The size of edge-critical uniquely 3-colorable planar graphs

Naoki Matsumoto

Graduate School of Environment and Information Sciences
Yokohama National University, Japan

E-mail:matsumoto-naoki-gn@ynu.ac.jp

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Abstract

A graph G is *uniquely k-colorable* if the chromatic number of G is k and G has only one k-coloring up to permutation of the colors. A uniquely k-colorable graph G is *edge-critical* if $G-e$ is not a uniquely k-colorable graph for any edge $e \in E(G)$. In this paper, we prove that if G is an edge-critical uniquely 3-colorable planar graph, then $|E(G)| \leqslant \frac{8}{3}|V(G)| - \frac{17}{3}$. On the other hand, there exists an infinite family of edge-critical uniquely 3-colorable planar graphs with n vertices and $\frac{9}{4}n - 6$ edges. Our result gives a first non-trivial upper bound for $|E(G)|$.

Keywords: uniquely colorable; edge-critical; planar graph

1 Introduction

In this paper, we only deal with finite undirected simple graphs, and for any vertex subset U in a graph G, $\langle U \rangle$ means the subgraph of G induced by U.

A *k-coloring* of a graph G is a map $c : V(G) \to \{1, 2, \ldots, k\}$ such that for any $uv \in E(G)$, $c(u) \neq c(v)$. A graph G is *k-colorable* if there exists a k-coloring of G, and the *chromatic number* of G, denoted by $\chi(G)$, is the minimum number k such that G is k-colorable. A graph G is *uniquely k-colorable* if $k = \chi(G)$ and G has only one k-coloring up to permutation of the colors, where the coloring is called a *unique k-coloring* (note that if G is uniquely k-colorable, then it is clear that $|V(G)| \geqslant k$ by the definition). In other words, every two k-colorings of G produce the same partition of $V(G)$ into k independent subsets (color classes). Then, two k-colorings c and c' are said to be *distinct*, denoted by $c \neq c'$, if they produce two distinct partitions of $V(G)$ into k color classes. Moreover, we denote the set of uniquely k-colorable graphs by UC_k. For two distinct colors $i, j \in \{1, 2, \ldots, k\}$ in a k-coloring c of a graph G, define $G_{i,j} = \langle c^{-1}(i) \cup c^{-1}(j) \rangle$. For uniquely k-colorable graphs, Harary et al. proved the following theorem.

Theorem 1 (Harary et al. [4]) *If $c : V(G) \to \{1, 2, \dots, k\}$ is a unique k-coloring of $G \in UC_k$, then the graph $G_{i,j}$ is connected for all $i \neq j$ ($i, j \in \{1, 2, \dots, k\}$).*

If a graph G is uniquely 1-colorable, then G has no edges. Hence, throughout this paper, we only consider $k \geqslant 2$ for any uniquely k-colorable graphs. Moreover, the following corollary holds by Theorem 1. (For other results and related topics, see [3].)

Corollary 2 *If $G \in UC_k$ with $|V(G)| \geqslant n$, then G has at least $(k-1)n - \binom{k}{2}$ edges.*

In this paper, we consider the size of edge-critical uniquely k-colorable planar graphs. For a graph $G \in UC_k$, G is *edge-critical* if $G - e \notin UC_k$ for any edge $e \in E(G)$. However, since uniquely 5-colorable planar graphs do not exist [2], we only consider edge-critical uniquely k-colorable planar graphs for $k \in \{2, 3, 4\}$.

By Corollary 2, if a uniquely k-colorable planar graph G has exactly $(k-1)|V(G)| - \binom{k}{2}$ edges, then G is edge-critical. Moreover, it is not difficult to see that any edge-critical uniquely 2-colorable planar graph G has at most $|V(G)| - 1$ edges (that is, G is a tree). On the other hand, since every planar graph G has at most $3|V(G)| - 6$ edges by Euler's formula, we have Table 1. (Following this, we denote the upper bound of the size of any edge-critical uniquely 3-colorable planar graph by $f(n)$, where n is the number of vertices.) In Table 1, it is clear that $f(n)$ is at most $3n - 6$ by the planarity, but this is not a "good" bound.

k	Lower bound (Corollary 2)	Upper bound
2	$n-1$	$n-1$
3	$2n-3$	$f(n)$
4	$3n-6$	$3n-6$

Table 1: The upper (or lower) bound of the size of any edge-critical uniquely k-colorable planar graph G, where $|V(G)| = n$ and $k \in \{2, 3, 4\}$.

