# Recognition and Characterization of Chronological Interval Digraphs

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**Abstract**

Interval graphs admit elegant ordering and structural characterizations as well as linear time recognition algorithms. On the other hand, the usual interval digraphs lack all three of these characteristics. In this paper we identify another natural digraph analogue of interval graphs that we call "chronological interval digraphs". By contrast, the new class admits an ordering characterization, several forbidden substructure characterizations, as well as a linear time recognition algorithm. Chronological interval digraphs arise by interpreting the standard definition of an interval graph with a natural orientation of its edges: $G$ is a chronological interval digraph if there exists a family of closed intervals $I_v, v \in V(G)$, such that $uv$ is an arc of $G$ if and only if $I_u$ contains the left endpoint of $I_v$.

We characterize chronological interval digraphs in terms of vertex orderings, and in terms of two kinds of forbidden substructures. These characterizations exhibit strong similarity with the corresponding characterizations of interval graphs, and lead to a linear time recognition algorithm.

## 1 Background

All graphs and digraphs considered in this paper may contain loops but no multiple edges or arcs; they are called *reflexive* if there is a loop at every vertex. The vertex set of a graph or digraph $G$ is always denoted by $V(G)$. If $G$ is a graph, we denote by $E(G)$ its edge set. If $G$ is a digraph, we denote by $A(G)$ its arc set, and by $S(G)$ the set of

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all symmetric arcs (i.e., arcs $uv$ such that $vu$ is also an arc) in $A(G)$. Note that $S(G)$ contains all loops as each loop is symmetric.

A graph $G$ is an *interval graph* if there exists a family of closed intervals $I_v, v \in V(G)$, called an *interval representation* of $G$, such that $uv$ is an edge of $G$ if and only if $I_u \cap I_v \neq \emptyset$. Interval graphs have been extensively studied, cf. e.g. [11]. The beautiful theory surrounding their study includes natural applications, elegant characterizations in terms of orderings and forbidden substructures, e.g., [9, 10, 13], as well as linear time recognition algorithms, e.g. [2, 4].

A digraph analogue of interval graphs, called interval digraphs, was pioneered in [17]. A digraph $G$ is an *interval digraph* if there exists a family of ordered pairs of closed intervals $(I_v, J_v), v \in V(G)$, also called an *interval representation* of $G$, such that $uv$ is an arc in $G$ if and only if $I_u \cap J_v \neq \emptyset$. Interval digraphs have also been much studied [3, 6, 5, 7, 12, 15, 18, 17, 22]. In particular, there are characterizations of interval digraphs in terms of matrices, as well as a polynomial time recognition algorithm. However, the most attractive aspects of interval graphs are absent, namely, an ordering characterization, a forbidden substructure characterization, and a linear time time recognition algorithm. (The only known polynomial recognition algorithm [15] has complexity $O(nm^6(n+m)\log n)$.)

In this paper we propose a natural alternative digraph analogue of interval graphs. A digraph $G$ is a *chronological interval digraph* if there exists a family of closed intervals $I_v, v \in V(G)$, called a *chronological interval representation* of $G$, such that $uv$ is an arc of $G$ if and only if $I_u$ contains the left endpoint of $I_v$. (Equivalently, $uv$ is an arc of $G$ if and only if $I_u$ intersects $I_v$ and the left endpoint of $I_u$ is not greater than the left endpoint of $I_v$.) Since every interval contains its own left endpoint, the digraph $G$ is reflexive. For the same reason, every interval graph is reflexive. In fact, an undirected graph is an interval graph if and only if it admits an orientation which is a chronological interval digraph. In this sense, our results provide new characterizations of interval graphs in terms of their orientations, see Corollary 9.

The possible orderings of endpoints of intervals representing an interval graph have been investigated under the name "chronological orderings" [20]. (In fact, "chronological orderings" of interval bigraphs and digraphs have also been investigated in [19].) We adopt the adjective "chronological" for our digraphs.

Let $G$ be an interval digraph with an interval representation $(I_v, J_v), v \in V(G)$. If each interval $J_v$ is a single point, then $G$ is called an *interval catch digraph* [16]. If each pair $I_v, J_v$ have the same left endpoint then $G$ is called an *adjusted interval digraph*, cf. [7, 8]. If each $J_v$ is the single point that is the left endpoint of $I_v$ then $I_v, v \in V(G)$, is a chronological interval representation of $G$. Consequently, every chronological interval digraph is an interval catch digraph as well as an adjusted interval digraph. However, the converse is not true: for instance the digraph $Z$ in Figure 1 has one representation with each interval $J_v$ being a single point, and another representation with each pair $I_v, J_v$ having the same left endpoint; however, no representation satisfies both properties simultaneously.

We shall give three characterizations of chronological interval digraphs - an ordering characterization, a structural characterization similar to the theorem of Lekkerkerker and

Boland [13], and a novel characterization in terms of so-called parallel vertices. The last characterization leads to a linear time recognition algorithm for the class. We note that in the special case of acyclic digraphs, a different forbidden subgraph characterization of chronological interval digraphs has been described (without proof) in [14].

