On the characteristic polynomial of n-Cayley digraphs

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Abstract

A digraph Γ is called n-Cayley digraph over a group G, if there exists a semiregular subgroup R_G of $\mathrm{Aut}(\Gamma)$ isomorphic to G with n orbits. In this paper, we represent the adjacency matrix of Γ as a diagonal block matrix in terms of irreducible representations of G and determine its characteristic polynomial. As corollaries of this result we find: the spectrum of semi-Cayley graphs over abelian groups, a relation between the characteristic polynomial of an n-Cayley graph and its complement, and the spectrum of Cayley graphs over groups having cyclic subgroups. Finally we determine the eigenspace of n-Cayley digraphs and their main eigenvalues.

Keywords: n-Cayley digraph; linear representations of groups; characteristic polynomial

1 Introduction

Graphs come in two principle types: directed graphs and undirected graphs. We shall refer to directed graphs as digraphs and use the term graph to refer to undirected graphs. A digraph Γ is a pair (V, E) of vertices V and edges E where $E \subseteq V \times V$; the digraph Γ is said to be finite if V is finite. A graph is a digraph with no edges of the form (α, α) and with the property that $(\alpha, \beta) \in E$ implies $(\beta, \alpha) \in E$. The set of all permutations of V which preserve the adjacency structure of Γ is called the automorphism group of Γ; it is denoted by $\mathrm{Aut}(\Gamma)$. In this paper all digraphs have no loops. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [9, 2].

Let G be a group and S a subset of G not containing the identity element 1. Recall that the Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ of G with respect to S has vertex set G and edge set $\{(g, sg) \mid g \in G, s \in S\}$. If $S = S^{-1}$, then $\mathrm{Cay}(G, S)$ can be viewed as an undirected graph, identifying an undirected edge with two directed edges (g, h) and (h, g). This graph is called Cayley graph of G with respect to S. By a theorem of Sabidussi [8], a digraph Γ is a Cayley digraph over a group G if and only if there exists a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. There is a natural generalization of the Sabidussi's Theorem. A digraph Γ is called an n-Cayley digraph over a group G if there exists an n-orbit semiregular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. 2-Cayley graphs are called by some authors semi-Cayley graph, see for example [3], and also bi-Cayley graph [5]. Also a special case of 2-Cayley graphs are also called bi-Cayley graph by some authors, see for example [10].

The spectrum of a finite digraph Γ is the spectrum of its adjacency matrix A, that is, the set of eigenvalues together with their multiplicities. The characteristic polynomial of Γ is the characteristic polynomial of A, that is the polynomial defined by $\chi_A(\lambda) = \det(\lambda I - A)$. It is known that numerous proofs in graph theory depend on the spectrum of graphs and the spectrum of a graph is one of the most important algebraic invariants. The basic relationships between algebraic properties of these eigenvalues and the usual properties of graphs are available in [2].

Lovász [7] showed that determination of the spectrum of a graph with transitive automorphism group can be reduced to the same task for some Cayley graph and found a formula for certain power sums of the eigenvalues in terms of irreducible characters of automorphism group of the graph. Babai [1] succeeded in simplifying Lovasz' formula for the power sums of the eigenvalues of Cayley graphs.

Spectrum of 2-Cayley graphs over abelian groups is computed by Gao and Luo in [3] using matrix theory. It seems that their arguments cannot be extended to non-abelian groups or n-Cayaley graphs, $n \geqslant 3$. In this paper we find a factorization of characteristic polynomial of n-Cayley digraphs over an arbitrary group in terms of linear representations of the group. We prove that every Cayley graph over a group having a subgroup of index n can be regarded as an n-Cayley graph; and compute the spectrum of Caylay graphs over groups having a cyclic subgroup of index 2. Finally we find the eigenvectors of n-Cayley (di)graphs.

2 Main Results

In this paper all vector spaces are over the complex field $\mathbb{C}$ and have finite dimension. Throughout the paper G denotes a finite group. We denote by $\mathbb{C}[G]$, the complex vector space of dimension $|G|$. Let $\mathscr{A} = \{e_g \mid g \in G\}$ be an arbitrary basis of $\mathbb{C}[G]$. We identify $\mathbb{C}[G]$ with the vector space of all complex-valued functions on G. Thus a function $\varphi : G \to \mathbb{C}$ corresponds to the vector $\varphi = \sum_{g\in G} \varphi(g)e_g$ and vice versa. In particular, the vector e_g of the standard basis $\mathscr{A}$ corresponds to a function, also denoted by e_g, where

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

The (left) regular representation ρ_{reg} of G on $\mathbb{C}[G]$ is defined by its action on the basis $\{e_h \mid h \in G\}$; that is for all $g, h \in G$, $\rho_{\mathrm{reg}}(g)e_h = e_{gh}$. The regular representation has degree $|G|$.

Let $\mathrm{Irr}(G) = \{\rho_1, \ldots, \rho_m\}$ be the set of all irreducible inequivalent $\mathbb{C}$-representations of G. Let d_k and $\varrho^{(k)}$ be the degree and a unitary matrix representation of ρ_k, $k = 1, \ldots, m$, respectively. We keep these notations throughout the paper.

In the following lemma, which seems to be well-known, we construct an orthogonal basis for $\mathbb{C}[G]$ using the matrix representations $\varrho^{(k)}$, $1 \leqslant k \leqslant m$.