In 1977, Aksionov [1] conjectured that $f(n) = 2n - 3$. However, in the same year, Mel'nikov and Steinberg [5] disproved the conjecture by constructing a counterexample shown in Figure 1. Hence $f(n)$ is greater than $2n - 3$. However, we have not yet known any reasonable upper bound.

Our main results are the followings.

Theorem 3 *If G is an edge-critical uniquely 3-colorable planar graph, then,*

$$|E(G)| \leqslant \frac{8}{3}|V(G)| - \frac{17}{3}.$$

Theorem 4 *For any integer $n \geqslant 12$ such that $n \equiv 0 \pmod 4$, there exists an edge-critical uniquely 3-colorable planar graphs with n vertices and $\frac{9}{4}n - 6$ edges.*

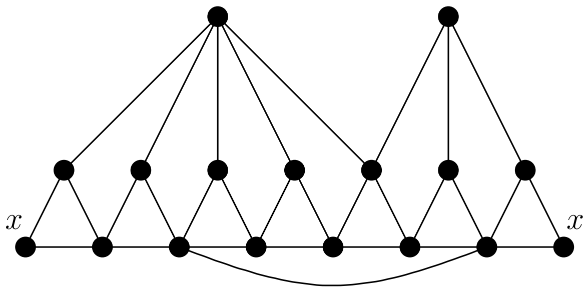

Figure 1: This graph is a counterexample for Aksionov's conjecture with $n = 16$ vertices and 30 edges, where $2n - 3 \neq 30$ and two $x's$ are the same vertices.

By our results, we have the following corollary.

Corollary 5 *For any $n \geqslant 12$ such that $n \equiv 0 \pmod 4$, we have*

$$\frac{9}{4}n - 6 \leqslant f(n) \leqslant \frac{8}{3}n - \frac{17}{3}.$$

In the next section, we prove Theorem 3. In Section 3, we construct an infinite family of edge-critical uniquely 3-colorable planar graphs with n vertices and $\frac{9}{4}n - 6$ edges.

2 Proof of Theorem 3

For a plane graph G, a Δ-*face cycle* $C = T_1T_2 \ldots T_k$ is a subgraph of G which consists of the vertices and edges of T_i's, where T_i is a triangular face and T_i and T_{i+1} share an edge for any $1 \leqslant i \leqslant k$ ($T_{k+1} = T_1$), see Figure 2. Similarly to a Δ-face cycle, we define a Δ-*face path* $P = T_0T_1 \ldots T_{l-1}$, where T_0 and T_{l-1} do not share an edge. Note that any 3-coloring of a Δ-face cycle and a Δ-face path is unique.

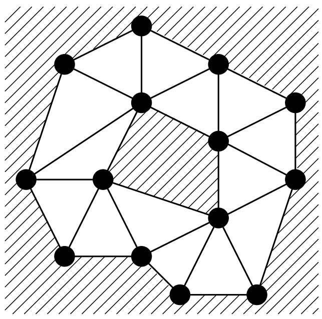

Figure 2: A Δ-face cycle C

Lemma 6 *Let G be an edge-critical uniquely 3-colorable plane graph. Then G has no Δ-face cycle.*

Proof. Suppose that G has a Δ-face cycle $C = T_1T_2\dots T_k$, and let $T_1 = e_1e_ke$ with $e_1 \in E(T_1)\cap E(T_2)$ and $e_k \in E(T_1)\cap E(T_k)$, where $e = uv$ (see Figure 3).