## 2 Preliminary structures

We assume from now on that all digraphs $G$ are reflexive. When $uv \in A(G)$, we say that $v$ is an *out-neighbour* of $u$ and that $u$ is an *in-neighbour* of $v$. Note that each vertex $v$ is an out-neighbour as well as an in-neighbour of itself. We use the symbol $N^+(v)$, respectively $N^-(v)$, to denote the set of out-neighbours, respectively in-neighbours, of a vertex $v$. The sizes $|N^+(v)|$ and $|N^-(v)|$ are respectively the *outdegree* and *indegree* of $v$.

A digraph is *complete* if any two vertices are joined by symmetric arcs and is *semi-complete* if any two vertices are joined by at least one arc (which may or may not be symmetric). A digraph is *in-semicomplete* [1] if the in-neighbours of each vertex induce a semicomplete subdigraph. A digraph $G$ is *clustered* if it is in-semicomplete and for any two vertices $u, v$ in the same strong component of $G$, we have $N^-(u) = N^-(v)$, and $N^+(u) \subseteq N^+(v)$ or $N^+(v) \subseteq N^+(u)$. Note that the condition $N^-(u) = N^-(v)$ implies that $uv \in S(G)$, that is, every strong component of a clustered digraph is complete.

**Proposition 1.** *Every chronological interval digraph is clustered.*

**Proof:** Let $G$ be a chronological interval digraph and $I_v, v \in V(G)$ be a chronological interval representation of $G$. If $u, u'$ are both in-neighbours of a vertex $v$ in $G$, then $I_u, I_{u'}$ must intersect as they both contain the left endpoint of $I_v$, so there is at least one arc between $u, u'$. Hence $G$ is in-semicomplete.

Suppose that $u, v$ are in the same strong component of $G$. Then there exist a directed path from $u$ to $v$ and a directed path from $v$ to $u$. It follows from the definition that the left endpoint of $I_u$ is not greater than the left endpoint of $I_v$ and vice versa. Consequently, the left endpoints of $I_u$ and $I_v$ are the same which means that $uv$ is a symmetric arc in $G$. Hence every strong component of $G$ is complete.

Since $I_u$ and $I_v$ have the same left endpoint, $N^-(u) = N^-(v)$. Furthermore, $N^+(u) \subseteq N^+(v)$ if the right endpoint of $I_u$ is not greater than the right endpoint of $I_v$ and $N^+(v) \subseteq N^+(u)$ otherwise. Therefore $G$ is clustered. □

The reflexive digraph $Z$ in Figure 1 is not clustered as $N^+(a) \not\subseteq N^+(b)$ and $N^+(b) \not\subseteq N^+(a)$. Hence it cannot be an induced subdigraph of a chronological interval digraph.

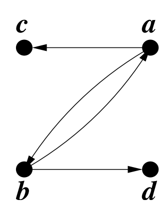


Figure 1: The reflexive digraph $Z$ where the loops are not drawn

**Corollary 2.** *No chronological interval digraph contains $Z$ as an induced subdigraph.* $\square$

A digraph is *connected* if its underlying undirected graph is connected. Clearly, a digraph is a chronological interval digraph if and only if this is so for each of its connected components. Therefore, we assume from now on that all digraphs considered are connected; this will not be repeated.

The following ordering characterization of interval graphs is well known. An undirected graph $G$ is an interval graph if and only if $V(G)$ can be linearly ordered by $<$ so that for $u, v, w \in V(G)$ with $u < v < w$ we have that $uw \in E(G)$ implies $uv \in E(G)$, cf. [20]. Similarly, a digraph is an interval catch digraph if and only if $V(G)$ can be linearly ordered by $<$ so that for $u, v, w \in V(G)$ with $u < v < w$ we have that $uw \in A(G)$ implies $uv \in A(G)$, and $wu \in A(G)$ implies $wv \in A(G)$ cf. [16]. A digraph is an adjusted interval digraph if and only if $V(G)$ can be linearly ordered by $<$ so that for $u, v, w \in V(G)$ with $u < v < w$ we have that $uw \in A(G)$ implies $uv \in A(G)$, and $wu \in A(G)$ implies $vu \in A(G)$ cf. [7].

There is an ordering characterization also for chronological interval digraphs. We say that a linear ordering $<$ of the vertices of a digraph $G$ is a *chronological ordering* of $G$ if it satisfies the following four properties, for any $u < v$ (for $P_1$) and any $u < v < w$ (for $P_2 - P_4$).

($P_1$) $vu \notin A(G) - S(G)$

($P_2$) $uw \in S(G)$ implies $uv, vw \in S(G)$

($P_3$) $uw \in A(G) - S(G)$ implies either $uv \in A(G) - S(G)$ or both $uv \in S(G)$ and $vw \in A(G) - S(G)$

($P_4$) $uw \notin A(G)$ implies $uv \notin A(G)$ or $vw \notin S(G)$.

The ordering can be intuitively described as follows. Each vertex $v$ forms a maximal complete digraph $C(v)$ with some (possibly none) of its immediately preceding and some (possibly none) of its immediately following vertices. These maximal complete digraphs comprise the strong components of $G$. If a vertex $u$ in $C(v)$ has an arc to a vertex in a $C(w)$, with $v < w$, then $u$ has an arc to each vertex of $C(w)$.

