A note on forbidding clique immersions

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Submitted: July 10, 2012; Accepted: Sep 23, 2013; Published: Oct 7, 2013
Mathematics Subject Classifications: 05C75

Abstract

Robertson and Seymour proved that the relation of graph immersion is well-quasi-ordered for finite graphs. Their proof uses the results of graph minors theory. Surprisingly, there is a very short proof of a corresponding rough structure theorem for graphs without K_t-immersions; it is based on the Gomory-Hu theorem. The same proof also works to establish a rough structure theorem for Eulerian digraphs without $\vec{K}_t$-immersions, where $\vec{K}_t$ denotes the complete digraph of order t.

In this paper all graphs and digraphs are finite and may have loops and multiple edges, unless explicitly stated otherwise.

A pair of distinct adjacent edges uv and vw in a graph are *split off* from their common vertex v by deleting the edges uv and vw, and adding the edge uw (possibly in parallel to an existing edge, and possibly forming a loop if $u = w$). A graph H is said to be *immersed* in a graph G if a graph isomorphic to H can be obtained from a subgraph of

*Supported in part by an NSERC Discovery Grant (Canada) and a Sloan Fellowship.
†Previously supported by an NSERC Postdoctoral Fellowship (Canada).
‡Supported in part by an NSERC Discovery Grant (Canada), by the Canada Research Chair program, and by the Research Grant J1–4106 of ARRS (Slovenia). On leave from: IMFM & FMF, Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia.

G by splitting off pairs of edges (and removing isolated vertices). If H is immersed in a graph G, then we also say that G has an *H-immersion*. An alternative definition is that H is immersed in G if there is a 1-1 function $\phi : V(H) \to V(G)$ such that for each edge $uv \in E(H)$, if $u \neq v$ there is a path P_{uv} in G joining vertices $\phi(u)$ and $\phi(v)$, if $u = v$ there is a cycle C_{uv} in G containing $\phi(u) = \phi(v)$, and the paths P_{uv} and cycles C_{uv} are pairwise edge-disjoint for all $uv \in E(H)$.

Robertson and Seymour [6] proved that the relation of graph immersion is a well-quasi-ordering, that is, for every infinite set of graphs, one of them can be immersed in another one. Their proof is based on a significant part of the graph minors project. It is perhaps surprising then that there is a very short proof of a corresponding rough structure theorem for graphs without K_t-immersions. Moreover, the same proof technique also works to prove a rough structure theorem for Eulerian digraphs without $\vec{K}_t$-immersions. Here, by $\vec{K}_t$ we mean the *complete digraph* of order t, having t vertices and a *digon* (pair of oppositely oriented edges) between each pair of vertices. By immersion in digraphs we mean the natural directed analogue of immersion in undirected graphs. That is, we say that a digraph F is *immersed* in a digraph D if there is a 1-1 function $\phi : V(F) \to V(D)$ such that for each edge $uv \in E(F)$, if $u \neq v$ there is a directed path P_{uv} in D from $\phi(u)$ to $\phi(v)$, if $u = v$ there is a directed cycle C_{uv} in D containing $\phi(u) = \phi(v)$, and the paths P_{uv} and cycles C_{uv} are pairwise edge-disjoint for all $uv \in E(F)$. Given a directed walk uvw of length two in a digraph , the pair of edges uv, vw are *split off* from v by deleting the edges uv and vw, and adding the edge uw.

To state the two rough structure theorems explicitly, we first need the definition of a laminar family of edge-cuts. Given a graph or digraph G and a vertex-set $X \subseteq V(G)$, we denote by $\delta(X)$ the edge-cut of G consisting of all edges between X and $V(G) \setminus X$ (in both directions). Two edge-cuts $\delta(X)$ and $\delta(Y)$ in a connected graph or digraph are *uncrossed* if either X or $V(G) \setminus X$ is contained in either Y or $V(G) \setminus Y$. Two edge-cuts in a general graph or digraph are *uncrossed* if this holds in each component. A family of pairwise uncrossed edge-cuts is called *laminar*. A laminar family of edge-cuts in a graph or digraph induces a partition of the entire vertex set; we call the parts of such a partition *blocks*.