Lemma 1. *Let $\varrho_{ij}^{(k)}(g)$ be the ijth entry of $\varrho^{(k)}(g)$, $1 \leqslant i, j \leqslant d_k$, and $\bar{\varrho}_{ij}^{(k)} = \sum_{g \in G} \overline{\varrho_{ij}^{(k)}(g)} e_g$, where $\overline{z}$ denotes the complex conjugate of a complex number z. Then*

(i) $\{\bar{\varrho}_{ij}^{(k)} \mid 1 \leqslant k \leqslant m, 1 \leqslant i, j \leqslant d_k\}$ form an orthogonal basis for $\mathbb{C}[G]$,

(ii) $\rho_{\mathrm{reg}}(g)\bar{\varrho}_{ij}^{(k)} = \sum_{l=1}^{d_k} \varrho_{li}^{(k)}(g)\bar{\varrho}_{lj}^{(k)}$, for all $g \in G$ and $1 \leqslant i, j \leqslant d_k$, $1 \leqslant k \leqslant m$,

(iii) $\mathbb{C}[G] = \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{d_k} W_j^{(k)}$, where $W_j^{(k)} = \langle \bar{\varrho}_{ij}^{(k)} \mid 1 \leqslant i \leqslant d_k \rangle$ which is a ρ_{reg}-invariant subspace of $\mathbb{C}[G]$ of dimension d_k.

Proof. Let $1 \leqslant k, k' \leqslant m$. Then by [9, Corollaries 2, 3, p. 14],

$$\frac{1}{|G|} \sum_{g \in G} \varrho_{ij}^{(k)}(g) \varrho_{i'j'}^{(k')}(g^{-1}) = \frac{\delta_{ij'}\delta_{ji'}\delta_{kk'}}{d_k}, \tag{1}$$

for all $1 \leqslant i, j \leqslant d_k$ and $1 \leqslant i', j' \leqslant d_{k'}$. On the other hand the matrices $\varrho^{(k)}(g)$ are unitary for all $g \in G$ and $1 \leqslant k \leqslant m$ and so $\varrho_{ij}^{(k)}(g)^{-1} = \overline{\varrho_{ji}^{(k)}(g)}$. Hence (1) yields that

$$\frac{1}{|G|} \sum_{g \in G} \varrho_{ij}^{(k)}(g) \overline{\varrho_{j'i'}^{(k')}(g)} = \frac{\delta_{ij'}\delta_{ji'}\delta_{kk'}}{d_k}.$$

Now

$$\langle \bar{\varrho}_{ij}^{(k)}, \bar{\varrho}_{j'i'}^{(k')} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varrho_{ij}^{(k)}(g)} \varrho_{j'i'}^{(k')}(g) = \frac{\delta_{ij'}\delta_{ji'}\delta_{kk'}}{d_k},$$

which shows that $\bar{\varrho}_{ij}^{(k)}, \bar{\varrho}_{j'i'}^{(k')}$ are mutually orthogonal vectors of $\mathbb{C}[G]$ and hence independent. On the other hand the cardinality of $\{\bar{\varrho}_{ij}^{(k)} \mid 1 \leqslant k \leqslant m, 1 \leqslant i, j \leqslant d_k\}$ is $\sum_{k=1}^{m} d_k^2$ which is equal to $|G|$, by [9, Corollary 2, p. 18]. Since the dimension of $\mathbb{C}[G]$ as a $\mathbb{C}$-vector space is $|G|$, we conclude that $\{\bar{\varrho}_{ij}^{(k)} \mid 1 \leqslant k \leqslant m, 1 \leqslant i, j \leqslant d_k\}$ is a basis for $\mathbb{C}[G]$. This proves (i).

Now we prove (ii). First note that for every $\varphi \in \mathbb{C}[G]$, and $g, h \in G$ we have $\rho_{\mathrm{reg}}(g)\varphi(h) = \varphi(g^{-1}h)$, since

$$\begin{aligned}
\rho_{\mathrm{reg}}(g)\varphi(h) &= \rho_{\mathrm{reg}}(g) \sum_{x \in G} \varphi(x) e_x(h) \\
&= \sum_{x \in G} \varphi(x) \rho_{\mathrm{reg}}(g) e_x(h)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in G}\varphi(x)e_{gx}(h)\\
&= \varphi(g^{-1}h).
\end{aligned}
$$

Hence for each $g,h\in G$ we have (here we denote by $[A]_{ij}$ the ijth entry of a matrix A)

$$
\begin{aligned}
\rho_{\mathrm{reg}}(g)\bar{\varrho}_{ij}^{(k)}(h) &= \bar{\varrho}_{ij}^{(k)}(g^{-1}h)\\
&= [\bar{\varrho}^{(k)}(g)^{-1}\bar{\varrho}^{(k)}(h)]_{ij}\\
&= \sum_{l=1}^{d_k}\varrho_{li}^{(k)}(g)\bar{\varrho}_{lj}^{(k)}(h) \quad (\text{ since } \varrho^{(k)}(g) \text{ is a unitary matrix})
\end{aligned}
$$

as desired. The statement (iii) is an immediate consequence of (i) and (ii). □

If a group G acts on a set Ω and $\alpha\in\Omega$, we denote by α^G and G_α, the orbit of G on Ω with representative α and the stabilizer of α in G, respectively. Now we will find a useful decomposition of the characteristic polynomial of n-Cayley (di)graphs. First we state a very useful and well known equivalent definition of n-Cayley (di)graphs.