**Proposition 3.** *A digraph $G$ is a chronological interval digraph if and only if it admits a chronological ordering.*

**Proof:** Suppose $G$ is a chronological interval digraph. Let $I_v = [l_v, r_v], v \in V(G)$ be a chronological interval representation of $G$. Assume without loss of generality that no two intervals are the same. Then the vertices of $G$ can be ordered lexicographically according to the ordered pairs of endpoints, that is, $u < v$ if and only if either $l_u < l_v$ or $l_u = l_v$ and $r_u < r_v$. It is a routine exercise to check that this ordering $<$ is chronological. (Note that $uv \in S(G)$ if and only if $I_u$ and $I_v$ share the same left endpoint.)

Conversely, suppose that $v_1 < v_2 < \cdots < v_n$ is a chronological ordering of $G$. Then, for each $i = 1, 2, \ldots, n$, there exist subscripts $a_i \leqslant i \leqslant b_i \leqslant c_i$ such that $N^-(v_i) \cap N^+(v_i) =$

$\{v_{a_i}, v_{a_i+1}, \dots, v_{b_i}\}$ and $N^+(v_i) = \{v_{a_i}, v_{a_i+1}, \dots, v_{c_i}\}$. It is easy to verify that the intervals $I_{v_i} = [a_i, c_i]$, $i = 1, 2, \dots, n$, form a chronological interval representation of $G$. $\square$

We pursue the analogy with interval graphs further as follows. A natural obstruction to being an interval graph is the concept of an asteroidal triple. Specifically, an *asteroidal triple* in a graph $G$ is a triple of vertices $a, b, c$, such that any two of them are joined by a path that does not contain any neighbours of the third vertex. It is natural to view an asteroidal triple as an obstruction to having an ordering $<$ such that, for $u < v < w$, the edge $uw \in E(G)$ implies the edge $uv \in E(G)$. To make this explicit, we state the following equivalent definition of an asteroidal triple. An *asteroidal triple* in a graph $G$ is a triple of vertices $a, b, c$, such that for every ordering $u < v < w$ of $a, b, c$, there exists a sequence $u = u_1, u_2, \dots, u_p = w$ of vertices of $G$, such that for every $i = 1, 2, \dots, p-1$ the triple $u_i, v, u_{i+1}$ is "bad", in the sense that

$$u_i u_{i+1} \in E(G) \text{ but } u_i v \notin E(G) \text{ and } v u_{i+1} \notin E(G).$$

Thus an asteroidal triple makes it impossible to find a suitable ordering $<$. (It is worth noting that in this case the sequence $u = u_1, u_2, \dots, u_p = w$ is a path, a property we will lose in our analogue, due to the difference in the definition of being "bad".)

In a similar spirit we introduce a concept analogous to asteroidal triples. An *asynchronous triple* in a digraph $G$ is a triple of vertices $a, b, c$, such that for every ordering $u < v < w$ of $a, b, c$, there exists a sequence $u = u_1, u_2, \dots, u_p = w$ of vertices of $G$, such that for every $i = 1, 2, \dots, p-1$ the triple $u_i, v, u_{i+1}$ is *bad*, in the sense that it violates one of the properties $P_1 - P_4$ above (with $u, w$ replaced by $u_i, u_{i+1}$ respectively), that is, at least one of the following four properties holds.

($Q_1$) one of $vu_i, u_{i+1}v, u_{i+1}u_i$ is in $A(G) - S(G)$

($Q_2$) $u_i u_{i+1} \in S(G)$ and at least one of $u_i v \notin S(G), v u_{i+1} \notin S(G)$

($Q_3$) $u_i u_{i+1} \in A(G) - S(G)$ and either $u_i v \notin A(G)$ or both $u_i v \in S(G)$ and $v u_{i+1} \notin A(G) - S(G)$

($Q_4$) $u_i u_{i+1} \notin A(G)$, $u_i v \in A(G)$ and $v u_{i+1} \in S(G)$.

Since each property $Q_i, i = 1, 2, 3, 4$, is obtained by negation from the corresponding property $P_i$ (with $u, w$ replaced by $u_i, u_{i+1}$ respectively), we obtain the following corollary of Proposition 3.

**Corollary 4.** *No chronological interval digraph contains an asynchronous triple.* $\square$

For a graph $G$ to be an interval graph, there are two natural necessary conditions: $G$ has to be chordal (i.e., must not contain a chordless cycle of length greater than three), and it must not have any asteroidal triples. The celebrated theorem of Lekkerkerker and Boland [13] claims that these two conditions together are also sufficient, i.e., a chordal graph without asteroidal triples is an interval graph.

We have also found two necessary conditions for a digraph $G$ to be a chronological interval digraph: $G$ has to be clustered, and it must not have any asynchronous triples. It is our goal to prove that these two conditions together are again sufficient.

Before we proceed with the proof, we point out that neither of the two conditions alone is sufficient. For instance, the reflexive digraph $Y$ in Figure 2 is clustered and contains an asynchronous triple. Indeed, the triple $a, b, c$ is asynchronous, since for each ordering $u < v < w$ of $a, b, c$, the unique path $u = u_1, u_2, \ldots, u_5 = w$ in the underlying graph of $Y$ has, for each $i = 1, 2, \ldots, 4$, the triple $u_i, v, u_{i+1}$ satisfy one of properties $Q_1$ or $Q_3$ above. Moreover, the digraph $Z$ in Figure 1, which is not clustered, has no asynchronous triple. To show it does not contain an asynchronous triple, we may focus without loss of generality on the triples $a, b, c$ and $a, c, d$. It is easy to see that there is no sequence from $b$ to $c$ which satisfies the property that each pair of consecutive vertices with $a$ in the middle is a bad triple. So $a, b, c$ is not an asynchronous triple. Similarly, there is no such sequence from $d$ to $c$, so $a, c, d$ is not an asynchronous triple.