Theorem 1. *For every graph which does not contain an immersion of the complete graph K_t, there exists a laminar family of edge-cuts, each with size $< (t-1)^2$, so that every block of the resulting vertex partition has size less than t.*

Theorem 2. *For every Eulerian digraph which does not contain an immersion of $\vec{K}_t$, there exists a laminar family of edge-cuts, each with size $< 2(t-1)^2$, so that every block of the resulting partition has size less than t.*

We will show that the Gomory-Hu Theorem (stated shortly) yields very easy proofs of Theorems 1 and 2. We originally had somewhat weaker bounds for both of these results. In fact, Theorem 1 as stated above is due to Wollan [7], whose work we learned of later. We were helped by the following observation of Wollan [7], and its directed analog.

Observation 3. *Let H_t be the graph obtained from $K_{1,t-1}$ by replacing each edge with $t-1$ parallel edges. Then H_t has a K_t-immersion.*

Proof. Let v_1 be the vertex of degree $(t-1)^2$ and let $v_2, \dots v_t$ be the vertices of degree $t-1$. Label the $t-1$ edges between v_1 and v_j by $\{e_{j,1}, e_{j,2}, \dots e_{j,t}\} \setminus e_{j,j}$ for $2 \leqslant j \leqslant t$. Then we have an immersion of K_t on the vertices $v_1, \dots, v_t$, where the requisite paths between v_1 and $v_2, \dots v_t$ are given by $e_{2,1}, \dots e_{t,1}$, and for every pair of vertices v_i, v_j with $1 < i < j \leqslant t$, the path between v_i and v_j is given by $e_{i,j}e_{j,i}$. □

Observation 4. *Let $\vec{H}_t$ be the digraph obtained from $K_{1,t-1}$ by replacing each edge with $t-1$ digons. Then $\vec{H}_t$ has a $\vec{K}_t$-immersion.*

Proof. Modify the proof of Observation 3 by labelling digons as opposed to labelling edges. □

The bounds of Theorems 1 and 2 provide rough structure to the same extent. Namely, while the laminar family of Theorem 1 does not prohibit a K_t-immersion, it does indeed prohibit a K_{t^2}-immersion. Analogously, the laminar family of Theorem 2 also prohibits a $\vec{K}_{t^2}$-immersion.

That we restrict ourselves to Eulerian digraphs in Theorem 2 should not be surprising. First of all, let us observe that digraph immersion is not a well-quasi-order in general. Furthermore, the present authors have exhibited in [2] that there exist simple non-Eulerian digraphs with all vertices of arbitrarily high in- and outdegree which do not contain even a $\vec{K}_3$-immersion. (Here, by simple digraph we mean a digraph D with no loops and at most one edge from x to y for any $x, y \in V(D)$, but where digons are allowed). On the positive side, it has been shown by Chudnovsky and Seymour [1] that digraph immersion is a well-quasi-order for tournaments. That Eulerian digraphs of maximum outdegree 2 are well-quasi-ordered by immersion was proved (although not written down) by Thor Johnson as part of his PhD thesis [5].

In [3], the present authors along with Fox and Dvořák, proved that (undirected) simple graphs with minimum degree at least $200t$ contain a K_t-immersion. In another paper [2], the following positive result is obtained for digraphs when the Eulerian condition is added.

Theorem 5. [2] *Every simple Eulerian digraph with minimum outdegree at least $t(t-1)$ contains an immersion of $\vec{K}_t$.*

Theorem 5 is directly implied by Theorem 2, providing an alternate proof of Theorem 5 to that given in [2]. In fact, we obtain a slightly stronger version, as follows.

Theorem 6. *Every simple Eulerian digraph with minimum outdegree at least $(t-1)^2$ contains an immersion of $\vec{K}_t$.*