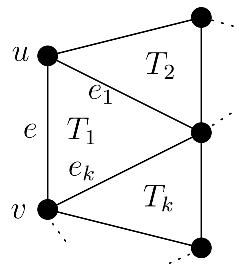

Figure 3: Surrounding of T_1

Since $G \in UC_3$, there exists a unique coloring $c : V(G) \to \{1, 2, 3\}$. Moreover, since $G-e \notin UC_3$ by the assumption, there exists another coloring $c' : V(G-e) \to \{1, 2, 3\}$ such that $c \neq c'$. In this case, since the Δ-face path $P = T_2T_3\dots T_k$ is uniquely 3-colorable, we now have $c'(u) \neq c'(v)$ (otherwise, it contradicts to G is 3-colorable since $e \in E(G)$). Therefore, we can obtain the coloring c' of G by adding e to $G - e$, that is, there exist two distinct 3-colorings c and c' of G. However, this contradicts $G \in UC_3$. ■

Lemma 7 *Let G be a plane graph with n vertices. If G has no Δ-face cycle, then $|E(G)| \leqslant \frac{8}{3}n - \frac{17}{3}$, where this estimation is best possible.*

Proof. It is well-known that any plane graph can be transformed into a *triangulation* (which is a plane graph such that each face is bounded by a cycle of length three) only by adding edges preserving the simpleness. Hence we regard G as a plane graph obtained from a triangulation T by removing k edges. (Since $|E(G)| \leqslant 3n-6-k$ by $|E(T)| = 3n-6$, we finally show $k \geqslant \frac{n-1}{3}$.)

We consider the dual graph of G, denoted by G^*. Let V_3 be the set of vertices of degree 3 in G^* (which is the set of triangular faces in G) and let $V_{\geqslant 4} = V(G^*) \setminus V_3$. We now have $|E(G^*)| = 3n - 6 - k$ and $|V_3| \geqslant 2n - 4 - 2k$, since removing a single edge decreases the number of triangular faces by at most two in G and any triangulation with n vertices has $2n - 4$ triangular faces by Euler's formula. We may suppose that $V_3 \neq \emptyset$, otherwise, the lemma holds since we have $|E(G)| \leqslant 2n - 4$ by $k \geqslant n - 2$. Moreover, by the assumption, $\langle V_3\rangle$ is a forest (otherwise, G has a Δ-face cycle). Then, we let $\langle V_3\rangle = T_1 \cup T_2 \cup \cdots \cup T_m$, where each T_i is a tree. Then we now have

$$\sum_i^m |V(T_i)| = |V_3|, \quad |E(\langle V_3\rangle)| = \sum_i^m |E(T_i)| = \sum_i^m (|V(T_i)| - 1) = |V_3| - m.$$

Let $e(V_3, V_{\geqslant 4}) = E(G^*) \setminus (E(\langle V_3\rangle) \cup E(\langle V_{\geqslant 4}\rangle))$ (see Figure 4, for example). Since each vertex in V_3 has degree exactly three, we have

$$|e(V_3, V_{\geqslant 4})| = 3\sum_i^m |V(T_i)| - 2\sum_i^m |E(T_i)| = |V_3| + 2m.$$

Here, let us count $|E(\langle V_{\geqslant 4}\rangle)|$. By the above equations, we have

$$\begin{aligned}|E(\langle V_{\geqslant 4}\rangle)| &= 3n-6-k-|E(\langle V_3\rangle)|-|e(V_3,V_{\geqslant 4})| \\ &= 3n-6-k-(|V_3|-m)-(|V_3|+2m) \\ &= 3n-6-k-m-2|V_3| \\ &\leqslant 3n-6-k-m-2(2n-4-2k) \\ &= -n+2+3k-m.\end{aligned}$$

Then, by $m \geqslant 1$ and $|E(\langle V_{\geqslant 4}\rangle)| \geqslant 0$, we have

$$n-1 \leqslant 3k.$$

Therefore, we have $k \geqslant \frac{n-1}{3}$, and hence, $|E(G)| \leqslant 3n-6-k \leqslant \frac{8}{3}n - \frac{17}{3}$.