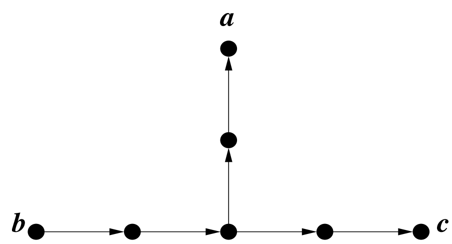


Figure 2: The reflexive digraph $Y$; the loops are not drawn

There is another obstruction we can use in place of asynchronous triples. It takes advantage of the special structure of clustered digraphs, and is reminiscent of the linear ordering of max cliques, that characterizes interval graphs [9].

An *out-branching* rooted at a vertex $h$ in a digraph $G$ is a spanning oriented tree $T$ of $G$ in which each vertex $v \neq h$ has in-degree one (and $h$ has in-degree zero). One of the prominent features of an out-branching $T$ is that there is a unique directed path in $T$ from $h$ to every vertex $v$. For each vertex $v \neq h$, we use $v^-$ to denote the unique in-neighbour of $v$ in $T$. We call $v^-$ the *parent* of $v$ and call $v$ a *child* of $v^-$. Two vertices which have the same parent are *siblings*. If $u \neq v$ and there is a directed path in $T$ from $u$ to $v$, we say that $u$ is an *ancestor* of $v$ and that $v$ is a *descendant* of $u$ in $T$. Thus the root $h$ is an ancestor of every other vertex.

Clustered digraphs contain special out-branchings. Let $G$ be a clustered digraph. Since $G$ is in-semicomplete, it has a unique initial strong component (i.e., having no incoming arcs from any other strong component). If a strong component $C$ of $G$ has an incoming arc from a vertex $x$ in another strong component, then every vertex of $C$ is an out-neighbour of $x$ since all vertices of $C$ have the same in-neighbourhood. It follows that if a strong component $C$ has incoming arcs from two other strong components $C', C''$ then there is an arc between $C'$ and $C''$. Hence, if strong component $C$ is not initial then amongst the strong components that have an arc to $C$ there is a unique strong component $C^-$ such that any other strong component that has an arc to $C$ also has an arc to $C^-$.

Since $G$ is clustered, each strong component $C$ is a complete digraph and the vertices of $C$ can be ordered, $v_1, v_2, \ldots, v_k$, such that $N^+(v_i) \subseteq N^+(v_j)$ for all $i < j$. Note that vertices of $C$ of the same out-neighbourhood appear consecutively in the ordering so such an ordering is unique up to relabelings of vertices of the same out-neighbourhood. Clearly, $v_1 v_2 \ldots v_k$ is a hamiltonian path of $C$. We call it a canonical path of $C$ and $v_1$ (respectively, $v_k$) the first (respectively, last) vertex of the path. It is easy to see that if $C$ is not initial, then the first vertex of a canonical path of $C$ is an out-neighbour of the last vertex of a canonical path of $C^-$. Arbitrarily choose a canonical path for each strong component. Let $T$ be the spanning subdigraph of $G$ consisting of all arcs in canonical paths and the arc from the last vertex of the path of $C^-$ to the first vertex of the path of $C$ for each strong component $C$ which is not initial. It is easy to see that $T$ is an out-branching of $G$ rooted at the first vertex of the path of the initial component. We call $T$ a *canonical out-branching* of $G$.

**Proposition 5.** *Suppose that $G$ is a clustered digraph and $T$ is a canonical out-branching of $G$. Then for any two vertices $u, v$ the following three properties hold:*

1. *if $u$ is an ancestor of $v$ in $T$ and $uv$ is an arc of $G$, then for each vertex $x$ in the unique directed path from $u$ to $v$ in $T$, $ux$ is an arc of $G$;*

2. *if $u$ is an ancestor of $v$ in $T$ and $uv$ is a symmetric arc of $G$, then for each vertex $x$ in the directed path $P$ from $u$ to $v$ in $T$, $ux$ is a symmetric arc of $G$, that is, all vertices in $P$ are in the same strong component of $G$; furthermore, $N^+(u) \subseteq N^+(v)$;*

3. *if neither of $u, v$ is an ancestor of the other in $T$, then there is no directed path in $G$ between $u$ and $v$.*

**Proof:** Suppose that $u$ is an ancestor of $v$ in $T$ and $uv$ is an arc in $G$. Let $x_1 x_2 \ldots x_k$ be the directed path in $T$ from $u$ to $v$. Let $i \geqslant 1$ be the smallest subscript such that $x_1 x_j$ is an arc of $G$ for each $i = j, \ldots, k$. If $i > 1$, then both $x_1, x_{i-1}$ are in-neighbours of $x_i$. Since $G$ is in-semicomplete there is an arc between $x_1$ and $x_{i-1}$ in $G$. The minimality of $i$ implies that $x_{i-1} x_1$ is an arc in $G$. Thus $x_1 \ldots x_{i-1} x_1$ is a directed cycle in $G$ and hence $x_1 x_{i-1}$ is a symmetric arc in $G$, a contradiction to the minimality of $i$. So $i = 1$, i.e., $T$ satisfies property 1. If $uv = x_1 x_k$ is a symmetric arc, then $x_1 x_2 \ldots x_k x_1$ is a directed cycle in $G$. Hence $x_1 x_i$ is a symmetric arc for each $i = 1, 2, \ldots, x_k$, i.e., $T$ satisfies property 2.