Lemma 2. *A digraph Γ is n-Cayley digraph over G if and only if there exist subsets T_{ij} of G, where $1\leqslant i,j\leqslant n$, such that Γ is isomorphic to a digraph X with*

$$
V(X)=G\times\{1,2,\ldots,n\},\quad E(X)=\bigcup_{1\leqslant i,j\leqslant n}\big\{((g,i),(tg,j))\mid g\in G \text{ and } t\in T_{ij}\big\}. \tag{2}
$$

Proof. " $\Rightarrow$ " Let α_i^G, $1\leqslant i\leqslant n$, be the orbits of G on $V(\Gamma)$. Let $T_{ij}:=\{g\in G \mid (\alpha_i,\alpha_j^g)\in E(\Gamma)\}$, $1\leqslant i,j\leqslant n$, and $\varphi: V(\Gamma)\longrightarrow V(X)$, where $\alpha_i^g\mapsto(g,i)$ and X is defined in (2). We show that φ is a digraph isomorphism. Since $V(\Gamma)=\bigcup_{i=1}^n\alpha_i^G$, every vertex $v\in V(\Gamma)$ is of the form α_i^g for some $g\in G$ and $1\leqslant i\leqslant n$. Also

$$
\alpha_i^g=\alpha_j^h \Longleftrightarrow i=j,\ gh^{-1}\in G_{\alpha_i}(=1)\Longleftrightarrow g=h,\ i=j.
$$

Hence φ is well-defined and one to one. Clearly φ is onto. Moreover

$$
\begin{aligned}
(\alpha_i^g,\alpha_j^h)\in E(\Gamma) &\Longleftrightarrow (\alpha_i,\alpha_j^{hg^{-1}})\in E(\Gamma)\\
&\Longleftrightarrow hg^{-1}\in T_{ij}\\
&\Longleftrightarrow ((g,i),(h,j))\in E(X)\\
&\Longleftrightarrow ((\alpha_i^g)^\varphi,(\alpha_j^h)^\varphi)\in E(X).
\end{aligned}
$$

Thus φ is a digraph isomorphism as desired.
" $\Leftarrow$ " Let X be the digraph defined in (2). We show that X is an n-Cayley digraph over G. Consider the action of G on $V(X)$ as $(g,i)^h:=(gh,i)$. Let φ be the corresponding permutation representation. Then $G^\varphi\leqslant\mathrm{Sym(V(X))}$. Let $(g,i),(h,j)\in V(X)$. Then for each $a\in G$, we have

$$
((g,i),(h,j))\in E(X) \quad\Longleftrightarrow\quad hg^{-1}\in T_{ij}
$$

$$\begin{aligned}&\iff (ha)(ga)^{-1} \in T_{ij}\\ &\iff ((ga,i),(ha,j)) \in E(X)\\ &\iff ((g,i)^{a^\varphi},(h,j)^{a^\varphi}) \in E(X).\end{aligned}$$

Therefore $G^\varphi \leqslant \mathrm{Aut}(X)$. Thus $G \cong G^\varphi \leqslant \mathrm{Aut}(X)$ and also G^φ acts on $V(X)$ semiregularly. Note that $V(X) = \bigcup_{i=1}^n (g,i)^{G^\varphi}$ is a partition of $V(X)$ to G^φ-orbits for any $g \in G$. Thus the proof is complete. $\square$

By Lemma 2, an n-Cayley (di)graph is characterized by a group G and n^2 subsets T_{ij} of G (some subsets may be empty). So we denote an n-Cayley (di)graph with respect to n^2 subsets T_{ij} by $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leqslant i,j \leqslant n)$. Note that this representation is not unique. By Lemma 2, $V(\Gamma) = G \times \{1,\dots,n\}$, $(g,i) \sim (h,j)$ if and only if $hg^{-1} \in T_{ij}$ and Γ is undirected if and only if for all $1 \leqslant i,j \leqslant n$, $T_{ij} = T_{ji}^{-1}$. Note also that Γ is a (di)graph without loops if and only if $T_{ii} \subseteq G \setminus \{1\}$, for all $1 \leqslant i \leqslant n$. Let $A = [a_{(g,i)(h,j)}]_{g,h\in G, 1\leqslant i,j\leqslant n}$ be the adjacency matrix of Γ. For a $1 \times m$ vector v and $1 \leqslant i \leqslant n$, we define v^i to be a $1 \times nm$ vector with n blocks, whose the ith block is v and other blocks are $0_{1\times m}$. Let e_g^i be the $1 \times n|G|$ vector with n blocks, where ith block is e_g, as defined, and other blocks are $0_{1\times|G|}$ vectors. Let V be the vector space with basis $\{e_g^i \mid g \in G, 1 \leqslant i \leqslant n\}$. Clearly $V \cong \underbrace{\mathbb{C}[G] \oplus \mathbb{C}[G] \oplus \cdots \oplus \mathbb{C}[G]}_{n-\text{times}}$, as $\mathbb{C}[G] = \langle e_g \mid g \in G\rangle$. So $\dim_{\mathbb{C}} V = n \dim_{\mathbb{C}} \mathbb{C}[G] = n|G|$. Hence we can view A as the linear map

$$\begin{aligned}A : V &\to V\\ e_g^i &\mapsto \sum_{j=1}^n \sum_{h\in G} a_{(h,j)(g,i)} e_h^j, \qquad 1 \leqslant i \leqslant n,\ g \in G.\end{aligned}$$

For an element $g \in G$, we define $\widehat{\rho}_{\mathrm{reg}}(g) : V \to V$ with

$$e_h^i \mapsto e_{gh}^i, \qquad 1 \leqslant i \leqslant n,\ h \in G.$$

Then $g \mapsto \widehat{\rho}_{\mathrm{reg}}(g)$ induces a representation $\widehat{\rho}_{\mathrm{reg}} : G \to GL(V)$.