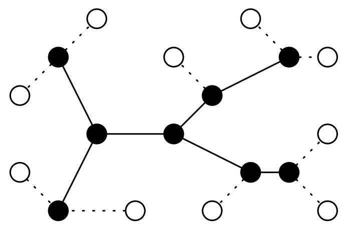

Figure 4: The black vertices and white ones are members in V_3 and $V_{\geqslant 4}$, respectively, and the dotted segments are members in $e(V_3, V_{\geqslant 4})$.

Next, we show that the estimation is best possible by constructing an infinite family of plane graphs which have no Δ-face cycle with n vertices and $\frac{8}{3}n - \frac{17}{3}$ edges as follows:

We prepare two graphs R and X shown in Figures 5 and 6, respectively. Then, as shown in Figure 7, we glue R to X by identifying a, b and c and a', b' and c', respectively. Moreover, we repeatedly apply the above operation to the resulting graph (the right hand of Figure 7), but from the second step, we identify a, d and c and a', b' and c', respectively.

Then, it is easy to check that plane graphs constructed by the above operation have no Δ-face cycle with n vertices and $\frac{8}{3}n - \frac{17}{3}$ edges. Therefore, the lemma holds. ■

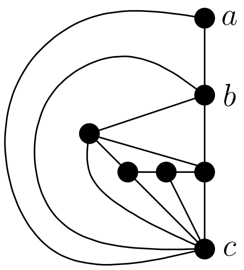

Figure 5: A graph R (7 vertices and 13 edges)

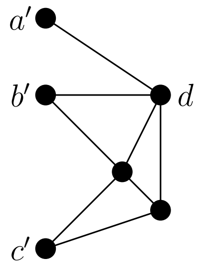

Figure 6: A graph X (6 vertices and 8 edges)

Proof of Theorem 3. By combining Lemmas 6 and 7, we have $|E(G)| \leqslant \frac{8}{3}|V(G)| - \frac{17}{3}$ for any edge-critical uniquely 3-colorable planar graph G. Therefore, this completes the proof of Theorem 3. ■

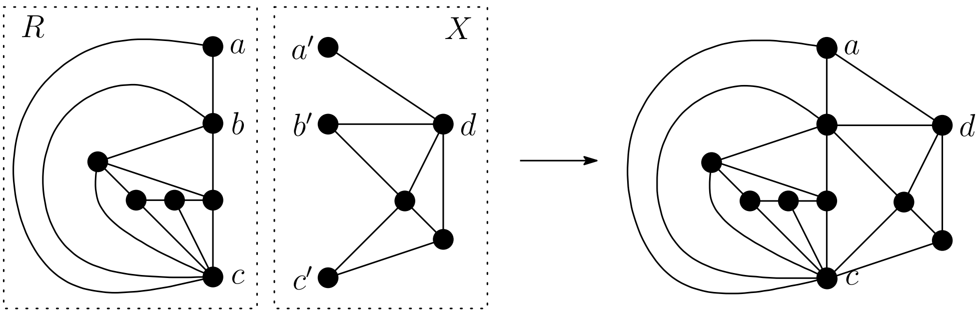

Figure 7: A construction of a plane graph which has no Δ-face cycle with n vertices and $\frac{8}{3}n - \frac{17}{3}$ edges

In this paper, we estimate the size of plane graphs with no Δ-face cycle, since such graphs are edge-critical uniquely 3-colorable planar graphs. However, since Lemma 7 is best possible, we have to find another structure of uniquely 3-colorable planar graphs to improve Theorem 3.

3 Proof of Theorem 4

In [5], Mel'nikov and Steinberg stated that an infinite family of edge-critical uniquely 3-colorable planar graphs with n vertices and more than $2n-3$ edges can be obtained by combining the graph K shown in Figure 8 several times. However, they did not clarify its structure. Hence, we give a construction of an infinite family of edge-critical uniquely 3-colorable planar graphs with n vertices and $\frac{9}{4}n - 6$ edges.