For property 3, suppose that neither of $u, v$ is an ancestor of the other. Let $w_1 \ldots w_t u_1 \ldots u_r$ and $w_1 \ldots w_t v_1 \ldots v_s$ be the unique paths from the root of $T$ to $u$ and $v$ respectively where $u_i \neq v_j$ for all $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$. First we show that $w_t u_1$ is not symmetric. Otherwise $w_t, u_1$ are in the same strong component. Since $w_t$ is the parent of $u_1$, it is not the last vertex in the strong component and so cannot be the parent of two children. This is a contradiction, as it is also the parent of $v_1$. Hence $w_t u_1 \in A(G) - S(G)$. Similarly, $w_t v_1 \in A(G) - S(G)$.

Suppose to the contrary that there is a directed path in $G$ between $u$ and $v$. Without loss of generality assume $u = u_r u_{r+1} \ldots u_{r+t} v_s = v$ is such a path. Note that the two paths $u_1 \ldots u_r$ and $u_{r+1} \ldots u_{r+t}$ may have vertices in common (and thus $u_1 \ldots u_r u_{r+1} \ldots u_{r+t}$ is a walk but not necessarily a path). Note also that there is an arc (e.g., $v_{r+t} v_s$) between

$u_1 \dots u_r u_{r+1} \dots u_{r+t}$ and $v_1 \dots v_s$ Consider such an arc joining $u_i$ and $v_j$ with the minimum sum $i+j$. The minimality of $i+j$ and the fact that $G$ is in-semicomplete imply that $i=1$ and $v_j u_1$ is an arc of $G$, or $j=1$ and $u_i v_1$ is an arc of $G$. In the former case $v_1 \dots v_j u_1$ is a directed path from $v_1$ to $u_1$ and in the latter case the directed walk $u_1 \dots u_i v_1$ contains a directed path from $u_1$ to $v_1$. Hence there is a directed path in $G$ between $u_1$ and $v_1$.

We consider without loss of generality the case when there is a directed path from $u_1$ to $v_1$. Let $u_1 = z_1 z_2 \dots z_k = v_1$ be such a path. Since $G$ is in-semicomplete, there is an arc between $w_t$ and $z_{k-1}$. If $z_{k-1} w_t \in A(G)$, then the directed cycle $w_t z_1 \dots z_{k-1} w_t$ shows that all the vertices in the cycle are in the same strong component. In particular, $w_t u_1 = w_t z_1$ is symmetric, a contradiction. So $w_t z_{k-1} \in A(G) - S(G)$. This implies that $w_t$ and $z_{k-1}$ can not be in the same strong component. However both are in-neighbours of $v_1$. This contradicts the fact $w_t$ is the parent of $v_1$. □

Two vertices $x, y$ in a digraph $G$ are said to be *parallel* if there exist in $G$ two directed paths ending in $x$ and $y$, say $x_1 x_2 \dots x_a = x$ and $y_1 y_2 \dots y_b = y$ where $a, b \geqslant 3$, such that

- $x_1 x_a, y_1 y_b \notin A(G)$, and
- for some $p, q$ with $1 < p \leqslant a$ and $1 < q \leqslant b$, $x_1 y_q, y_1 x_p \in A(G)$ and there is no directed path in $G$ between $x_p$ and $y_q$ in either direction.

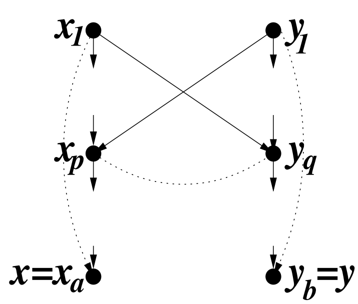


Figure 3:

**Proposition 6.** *If a clustered digraph $G$ contains parallel vertices, then it contains an asynchronous triple.*

**Proof:** Suppose that $x, y$ are parallel vertices in $G$. By definition there exist in $G$ directed paths $x_1 x_2 \dots x_a = x$ and $y_1 y_2 \dots y_b = y$, and subscripts $p$ and $q$, that satisfy the conditions in the definition above.

Let $T$ be a canonical out-branching of $G$. Then $T$ satisfies the properties of Proposition 5. These properties ensure that $T$ contains two directed paths from the root such that one contains $x_1, x_p, x_a$ and the other contains $y_1, y_q, y_b$. Let $w_1 \dots w_t u_1 \dots u_r$ and $w_1 \dots w_t v_1 \dots v_s$ where $u_i \neq v_j$ for all $i, j$ be such two paths. Since there is no directed path in $G$ between $x_p$ and $y_q$, we have $x_p = u_\alpha$ and $y_q = v_\beta$ for some $\alpha$ and $\beta$. Since there are directed paths in $G$ from $x_1$ to $y_q$ and from $y_1$ to $x_p$, we have $x_1 = w_\gamma$ and $y_1 = w_\eta$ for some $\gamma$ and $\eta$. If $x_a = w_i$ for some $i$, then $x_p \dots x_a \dots w_t v_1 \dots v_\beta$ is a directed walk in $G$ from $x_p$ to $y_q$ which contains a directed path from $x_p$ to $y_q$, a contradiction to the assumption. So $x_a = u_{\alpha'}$ for some $\alpha'$. A similar proof shows that $y_b = v_{\beta'}$ for some $\beta'$. Since $w_\eta u_\alpha, w_\gamma v_\beta \in A(G)$, we have $w_\eta u_1, w_\gamma v_1 \in A(G)$ according to Proposition 5.