In the following lemma, we find a relation between A and $\widehat{\rho}_{\mathrm{reg}}$.

Lemma 3. *Let A be the adjacency matrix of the digraph $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leqslant i,j \leqslant n)$. For all $1 \leqslant i \leqslant n$ and $g \in G$, we have $Ae_g^i = \sum_{j=1}^n \sum_{t\in T_{ij}} \widehat{\rho}_{\mathrm{reg}}(t) e_g^j$ (with the convention $\sum_{t\in T_{ij}} \widehat{\rho}_{\mathrm{reg}}(t) = 0$ if $T_{ij} = \emptyset$).*

Proof. We have $a_{(h,j)(g,i)} = 1$ if and only if $hg^{-1} \in T_{ij}$. So

$$\begin{aligned}Ae_g^i &= \sum_{j=1}^n \sum_{h\in G} a_{(h,j)(g,i)} e_h^j\\ &= \sum_{j=1}^n \sum_{\substack{h=tg\\ t\in T_{ij}}} e_h^j\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}\sum_{t\in T_{ij}} e^{j}_{tg} \\
&= \sum_{j=1}^{n}\sum_{t\in T_{ij}} \widehat{\rho}_{\mathrm{reg}}(t)e^{j}_{g}
\end{aligned}
$$

as desired. $\square$

For $i = 1, \ldots, n$, let V_i be a vector space with basis $\{e^i_g \mid g \in G\}$. Then for all $h \in G$, $\widehat{\rho}_{\mathrm{reg}}(h)e^i_g = e^i_{hg} \in V_i$. Thus V_i is a $\widehat{\rho}_{\mathrm{reg}}$-invariant subspace of V. Furthermore V_i, $1 \leqslant i \leqslant n$, is isomorphic to $\mathbb{C}[G]$ as a $\mathbb{C}$-vector space, $V = \bigoplus_{i=1}^{n} V_i$ and $\widehat{\rho}_{\mathrm{reg}} = \bigoplus_{i=1}^{n} \widehat{\rho}_{\mathrm{reg}}|_{V_i}$. Note that we can identify V_i with $\mathbb{C}[G]$ and $\widehat{\rho}_{\mathrm{reg}}|_{V_i}$ with ρ_{reg}.
If we denote $\widehat{\rho}_{\mathrm{reg}}|_{V_k}$ by $\widehat{\rho}_{\mathrm{reg},k}$, then by part (ii) of Lemma 1 we have the following result.

Lemma 4. *Let $\varphi_1, \ldots, \varphi_m$ be all inequivalent $\mathbb{C}$-irreducible representations of G with degrees $d_1, \ldots, d_m$, respectively. Let $\varrho^{(l)}$ be a unitary matrix representation of φ_l and $\bar{\varrho}^{(l),k}_{ij} := \sum_{g\in G} \overline{\varrho^{(l)}_{ij}(g)}e^k_g$, $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant m$ and $1 \leqslant i, j \leqslant d_l$. Then $\widehat{\rho}_{\mathrm{reg},k}(g)\bar{\varrho}^{(l),k}_{ij} = \sum_{r=1}^{d_l} \varrho^{(l)}_{ri}(g)\bar{\varrho}^{(l),k}_{rj}$.*

Proof. First note that $\bar{\varrho}^{(l),k}_{ij} = \sum_{g\in G} \overline{\varrho^{(l)}_{ij}(g)}e^k_g = \left(\sum_{g\in G} \overline{\varrho^{(l)}_{ij}(g)}e_g\right)^k = \left(\bar{\varrho}^{(l)}_{ij}\right)^k$, where $\bar{\varrho}^{(l)}_{ij}$ is defined in Lemma 1. Now we have

$$
\begin{aligned}
\widehat{\rho}_{\mathrm{reg},k}(g)\bar{\varrho}^{(l),k}_{ij} &= \widehat{\rho}_{\mathrm{reg},k}(g)\left(\bar{\varrho}^{(l)}_{ij}\right)^k \\
&= \left(\rho_{\mathrm{reg}}(g)\bar{\varrho}^{(l)}_{ij}\right)^k \\
&= \left(\sum_{r=1}^{d_l} \varrho^{(l)}_{ri}(g)\bar{\varrho}^{(l)}_{rj}\right)^k \\
&= \sum_{r=1}^{d_l} \varrho^{(l)}_{ri}(g)\bar{\varrho}^{(l),k}_{rj} \quad \text{(by Lemma 1}(ii)\text{)}
\end{aligned}
$$

as desired. $\square$

For the rest of this section we keep the notations of Lemma 4. Using the notations of this lemma we have the following corollary.