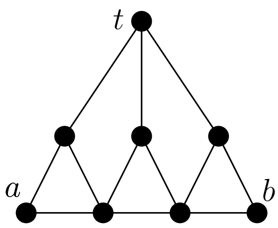

Figure 8: A graph K

Then we construct a graph H_k, as follows:

Prepare k triangles $u_1v_1u_2, u_2v_2u_3, \dots, u_{k-1}v_{k-1}u_k, u_kv_ku_1$, where $k \geqslant 5$ is an odd integer and for each i, u_i is shared by exactly two triangles $u_{i-1}v_{i-1}u_i$ and $u_iv_iu_{i+1}$ and add edges $u_3u_k, u_5u_k, \dots, u_{k-4}u_k$ (if $k = 5$, then we do not add u_1u_5 since there is a triangle $u_5v_5u_1$). Finally, we add two vertices x joined to $v_1, v_2, \dots, v_{k-3}, v_{k-2}$ and y joined to v_{k-2}, v_{k-1}, v_k. (For example, see Figure 9. If $k = 7$, then H_k is isomorphic to the graph of Figure 1.)

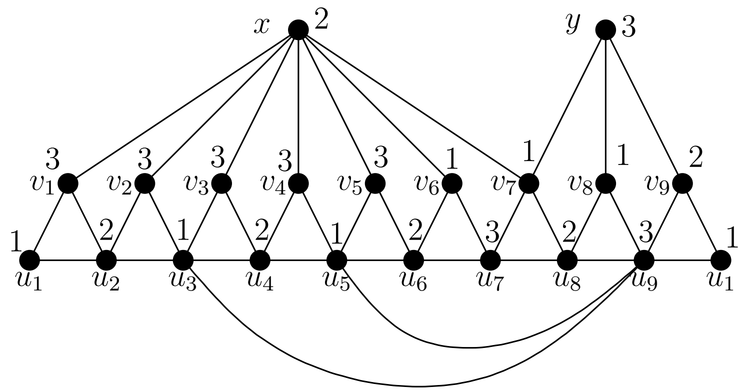

Figure 9: H_9

It is easy to see that H_k is planar and $|V(H_k)| = n = 2k+2 \geqslant 12$. Moreover, since $\deg(x) = k-2, \deg(y) = 3$ and there exist edges u_ik ($3 \leqslant i \leqslant k-4$, i : odd), we have

$$|E(H_k)| = 3k + (k-2) + 3 + \left(\frac{k-1}{2} - 2\right) = \frac{9}{2}k - \frac{3}{2}$$

Hence, we have $|E(H_k)| = \frac{9}{4}n - 6$ by $n = 2k+2$. Then, we shall prove the following theorem. By the above equation, this theorem implies Theorem 4.

Theorem 8 *For any odd integer $k \geqslant 5$, H_k is an edge-critical uniquely 3-colorable planar graph.*

Proof. We first show that H_k is uniquely 3-colorable. (finally, we have a unique 3-coloring of H_k shown in Figure 9). It is not difficult to see that for each i ($1 \leqslant i \leqslant k-4$), u_i, u_{i+3} and x correspond to a, b and t in the graph K shown in Figure 8, respectively. Hence, we use the following claim. Moreover, since it is not difficult to check that the theorem holds for H_5 using the following claim, we suppose $k \geqslant 7$.

Claim 9 [5] *Any 3-coloring of K assigns different colors to the vertices a and b.*

Let c be a 3-coloring of H_k, and by Claim 9, we now have $c(u_i) \neq c(u_{i+3})$ for each i ($1 \leqslant i \leqslant k-4$). Hence, without loss of generality, we may suppose that $c(u_{k-4}) = 1$ and $c(u_{k-1}) = 2$. Then we have $c(u_k) = 3$ and $c(v_{k-1}) = 1$.

Firstly, we show $c(x) = 2$. If $c(x) = 3$, then we have $c(v_{k-2}) = 1, c(u_{k-2}) = 3, c(u_{k-3}) = 2$ and $c(v_{k-4}) = 3$, but this is a contradiction since an edge $xv_{k-4} \in E(H_k)$. Hence we suppose that $c(x) = 1$. Then we now have $c(v_{k-2}) = 3, c(y) = 2, c(v_k) = 1, c(u_1) = 2, c(v_1) = 3$ and $c(u_2) = 1$. Since $u_3u_k \in E(H_k)$ and $c(u_k) = 3$, we have $c(u_3) = 2$ and then $c(v_3) = 3$ and $c(u_4) = 1$. Then we next consider $c(u_5)$ similarly to $c(u_3)$. By repeating the above argument, we have $c(u_{k-5}) = 1$. However, this is a contradiction since $u_{k-5}u_{k-4} \in E(H_k)$. Therefore, we have $c(x) = 2$.