Assume without loss of generality that $\eta \leqslant \gamma$. We claim that $w_t, u_{\alpha'}, v_{\beta'}$ form an asynchronous triple, which contradicts our assumption. Indeed, consider any ordering $<$ of $w_t, u_{\alpha'}, v_{\beta'}$. Suppose first that $u_{\alpha'} < v_{\beta'} < w_t$. Consider the sequence $u_{\alpha'}, u_{\alpha'-1}, \dots, u_1, w_t$. For each $1 \leqslant i \leqslant \alpha' - 1$, $u_{i+1}, v_{\beta'}, u_i$ is a bad triple because either $u_i u_{i+1} \in A(G) - S(G)$ or $u_i u_{i+1} \in S(G)$ and $v_{\beta'} u_i \notin S(G)$. Since $w_t u_1 \in A(G) - S(G)$, $u_1, v_{\beta'}, w_t$ is also a bad triple. The case $v_{\beta'} < u_{\alpha'} < w_t$ can be handled symmetrically. Suppose next that $v_{\beta'} < w_t < u_{\alpha'}$. Consider the sequence $v_{\beta'}, v_{\beta'-1}, \dots, v_1, u_{\alpha'}$. Note that $w_t v_i \notin S(G)$ for each $i = 1, 2, \dots, \beta'$. We see as above that $v_{i+1}, w_t, v_i$ is a bad triple for each $i = 1, 2, \dots, \beta' - 1$. Since $w_t v_1 \in A(G) - S(G)$, $v_1, w_t, u_{\alpha'}$ is also a bad triple. The case $u_{\alpha'} < w_t < v_{\beta'}$ is symmetric. Suppose now that $w_t < u_{\alpha'} < v_{\beta'}$. Consider the sequence $w_t, w_{t-1}, \dots, w_\gamma, v_1, \dots, v_{\beta'}$. Noting again that for each $i$, $u_{\alpha'} w_i \notin S(G)$, we see that $w_{i+1}, u_{\alpha'}, w_i$ is a bad triple. Since $w_\gamma v_1 \in A(G)$ and $w_\gamma u_{\alpha'} \notin A(G)$, the triple $w_\gamma, u_{\alpha'}, v_1$ is bad. A similar reasoning shows that $v_i, u_{\alpha'}, v_{i+1}$ is a bad triple for each $i = 1, 2, \dots, \beta' - 1$. Finally, for the case $w_t < v_{\beta'} < u_{\alpha'}$, we can use the sequence $w_t, w_{t-1}, \dots, w_\eta, u_1, \dots, u_{\alpha'}$. We see that each consecutive pair of vertices in the sequence with $v_{\beta'}$ in the middle forms a bad triple. $\square$

The following corollary follows from Proposition 1, Corollary 4, and Proposition 6.

**Corollary 7.** *No chronological interval digraph contains parallel vertices.* $\square$

# 3 Characterizations

The following theorem contains all our characterizations of chronological interval digraphs.

**Theorem 8.** *The following statements concerning a digraph $G$ are equivalent:*

*(i) $G$ is a chronological interval digraph;*

*(ii) $G$ admits a chronological ordering;*

*(iii) $G$ is clustered and contains no asynchronous triple;*

*(iv) $G$ contains no induced $Z$ and no asynchronous triple;*

*(v) $G$ is clustered and contains no parallel vertices.*

**Proof:** The equivalence of $(i)$ and $(ii)$ is established in Proposition 3. The implication $(i) \Rightarrow (iii)$ follows from Proposition 1 and Corollary 4. The implication $(iii) \Rightarrow (iv)$ follows from Corollary 2.

To prove $(iv) \Rightarrow (v)$, suppose that $G$ contains no induced $Z$ and no asynchronous triple. In view of Proposition 6, it suffices to prove that $G$ is clustered. We prove it by showing that if $G$ is not clustered then it contains an asynchronous triple or an induced copy of $Z$. So assume that $G$ is not clustered.

Suppose first that $G$ is not in-semicomplete. Then there are three vertices $x, y, z$ such that $xz, yz \in A(G)$ but $xy \notin A(G)$ and $yx \notin A(G)$. We claim that $x, y, z$ form an asynchronous triple in $G$. Indeed, consider an arbitrary ordering of the three vertices. If $x < y < z$, then for the sequence $x, z$ the triple $x, y, z$ satisfies property $Q_2$ or $Q_3$ and hence is bad. Similarly, if $y < x < z$, then we use the sequence $y, z$. Suppose that $x < z < y$. Then we can use the sequence $x, y$ as $x, z, y$ satisfies property $Q_1$ or $Q_4$. A similar argument applies to the case when $y < z < x$. Suppose that $z < x < y$. The sequence we use is $z, y$ as $z, x, y$ satisfies property $Q_1$ or $Q_2$. The last case $z < y < x$ is similar. Hence we may assume that $G$ is in-semicomplete.