Corollary 5. *Let A be the adjacency matrix of digraph $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leqslant i, j \leqslant n)$. Then $A\bar{\varrho}^{(l),k}_{ij} = \sum_{s=1}^{n}\sum_{t\in T_{ks}}\sum_{r=1}^{d_l} \varrho^{(l)}_{ri}(t)\bar{\varrho}^{(l),s}_{rj}$.*

Proof. We have

$$
A\bar{\varrho}^{(l),k}_{ij} = A\left(\sum_{g\in G} \overline{\varrho^{(l)}_{ij}(g)}e^k_g\right)
$$

$$
\begin{aligned}
&= \sum_{g\in G} \overline{\varrho_{ij}^{(l)}(g)} A e_g^k \\
&= \sum_{g\in G} \overline{\varrho_{ij}^{(l)}(g)} \sum_{s=1}^{n} \sum_{t\in T_{ks}} \widehat{\rho}_{\mathrm{reg}}(t) e_g^s \quad \text{(by Lemma 3)} \\
&= \sum_{s=1}^{n} \sum_{t\in T_{ks}} \sum_{g\in G} \overline{\varrho_{ij}^{(l)}(g)} \widehat{\rho}_{\mathrm{reg}}(t) e_g^s \\
&= \sum_{s=1}^{n} \sum_{t\in T_{ks}} \sum_{g\in G} \widehat{\rho}_{\mathrm{reg}}(t) \left(\overline{\varrho_{ij}^{(l)}(g)} e_g^s \right) \\
&= \sum_{s=1}^{n} \sum_{t\in T_{ks}} \sum_{g\in G} \widehat{\rho}_{\mathrm{reg},s}(t) \left(\overline{\varrho_{ij}^{(l)}(g)} e_g^s \right) \\
&= \sum_{s=1}^{n} \sum_{t\in T_{ks}} \widehat{\rho}_{\mathrm{reg},s}(t) \left(\sum_{g\in G} \overline{\varrho_{ij}^{(l)}(g)} e_g^s \right) \\
&= \sum_{s=1}^{n} \sum_{t\in T_{ks}} \widehat{\rho}_{\mathrm{reg},s}(t) \bar{\varrho}_{ij}^{(l),s} \\
&= \sum_{s=1}^{n} \sum_{t\in T_{ks}} \sum_{r=1}^{d_l} \varrho_{ri}^{(l)}(t) \bar{\varrho}_{rj}^{(l),s} \quad \text{(by Lemma 4)}
\end{aligned}
$$

as desired. □

Now we are ready to prove the main result of the paper.

Theorem 6. *Let $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leqslant i, j \leqslant n)$ be an n-Cayley digraph over a finite group G. For each $l \in \{1, \ldots, m\}$, we define $nd_l \times nd_l$ block matrix $A_l := \left[A_{ij}^{(l)}\right]$, where $A_{ij}^{(l)} = \sum_{t\in T_{ji}} \varrho^{(l)}(t)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, respectively. Then $\chi_A(\lambda) = \Pi_{l=1}^{m} \chi_{A_l}(\lambda)^{d_l}$.*

Proof. By the notations of Lemma 4 and using Lemma 1, if we put

$$
\mathscr{B}_j^{(l),k} = \left\{ \bar{\varrho}_{ij}^{(l),k} \mid 1 \leqslant i \leqslant d_l \right\} \quad \text{and} \quad V_j^{(l),k} = \left\langle \mathscr{B}_j^{(l),k} \right\rangle,
$$

then $\mathscr{B}^k = \bigcup_{l=1}^{m} \bigcup_{j=1}^{d_l} \mathscr{B}_j^{(l),k}$ is a basis for $V_k = \bigoplus_{l=1}^{m} \bigoplus_{j=1}^{d_l} V_j^{(l),k}$. So

$$
V = \bigoplus_{k=1}^{n} \bigoplus_{l=1}^{m} \bigoplus_{j=1}^{d_l} V_j^{(l),k} = \bigoplus_{l=1}^{m} \bigoplus_{j=1}^{d_l} \bigoplus_{k=1}^{n} V_j^{(l),k}
$$

and $\mathscr{B} = \{\mathscr{B}^k \mid 1 \leqslant k \leqslant n\}$ is a basis for V. Now put $\mathscr{C}_j^{(l)} = \left\{ \bar{\varrho}_{ij}^{(l),k} \mid 1 \leqslant k \leqslant n, 1 \leqslant i \leqslant d_l \right\}$ and $V_j^{(l)} = \left\langle \mathscr{C}_j^{(l)} \right\rangle$. Then $V_j^{(l)} = \bigoplus_{k=1}^{n} V_j^{(l),k}$. Hence $V = \bigoplus_{l=1}^{m} \bigoplus_{j=1}^{d_l} V_j^{(l)}$.

On the other hand, by Corollary 5, $V_j^{(l)}$ is A-invariant subspace of V and $\left[A|_{V_j^{(l)}}\right]_{\mathscr{C}_j^{(l)}} = \left[A|_{V_{j'}^{(l)}}\right]_{\mathscr{C}_{j'}^{(l)}}$ for all $1 \leqslant j, j' \leqslant d_l$, where $[T]_{\mathscr{B}}$ is the matrix of a linear transformation T with respect to the basis $\mathscr{B}$. So by the primary decomposition Theorem, $[A]_{\mathscr{B}} = \mathrm{diag}(I_{d_1} \otimes A_1, I_{d_2} \otimes A_2, \ldots, I_{d_m} \otimes A_m)$, where $A_l = \left[A|_{V_1^{(l)}}\right]_{\mathscr{C}_1^{(l)}}$, $1 \leqslant l \leqslant m$. We consider the ordering

$$\varrho_{1j}^{(l),1}, \varrho_{2j}^{(l),1}, \ldots, \varrho_{d_lj}^{(l),1}, \varrho_{1j}^{(l),2}, \ldots, \varrho_{d_lj}^{(l),2}, \ldots, \varrho_{1j}^{(l),n}, \ldots, \varrho_{d_lj}^{(l),n}$$

for the elements of $\mathscr{C}_j^{(l)}$. Now by Corollary 5, $A_l = \left[A_{ij}^{(l)}\right]$ is $nd_l \times nd_l$ block matrix with $d_l \times d_l$ blocks $A_{ij}^{(l)}$, $1 \leqslant i, j \leqslant n$, where $A_{ij}^{(l)} = \sum_{t \in T_{ji}} \varrho^{(l)}(t)$. Now the result is clear. $\square$