Then, since $c(u_{k-4}) = 1$ and $c(x) = 2$, we now have $c(v_{k-5}) = c(v_{k-4}) = 3$ and $c(u_{k-5}) = c(u_{k-3}) = 2$. By $u_{k-6}u_k \in E(H_k)$ and $c(u_k) = 3$, we have $c(u_{k-6}) = 1$, and hence, $c(v_{k-6}) = c(v_{k-7}) = 3$ and $c(u_{k-7}) = 2$. By repeating this, we have the following coloring for each i $(2 \leqslant i \leqslant k-5)$;

$$c(u_i) = \left\{ \begin{array}{ll} 1 & (i : odd) \\ 2 & (i : even) \end{array} \right\}, \quad c(v_i) = 3.$$

Finally, since we have $c(u_1) = 1$ by $c(u_k) = 3$ and $c(u_2) = 2$, we can obtain $c(v_1) = 3, c(v_k) = 2, c(y) = 3, c(v_{k-2}) = 1, c(u_{k-2}) = 3$ and $c(v_{k-3}) = 1$. Therefore, since the 3-coloring c is uniquely decided as shown in Figure 9, H_k is uniquely 3-colorable.

Next, we shall show that H_k is edge-critical. Let c be a unique coloring of H_k (for example, see Figure 9. Observe that $G_{1,2}$ and $G_{1,3}$ are trees. Hence, it suffices to prove that for any edge e which is contained in cycles in $G_{2,3}$, $H_k - e \notin UC_3$. In $G_{2,3}$, as shown in Figure 9 for example, there exists a 4-cycle $xv_{i-1}u_iv_i$ for each i $(2 \leqslant i \leqslant k-5,\ i:$ even). Hence, we need to consider the following two cases. For any edge e, let c' be a new 3-coloring of $H_k - e$.

Case 1. Remove xv_{i-1} or $v_{i-1}u_i$ (see Figure 10)

We re-color the vertices of $H_k - xv_{i-1}$ (or $v_{i-1}u_i$) as follows $(i \leqslant j \leqslant k-5)$;

$$c'(u_j) = \left\{ \begin{array}{ll} 2 & (j : odd) \\ 3 & (j : even) \end{array} \right\}, \quad c'(v_j) = 1,$$

$$c'(u_{k-4}) = c'(u_{k-2}) = c'(v_{k-1}) = 2, \quad c'(u_{k-3}) = c'(u_{k-1}) = c'(y) = 1,$$

$$c'(v_{k-3}) = c'(v_{k-2}) = 3.$$

Moreover, v_{i-1} can be re-colored by the third color since $\deg(v_{i-1}) = 2$. In this case, since we do not change the color of x, c' and c are distinct 3-colorings.