Let $C$ be any strong component of $G$. We show by contradiction that all arcs in $C$ are symmetric. Suppose that $C$ contains an arc that is not symmetric. Let $v_1v_2 \ldots v_kv_1$ be a directed cycle in $C$ where $v_1v_2$ is not symmetric. Such a cycle exists because every arc of $C$ is contained in a directed cycle. We may assume the cycle is chosen to be induced in $C$. We claim that $v_1, v_2, v_3$ form an asynchronous triple, which contradicts the assumption. Indeed, consider an arbitrary ordering of $v_1, v_2, v_3$. Suppose first that $v_1 < v_2 < v_3$. Consider the sequence $v_1, v_k, v_{k-1}, \ldots, v_3$. Then each pair of consecutive vertices with $v_2$ in the middle satisfies property $Q_1$ or $Q_2$. For every other ordering $u < v < w$ of $v_1, v_2, v_3$, we can use the sequence $u, w$ as at least one of properties $Q_1, Q_2$, and $Q_3$ is satisfied. Indeed, if $v_2 < v_1$, then $Q_1$ is satisfied. If $v_3 < v_1 < v_2$, then $Q_1$ or $Q_2$ is satisfied. Finally, if $v_1 < v_3 < v_2$, then $Q_3$ is satisfied.

Since $G$ is in-semicomplete but not clustered, there exist two vertices $u, v$ in the same strong component of $G$ such that $N^-(u) \neq N^-(v)$ or both $N^+(u) \not\subseteq N^+(v)$ and $N^+(v) \not\subseteq N^+(u)$. It follows from above that the strong component can only contain symmetric arcs and hence is complete since $G$ is in-semicomplete. The fact that $C$ is complete and $G$ is in-semicomplete implies $N^-(u) = N^-(v)$. So we have both $N^+(u) \not\subseteq N^+(v)$ and $N^+(v) \not\subseteq N^+(u)$. Let $u' \in N^+(u) - N^+(v)$ and $v' \in N^+(v) - N^+(u)$ be any two vertices. It is now easy to see that $u, v, u', v'$ induce a copy of $Z$ in $G$.

Finally, we prove $(v) \Rightarrow (ii)$. Suppose that $G$ is clustered and contains no parallel vertices. Let $T$ be a canonical out-branching of $G$ rooted at $h$. For each ancestor-descendant pair $(u, v)$ of vertices, define $f(u, v)$ as follows:

$f(u, v) = 0$ if $uv \notin A(G)$ (and $uv' \notin A(G)$ for every descendant $v'$ of $v$),
$f(u, v) = 1$ if $uv \in A(G)$ but $uv' \notin A(G)$ for some descendant $v'$ of $v$, and
$f(u, v) = 2$ if $uv \in A(G)$ and $uv' \in A(G)$ for every descendant $v'$ of $v$.

Note that $f(u, v) \geqslant 1$ if and only if $uv \in A(G)$ and that $f(u, v) \leqslant 1$ if and only $uv' \notin A(G)$ for a descendant $v'$ of $v$.

Let $x, y$ be siblings. Clearly, $x, y$ have the same set $U$ of ancestors. We show that either $f(u, x) \leqslant f(u, y)$ for all $u \in U$ or $f(u, x) \geqslant f(u, y)$ for all $u \in U$. Suppose to the contrary that $f(u, x) < f(u, y)$ and $f(u', x) > f(u', y)$ for some $u, u' \in U$. Then $uy \in A(G)$ and $ux' \notin A(G)$ for some descendant $x'$ of $x$; similarly, $u'x \in A(G)$ and $u'y' \notin A(G)$ for some descendant $y'$ of $y$. This means that $x', y'$ are parallel in $G$, a contradiction to our assumption.

For each vertex $v \neq h$, let $f(v) = \sum f(u,v)$ where the sum is taken over all ancestors $u$ of $v$. For a technical reason as seen in the next section, we also let $f(h) = 0$. It follows from the above that if $x, y$ are siblings then $f(x) \geqslant f(y)$ if and only if $f(u,x) \geqslant f(u,y)$ for all $u \in U$.

The out-branching $T$ may now be viewed as an ordered tree in such a way that, for any two siblings $x, y$, $x$ is left of $y$ if and only if $f(x) \geqslant f(y)$. We can then obtain a vertex ordering $<$ of $G$ by performing a depth first search (DFS) on $T$ by starting at $h$ and always choosing the left-most unvisited child. We show this ordering $<$ is a chronological ordering of $G$ by verifying the properties $P_1 - P_4$.

If $u < v$, then either $u$ is an ancestor of $v$ or the two vertices $u, v$ are in different branches of $T$ and in either case $vu \notin A(G) - S(G)$. If $u < v < w$ and $uw \in S(G)$, then $u, v, w$ are all in the same strong component and hence $uv, vw \in S(G)$. Suppose that $u < v < w$ and $uw \in A(G) - S(G)$. Then $u$ is an ancestor of both $v$ and $w$ but not in the same strong component as $w$. If $v$ is also an ancestor of $w$, then $uv \in A(G)$ and when $uv \in S(G)$, we must have $vw \in A(G) - S(G)$ as $N^+(v) \supseteq N^+(u)$. On the other hand, if $v$ is not an ancestor of $w$, then $v, w$ are in different branches of $T$. Let $x, y$ be the siblings contained respectively in the directed paths in $T$ from $h$ to $u, v$. Since $v < w$, $f(x) \geqslant f(y)$ and hence $f(u,x) \geqslant f(u,y)$. Since $uw \in A(G)$, we must have $uv \in A(G) - S(G)$. Finally suppose that $u < v < w$ and $uw \notin A(G)$. If $vw \in S(G)$, then $N^-(v) = N^-(w)$ as $G$ is clustered. Since $uw \notin A(G)$, $uv \notin A(G)$. Hence $uv \notin A(G)$ or $vw \notin S(G)$. Therefore the ordering $<$ is a chronological ordering. $\square$