Let $K_{r,r,\ldots,r}$ be the n-partite complete graph. The complement of this graph consists n components isomorphic with the complete graph K_r. By [2, p. 20], we have

$$\chi_{K_{r,r,\ldots,r}}(\lambda) = \lambda^{n(r-1)}(\lambda + r(1-n))(\lambda + r)^{n-1}.$$

Now we derive the latter formula using Theorem 6: To see this, suppose that G is a finite group of order r and $\Gamma = \mathrm{Cay}(G; T_{ij} \mid 1 \leqslant i, j \leqslant n)$ where $T_{ii} = \emptyset$ and $T_{ij} = G$, for $i \neq j$. Let $\varrho^{(1)}, \varrho^{(2)}, \ldots, \varrho^{(m)}$ be all inequivalent unitary irreducible representations of G. Let d_i and η_i be the degree and character of $\varrho^{(i)}$, $1 \leqslant i \leqslant m$, respectively. Also let η_1 be the trivial character. By Theorem 6, $\chi_A(\lambda) = \Pi_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$, where $A_l = [A_{ij}^{(l)}]_{1 \leqslant i,j \leqslant n}$ and $A_{ij}^{(l)} = \sum_{t \in T_{ji}} \varrho^{(l)}(t)$. By our convention $A_{ii}^{(l)} = 0$ (since $T_{ii} = \emptyset$). Now let $i \neq j$. Then for every $g \in G$, we have

$$\varrho^{(l)}(g) A_{ij}^{(l)} \varrho^{(l)}(g)^{-1} = \sum_{t \in G} \varrho^{(l)}(g)\varrho^{(l)}(t)\varrho^{(l)}(g^{-1}) = \sum_{t \in G} \varrho^{(l)}(gtg^{-1}) = \sum_{t \in G} \varrho^{(l)}(t) = A_{ij}^{(l)}.$$

Thus by Schur's Lemma, we have $A_{ij}^{(l)} = \frac{\sum_{g \in G} \eta_l(g)}{d_l} I_{d_l}$. Put $x_l = \frac{\sum_{g \in G} \eta_l(g)}{d_l}$. Then $A_l = B_l \otimes I_{d_l}$, where B_l is a circulant $n \times n$ matrix with first row $[0, x_l, x_l, \ldots, x_l]$. So by [2, p. 16], $\chi_{B_l}(\lambda) = (\lambda - (n-1)x_l)(\lambda + x_l)^{n-1}$ and therefore $\chi_{A_l}(\lambda) = (\lambda - (n-1)x_l)^{d_l}(\lambda + x_l)^{(n-1)d_l}$. On the other hand, $x_1 = |G|$ and $x_l = 0$ for $l \neq 1$. So

$$\begin{aligned}
\chi_A(\lambda) &= \Pi_{l=1}^m \Big((\lambda - (n-1)x_l)^{d_l^2} (\lambda + x_l)^{(n-1)d_l^2} \Big) \\
&= (\lambda - (n-1)|G|)(\lambda + |G|)^{n-1} \Pi_{l=2}^m (\lambda^{d_l^2} \lambda^{(n-1)d_l^2}) \\
&= (\lambda + (1-n)|G|)(\lambda + |G|)^{n-1} \lambda^{n \sum_{l=2}^m d_l^2} \\
&= (\lambda + (1-n)|G|)(\lambda + |G|)^{n-1} \lambda^{n(|G|-1)}.
\end{aligned}$$

Replacing $|G| = r$, we get $\Gamma \cong K_{r,r,\ldots,r}$ and now the result is clear.

In what follows we present some applications of Theorem 6. It is well-known that the diameter of any connected graph is less than the number of distinct eigenvalues of its adjacency matrix (See [2], Corollary 2.7). Thus by Theorem 6, if Γ is a connected n-Cayley graph over a finite group G, then the diameter of Γ is less than $n\sum_{i=1}^{m} d_i$, where $d_1, \ldots, d_m$ are character degrees of G. In particular the diameter of any Cayley graph over G is less than sum of character degrees of G.

Since an n-Cayley (di)graph over a group G is Cayley (di)graph over G if and only if $n = 1$, the following corollary is a direct consequence of Theorem 6.

Corollary 7. *(See [4, Corollary 5.3]) Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph over a finite group G with irreducible unitary matrix representations $\varrho^{(1)}, \ldots, \varrho^{(m)}$. Let d_l be the degree of $\varrho^{(l)}$. For each $l \in \{1, \ldots, m\}$, define a $d_l \times d_l$ block matrix $A_l := \left[A_S^{(l)}\right]$, where $A_S^{(l)} = \sum_{s\in S} \varrho^{(l)}(s)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, the adjacency matrix of Γ, respectively. Then $\chi_A(\lambda) = \Pi_{l=1}^{m}\chi_{A_l}(\lambda)^{d_l}$.*