Proof. Suppose, for a contradiction, that D is a simple Eulerian digraph with minimum outdegree at least $(t-1)^2$ that does not contain an immersion of $\vec{K}_t$. It must be the case that $t \geqslant 3$, so in particular D contains at least t vertices, and Theorem 2 implies that D contains a set $S \subseteq V(D)$ of $|S| = s < t$ vertices with $|\delta(S)| < 2(t-1)^2$. Note that by the minimum degree condition, we know that $s > 1$. Then

$$s(t-1)^2 \leqslant |E[S]| + \tfrac{1}{2}|\delta(S)| < s(s-1) + (t-1)^2$$

which implies that

$$s > (t-1)^2.$$

Since $t \geqslant 3$, this contradicts the fact that $s < t$. □

An edge-cut $\delta(X)$ in a graph G is said to *separate* a pair of vertices $x, y \in V(G)$ if $x \in X$ and $y \in V(G) \setminus X$ (or vice versa). Given a tree F and an edge $e \in E(F)$, there exists $X \subseteq V(F)$ such that $\delta(X) = e$; $\delta(X)$ is called a *fundamental cut* in F and is associated with the vertex partition $\{X, V(F) \setminus X\}$ of $V(F)$.

Theorem 7 (Gomory-Hu [4])**.** *For every graph G, there exists a tree F with vertex set $V(G)$ and a function $\mu : E(F) \to \mathbb{Z}$ so that the following hold.*

- *For every edge $e \in E(F)$ we have that $\mu(e)$ equals the size of the edge-cut of G given by the vertex partition associated with the fundamental cut of e in the tree F.*
- *For every $u, v \in V(G)$ the size of the smallest edge-cut of G separating u and v is the minimum of $\mu(e)$ over all edges e on the path in F from u to v.*

Proof of Theorem 1 (and Theorem 2): Let G be an arbitrary graph (or let D be an arbitrary Eulerian digraph and let G be underlying graph of D). Apply the Gomory-Hu Theorem to G to choose a tree F on $V(G)$ and an associated function μ. Let $\mathcal{C}$ be the family of edge-cuts of G that are associated with edges $e \in E(F)$ for which $\mu(e) < (t-1)^2$ ($\mu(e) < 2(t-1)^2$). We show that if any block of the resulting vertex partition has size $\geqslant t$, than G contains a K_t-immersion (D contains a $\vec{K}_t$-immersion). To this end, suppose we have such a block with distinct vertices $v_1, v_2 \ldots, v_t$. Then every edge-cut separating these t vertices has size $\geqslant (t-1)^2$ ($\geqslant 2(t-1)^2$). We claim that there exist $(t-1)^2$ edge disjoint (directed) paths starting at v_1 so that exactly $t-1$ of them end at each v_j for $2 \leqslant j \leqslant t$. To see this, consider adding an auxiliary vertex w to the graph with $t-1$ parallel edges ($t-1$ digons) between w and each of $v_2, \ldots, v_t$. Then apply Menger's Theorem to v_1 and w, noting that every cut separating v and w contains at least $(t-1)^2$ edges (in each direction). (Moreover, since G is Eulerian, if we delete these directed paths we can apply Menger's Theorem again to get $(t-1)^2$ edge disjoint directed paths ending at v_1 so that exactly $t-1$ of them start at each v_j for $2 \leqslant j \leqslant t$). Hence G immerses the graph H_t (D immerses the digraph $\vec{H}_t$) as in Observation 3 (Observation 4). □

Acknowledgements

The authors are very grateful to Paul Wollan for helpful discussions, and to an anonymous referee, whose comments simultaneously improved the bound of Theorem 2 and simplified its proof.

References

[1] M. Chudnovsky and P. D. Seymour. A well-quasi-order for tournaments. *J. Combin. Theory, Ser. B*, 101: 47–53, 2011.

[2] M. DeVos, J. McDonald, B. Mohar and D. Scheide. Immersing complete digraphs. *Europ. J. Combin.*, 33(6): 1294–1302, 2012.

[3] M. DeVos, Z. Dvořák, J. Fox, J. McDonald, B. Mohar and D. Scheide. A minimum degree condition forcing complete graph immersion. arXiv:1101.2630, 2011. To appear in *Combinatorica*.

[4] R. E. Gomory and T. C. Hu. Multi-terminal network flows. *J. Soc. Indust. Appl. Math.*, 9(4): 551–570, 1961.

[5] T. Johnson. Eulerian Digraph Immersion. Ph. D. thesis, Princeton University, 2002.

[6] N. Robertson and P. D. Seymour. Graph Minors XXIII, Nash-Williams' immersion conjecture. *J. Combin. Theory, Ser. B*, 100: 181–205, 2010.

[7] P. Wollan. The structure of graphs not admitting a fixed immersion. arXiv:1302.3867, 2013.