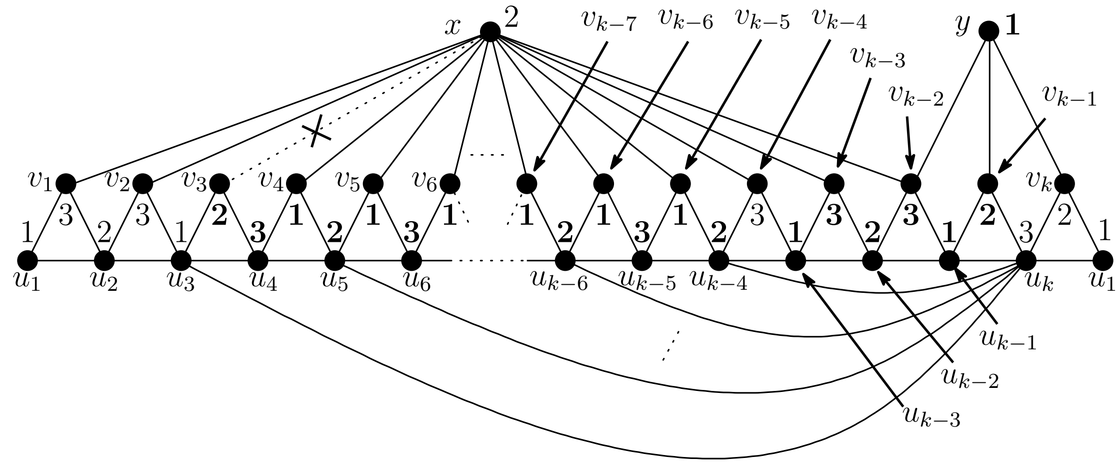

Figure 10: A new 3-coloring c' of $H_k - xv_{i-1}$ in Case 1 for fixed $i = 4$

Case 2. Remove xv_i or u_iv_i (see Figure 11)

Similarly to Case 1, we re-color the vertices as follows ($1 \leqslant j \leqslant i$);

$$c'(u_j) = \left\{ \begin{array}{ll} 2 & (j : odd) \\ 3 & (j : even) \end{array} \right\}, \quad c'(v_j) = 1, \quad c'(v_k) = 1.$$

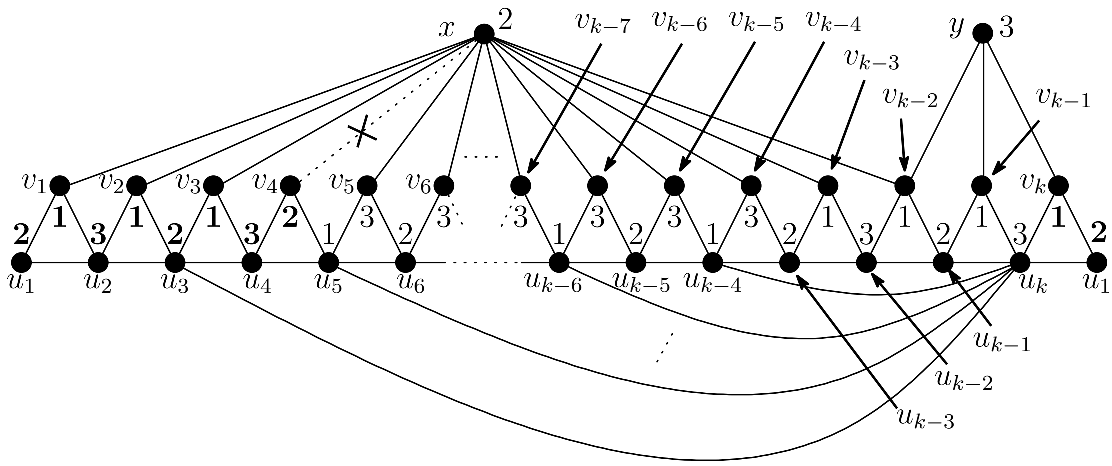

Figure 11: A new 3-coloring c' of $H_k - xv_i$ in Case 2 for fixed $i = 4$

Moreover, v_i can be re-colored by the third color since $\deg(v_i) = 2$. In this case, since we do not change the color of x, c' and c are distinct 3-colorings.

Therefore, since we can obtain distinct 3-colorings of the graph in both cases, H_k is edge-critical. Hence we complete the proof. ■

4 Concluding Remarks

In this section, we describe the size of edge-critical uniquely 3-colorable abstract graphs. Similarly to Lemma 6, it is not difficult to see that any edge-critical uniquely 3-colorable graph has no wheel as its subgraphs. In [6], the size of any graph G which has no wheel as its subgraphs is at most $\lfloor \frac{|V(G)|^2}{4} \rfloor + \lfloor \frac{|V(G)|+1}{4} \rfloor$. Hence, the size of any edge-critical uniquely 3-colorable abstract graph G is at most $\lfloor \frac{|V(G)|^2}{4} \rfloor + \lfloor \frac{|V(G)|+1}{4} \rfloor$.

We think that this estimation is not sharp similarly to our main results in this paper. However, since the estimation in [6] is best possible, we have to find a "good" structure to improve the estimation.

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