Since all irreducible characters of an abelian group have degree 1, Theorem 6 can be applied easily to compute the spectrum of n-Cayley (di)graphs over abelian groups. Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leqslant i, j \leqslant n)$, be an n-Cayley (di)graph over a finite abelian group G of order m with irreducible characters $\eta_1, \ldots, \eta_m$. By Theorem 6, $\chi_A(\lambda) = \Pi_{l=1}^{m}\chi_{A_l}(\lambda)$, where $A_l = \left[\sum_{t\in T_{ji}} \eta_l(t)\right]_{1\leqslant i,j\leqslant n}$, which generalizes Corollary 3.3 of [5]. In particular if $n = 2$, then $\chi_A(\lambda) = \Pi_{l=1}^{m}(\lambda - \lambda_+^{(l)})(\lambda - \lambda_-^{(l)})$, where

$$\lambda_\pm^{(l)} = \frac{c_{11}^{(l)} + c_{22}^{(l)} \pm \sqrt{(c_{11}^{(l)} - c_{22}^{(l)})^2 + 4c_{12}^{(l)}c_{21}^{(l)}}}{2}$$

and $c_{ij}^{(l)} = \sum_{t\in T_{ji}} \eta_l(t)$, which generalizes the main result of [3].

A Cayley digraph over a group with a subgroup of index n is an n-Cayley digraph, as the following result shows.

Lemma 8. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley (di)graph. Suppose that there exists a subgroup H of G with index n. If $\{t_1, \ldots, t_n\}$ is a left transversal to H in G, then $\Gamma \cong \mathrm{Cay}(H, T_{ij} \mid 1 \leqslant i, j \leqslant n)$, where $T_{ij} = \{h \in H \mid t_j^{-1}ht_i \in S\} = H \cap t_jSt_i^{-1}$.*

Proof. Let $\Sigma = \mathrm{Cay}(H, T_{ij} \mid 1 \leqslant i, j \leqslant n)$. Since $\{t_1, \ldots, t_n\}$ is a left transversal to H in G, every element of G is uniquely expressible in the form t_ih with $h \in H$ and $1 \leqslant i \leqslant n$. Define $\psi : G \to H \times \{1, \ldots, n\}$ where $(t_ih)^\psi = (h, i)$. Clearly ψ is a bijection from $V(\Gamma)$ to $V(\Sigma)$. Now $(t_ih_1, t_jh_2) \in E(\Gamma) \iff t_jh_2h_1^{-1}t_i \in S \iff h_2h_1^{-1} \in T_{ij} \iff ((h_1, i), (h_2, j)) \in E(\Sigma)$. Hence ψ is a (di)graph isomorphism from Γ to Σ. □

Corollary 9. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley digraph, $H = \langle a\rangle$ a cyclic subgroup of G of order n and of index 2 with left transversal $\{t_1, t_2\}$. Then the characteristic polynomial of the adjacency matrix of Γ is $\chi_A(\lambda) = \Pi_{k=0}^{n-1}(\lambda - \lambda_k^+)(\lambda - \lambda_k^-)$, where*

$$\lambda_k^+ = \frac{\lambda_k^{11} + \lambda_k^{22} + \sqrt{(\lambda_k^{11} - \lambda_k^{22})^2 + 4\lambda_k^{12}\lambda_k^{21}}}{2},\ \ \lambda_k^- = \frac{\lambda_k^{11} + \lambda_k^{22} - \sqrt{(\lambda_k^{11} - \lambda_k^{22})^2 + 4\lambda_k^{12}\lambda_k^{21}}}{2},$$

$\lambda_k^{ij} = \sum_{t\in T_{ji}} \omega_n^{kt}$ and $T_{ij} = \{t \mid 0 \leqslant t \leqslant n - 1, a^t \in t_jSt_i^{-1}\}$.

Proof. It is a direct consequence of Lemma 8 and Theorem 6. $\square$

Let Γ be a k-regular graph with n vertices and adjacency matrix A. Let A^c be the adjacency matrix of the complement of Γ. Then $(\lambda+k+1)\chi_{A^c}(\lambda) = (-1)^n(\lambda - n + k + 1)\chi_A(-\lambda-1)$, see [2, p. 20]. Clearly Cayley graphs are regular. If $n \geqslant 2$, n-Cayley graphs are not necessarily regular, but we have a similar relation between the characteristic polynomials of any n-Cayley graph and its complement which is given in the next theorem.

Theorem 10. *Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leqslant i, j \leqslant n)$ be an n-Cayley graph over a finite group G. Let Γ^c be the complement of Γ with adjacency matrix A^c. Then the characteristic polynomials of Γ and Γ^c are related with the following equation:*

$$\chi_{B_1}(\lambda)\chi_A(-\lambda-1) = (-1)^{|G|-1}\chi_{A_1}(-\lambda-1)\chi_{A^c}(\lambda),$$

where $B_1 = |G|J - I_n - A_1$, J is the all ones matrix of degree n, and $A_1 = [|T_{ji}|]_{1\leqslant i,j\leqslant n}$.

