between B and $A \cup L$. Thus, $\chi_D(\overline{G}) = m + max\{a + \ell, y\}$. Hence

$$\chi_D(G) + \chi_D(\overline{G}) = b + \max\{a, \ell, x\} + m + \max\{a + \ell, y\}. \tag{5}$$

We analyze the cases, depending on $max\{a, \ell, x\}$ and $max\{a + \ell, y\}$. Recall from Definition 11 that G is an NGD-graph iff $\chi_D(G) + \chi_D(\overline{G}) = a + b + \ell + m + D(G)$. Our proof will also show that the graphs in Cases (2) - (5) are not NGD-graphs.

Case (1) $max\{a, \ell, x\} = max\{a, \ell\}$ and $max\{a + \ell, y\} = a + \ell$.

Using Equation (5), we have

$$\chi_D(G) + \chi_D(\overline{G}) = b + max\{a, \ell\} + m + a + \ell = (a + b + \ell + m) + max\{a, \ell\}.$$

In this case G is an NGD-graph if and only if $D(G) = max\{a, \ell\}$. By Corollary 27, $D(G) \geqslant max\{a, \ell\}$, so if $D(G) = max\{a, \ell\}$, then G is an NGD-graph, and if $D(G) > max\{a, \ell\}$, then G is not an NGD-graph.

In Cases (2) - (5), we show that G is not an NGD-graph and that $D(G) > max\{a, \ell\}$.

Case (2) $max\{a, \ell, x\} = x$ and $max\{a + \ell, y\} = a + \ell$.

Using Equation (5), we have

$$\chi_D(G) + \chi_D(\overline{G}) = b + x + m + a + \ell = (a + b + \ell + m) + x.$$

Then G is an NGD-graph iff $D(G) = x$. By Lemma 30, $x < D(G)$, so G is not an NGD-graph, and indeed $D(G) > x \geqslant max\{a, \ell\}$ as desired.

Case (3) $max\{a, \ell, x\} = a$ and $max\{a + \ell, y\} = y$.

Using Equation 5, we have

$$\chi_D(G) + \chi_D(\overline{G}) = b + a + m + y = (a + b + m) + y.$$

Suppose that G were an NGD-graph. Then $y = \ell + D(G)$, by Lemma 29, $\ell + D(G) = y \leqslant x + y \leqslant D(G)$. So $\ell = 0$ and $y = D(G)$. By Lemma 31, when $\ell = 0$, we have $y < D(G)$, a contradiction, so G is not an NGD-graph.

Using Lemma 29 and the assumptions of this case, we have $D(G) \geqslant x + y \geqslant y \geqslant a + \ell$. We know $a > 0$ by Theorem 14. If $\ell > 0$, then $D(G) \geqslant a + \ell > max\{a, \ell\}$. If $\ell = 0$, then by Lemma 31, $D(G) > y \geqslant a + \ell = a \geqslant max\{a, \ell\}$. So we have shown $D(G) > max\{a, \ell\}$ as desired.

Case (4) $max\{a, \ell, x\} = \ell$ and $max\{a + \ell, y\} = y$.

Using Equation 5, we have

$$\chi_D(G) + \chi_D(\overline{G}) = b + \ell + m + y.$$

Using $a > 0$ from Theorem 14 and $y \leqslant D(G)$ from Lemma 29, we have $\chi_D(G) + \chi_D(\overline{G}) < (a + b + \ell + m) + y \leqslant (a + b + \ell + m) + D(G)$. Thus, G is not an NGD-graph.

Using the assumptions of this case, and Theorem 14, we have $\ell \geqslant a > 0$, thus $a + \ell > max\{a, \ell\}$. Now using Lemma 29, and the assumptions of this case, we have $D(G) \geqslant y \geqslant a + \ell > max\{a, \ell\}$ as desired.

Case (5) $max\{a, \ell, x\} = x$ and $max\{a + \ell, y\} = y$.

Using Equation 5, we have

$$\chi_D(G) + \chi_D(\overline{G}) = b + x + m + y.$$

Using $a > 0$ from Theorem 14 and $x + y \leqslant D(G)$ from Lemma 29, we have $\chi_D(G) + \chi_D(\overline{G}) < (a + b + \ell + m) + D(G)$. Hence G is not an NGD-graph.

Using $a > 0$ from Theorem 14 and the assumptions of this case, we have $x + y \geqslant a+a+\ell > max\{a, \ell\}$. However, $D(G) \geqslant x+y$ by Lemma 29, so we get $D(G) > max\{a, \ell\}$ as desired. □

If G is a Type 2 NG-graph, then its complement, $\overline{G}$, is a Type 1 NG-graph by Proposition 22, and applying Theorem 32 to $\overline{G}$ yields the following.

Corollary 33. *A Type 2 NG-graph G is an NGD-graph iff $D(G) = |A_G|$ or $D(G)$ equals the number of vertices in B_G that have no adjacencies to vertices in C_G.*

In our next example, we describe a Type 1 NG-graph which falls into Case 1 of the proof of Theorem 32, but is not an NGD-graph.

Example 34. Let G be a Type 1 NG-graph with $a = 1$, $b = 5$, $\ell = 0$, $m = 5$, and then define the edges between B_G and M_G so that each vertex in M_G has degree 1 and is adjacent to a different vertex in B_G. Then $D(G) = 3$, $x = 0$, $y = 0$, which fits in Case 1, except that G is not an NGD-graph, because $D(G) > max\{a, \ell\}$.

We conclude with two questions and acknowledgements.

Question 35. Theorem 32 characterizes those Type 1 NG-graphs that are NGD-graphs by their distinguishing number. Can the distinguishing number of the class of Type 1 NG-graphs be determined in polynomial time?

Question 36. In Theorem 32 and Corollary 33, we have characterized those NG-graphs which are NGD-graphs. Can this be extended to a characterization of the class of NGD-graphs?

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