To underscore the similarity with interval graphs, we mention corresponding similar characterizations of interval graphs: $G$ is an interval graph if and only if it admits a suitable ordering (corresponds to $(ii)$), if and only if it is chordal and contains no asteroidal triple (cf. $(iii)$), if and only if it contains no induced $C_4$ and no asteroidal triple (cf. $(iv)$), if and only if it is chordal and admits a clique tree representation that is a path (cf. $(v)$).

As noted in the first section, an undirected graph is an interval graph if and only if it admits an orientation that is a chronological interval digraph. Thus each of the above equivalent characterizations gives rise to a new characterization of interval graphs. For instance, we have the following fact.

**Corollary 9.** *A graph $G$ is an interval graph if and only if it has an orientation which is clustered and contains no asynchronous triple.* $\square$

# 4 A linear time algorithm for recognition and representation

The proof of Theorem 8 suggests a linear time algorithm to decide whether a given reflexive digraph is a chronological interval digraph and to find an interval representation if one exists.

Let $G$ be the given reflexive digraph. For the purposes of our algorithm we assume that $G$ is given by adjacency lists. By applying depth first search (DFS) we can find the strong components of $G$, ordered as $C_1, C_2, \ldots, C_p$, so that there is no arc from $C_j$

to $C_i$ if $i < j$, [21]. We then order the vertices inside each strong component according to their outdegrees, so that vertices with lower outdegrees precede vertices with higher outdegrees. (Thus the first vertex in each $C_i$ has the minimum outdegree, and the last vertex has the maximum outdegree.) This sorting can be done in linear time because the sum of degrees is linear with respect to the number of edges. We denote by $h$ the first vertex of $C_1$. For each vertex $v \neq h$, we define the parent $v^-$ of $v$ as follows. Suppose $v \in C_k$, for some $k$. If $v$ is not the first vertex in $C_k$, then $v^-$ is the previous vertex in the ordering of the vertices of $C_k$. Otherwise, if $v$ is the first vertex of $C_k$, then let $j$ be the largest subscript with $j < k$ such that there is an arc from $C_j$ to $C_k$; we let $v^-$ be the last vertex in $C_j$. Let $T$ be the digraph having vertex set $V(G)$ and having arcs $v^-v$. Note that $T$ is a tree rooted at $h$ in which every vertex except $h$ has indegree one; when $G$ is a clustered digraph $T$ is a canonical out-branching of $G$, as defined in Section 2.

Next we show how to compute the value $f(v)$ for each vertex $v$. For distinct vertices $u, v$, call $u$ an *ancestor* of $v$ and $v$ a *descendant* of $u$ if there is a directed path from $u$ to $v$ in $T$. Let $U_v$ be the set of all ancestors $u$ of $v$ such that $uv \in A(G)$. Let $F_v$ be the set of vertices $u \in U_v$ such that there is an arc from $u$ to each descendant of $v$. It is easy to see that the sets $U_v$ can be computed in linear time. Note that for each vertex $v$ that does not have a child, $F_v$ consists of all in-neighbours $u \neq v$ in $G$. Also $F_v = \cap F_x - \{v\}$ where the intersection is taken over all children $x$ of $v$ (i.e., $x^- = v$). Thus the sets $F_v$ can be computed in linear time. (Note that we maintain $U_v, F_v$ as ordered sets.) Let $f(v) = |U_v| + |F_v|$ for each $v$. Again we remark that when $G$ is a chronological interval digraph $f(v)$ is precisely the one defined in the proof of Theorem 8. Hence each vertex $v$ is equipped with value $f(v)$.

We can now proceed with a DFS on $T$ beginning with root $h$ and always choosing an unvisited child $x$ which has the maximum value $f(x)$. This gives us an ordering $v_1 < v_2 < \cdots < v_n$ of the vertices of $G$. In the case when $G$ is a chronological interval digraph, this ordering is a chronological ordering as shown in the previous section. Hence we only need to check for each vertex $v_i$ whether the out-neighbours and the in-neighbours of $v_i$ appear consecutively in the ordering. In particular, if there exist $a_i \leqslant i \leqslant b_i \leqslant c_i$ such that $N^-(v_i) \cap N^+(v_i) = \{v_{a_i}, v_{a_i+1}, \ldots, v_{b_i}\}$ and $N^+(v_i) = \{v_{a_i}, v_{a_i+1}, \ldots, v_{c_i}\}$ for each $i$, then we can find a chronological interval representation of $G$ as described in the proof of Proposition 3; otherwise $G$ is not a chronological interval digraph. It is clear that all these steps can be implemented in linear time. Therefore we have the following fact.

**Theorem 10.** *There is a linear time algorithm which decides whether a given reflexive digraph is a chronological interval digraph and finds an interval representation if it is.* □

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