Proof. Since $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma^c)$, Γ^c is an n-Cayley graph over G. Furthermore $\Gamma^c = \mathrm{Cay}(G, S_{ij} \mid 1 \leqslant i, j \leqslant n)$, where $S_{ii} = G \setminus (T_{ii} \cup \{1\})$ and $S_{ij} = G \setminus T_{ij}$, where $i \neq j$. By Theorem 6, $\chi_{A^c}(\lambda) = \prod_{l=1}^m \chi_{B_l}(\lambda)^{d_l}$, where $B_l = [B_{ij}^{(l)}]$ is an $nd_l \times nd_l$ matrix and $B_{ij}^{(l)} = \sum_{s\in S_{ji}} \varrho^{(l)}(s)$ and $\chi_A(\lambda) = \prod_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$ as A_l defined in Theorem 6.
For $i \neq j$ we have

$$B_{ij}^{(l)} = \sum_{s\in S_{ji}} \varrho^{(l)}(s) = \sum_{x\in G} \varrho^{(l)}(x) - \sum_{t\in T_{ji}} \varrho^{(l)}(t)$$

and for $i = j$ we have

$$B_{ii}^{(l)} = \sum_{x\in G} \varrho^{(l)}(x) - \sum_{t\in T_{ii}} \varrho^{(l)}(t) - I_{d_l}.$$

Put $X_l = \sum_{x\in G} \varrho^{(l)}(x)$. Then for every $g \in G$, we have

$$\varrho^{(l)}(g)X_l\varrho^{(l)}(g)^{-1} = \sum_{x\in G} \varrho^{(l)}(g)\varrho^{(l)}(x)\varrho^{(l)}(g^{-1}) = \sum_{x\in G} \varrho^{(l)}(gxg^{-1}) = \sum_{x\in G} \varrho^{(l)}(x) = X_l.$$

Therefore by Schur's Lemma, we have $X_l = \frac{\sum_{g\in G} \eta_l(g)}{d_l} I_{d_l}$. Hence $X_1 = |G|$ and $X_l = 0_{d_l}$ for $l \neq 1$, where 0_{d_l} is the $d_l \times d_l$ zero matrix. Therefore for all $l \neq 1$,

$$B_l = -A_l - \mathrm{diag}(I_{d_l}, I_{d_l}, \ldots, I_{d_l}),$$

and $B_1 = |G|J - I_n - A_1$, where J is the all ones matrix of degree n. Furthermore if $l \neq 1$ then $\chi_{B_l}(\lambda) = \prod_{\mu\in\mathrm{Spec}(A_l)}(\lambda+\mu+1)$. So

$$\chi_{A^c}(\lambda) = \chi_{B_1}(\lambda) \prod_{l=2}^m \prod_{\mu\in\mathrm{Spec}(A_l)} (\lambda+\mu+1)^{d_l}.$$

Also we have

$$\begin{aligned}
\chi_A(-\lambda-1) &= \chi_{A_1}(-\lambda-1)\prod_{l=2}^{m}\chi_{A_l}(-\lambda-1)^{d_l}\\
&= \chi_{A_1}(-\lambda-1)\prod_{l=2}^{m}\prod_{\mu\in\mathrm{Spec}(A_l)}(-\lambda-1-\mu)^{d_l}\\
&= (-1)^{d_2^2+d_3^2+\cdots+d_m^2}\chi_{A_1}(-\lambda-1)\prod_{l=2}^{m}\prod_{\mu\in\mathrm{Spec}(A_l)}(\lambda+1+\mu)^{d_l}\\
&= (-1)^{|G|-1}\chi_{A_1}(-\lambda-1)\prod_{l=2}^{m}\prod_{\mu\in\mathrm{Spec}(A_l)}(\lambda+1+\mu)^{d_l}
\end{aligned}$$

which implies that $\chi_{B_1}(\lambda)\chi_A(-\lambda-1) = (-1)^{|G|-1}\chi_{A_1}(-\lambda-1)\chi_{A^c}(\lambda)$ as desired. □

3 Eigenvectors of n-Cayley (di)graphs

In this section we determine the corresponding eigenspace of each eigenvalue of n-Cayley digraph Γ. We use the notations of Theorem 6.

Lemma 11. *Let $v_{(k)} = (v_1,\ldots,v_n)$ be an eigenvector of A_k, $1\leqslant k\leqslant m$, associated with λ. Then the following vectors are distinct linearly independent d_k eigenvectors of digraph Γ associated with λ:*

$$v_{(k)}^{j} := \sum_{s=1}^{n}\sum_{g\in G}\left[v_s\cdot\bar{\varrho}_j^{(k)}(g)\right]e_g^s,\quad 1\leqslant j\leqslant d_k$$

where $\cdot$ is the usual inner product and $\bar{\varrho}_j^{(k)}(g)$ is a vector whose coordinates are the complex conjugate of the coordinates of jth column of $\varrho^{(k)}(g)$.

Proof. By Corollary 5, we have $A\bar{\varrho}_{ij}^{(l),k} = \sum_{s=1}^{n}\sum_{t\in T_{ks}}\sum_{r=1}^{d_l}\varrho_{ri}^{(l)}(t)\bar{\varrho}_{rj}^{(l),s}$. For $i=1,2,\ldots,n$, let $v_i=(v_{i1},v_{i2},\ldots,v_{id_k})$. Then

$$\begin{aligned}
v_{(k)}^{j} &= \sum_{s=1}^{n}\sum_{g\in G}\left[v_s\cdot\bar{\varrho}_j^{(k)}(g)\right]e_g^s\\
&= \sum_{s=1}^{n}\sum_{t=1}^{d_k}v_{st}\sum_{g\in G}\overline{\varrho_{tj}^{(k)}(g)}e_g^s\\
&= \sum_{s=1}^{n}\sum_{t=1}^{d_k}v_{st}\bar{\varrho}_{tj}^{(k),s}.
\end{aligned}$$

Now we have

$$
\begin{aligned}
Av^j_{(k)} &= \sum_{s=1}^{n}\sum_{t=1}^{d_k} v_{st} A\bar{\varrho}^{(k),s}_{tj} \\
&= \sum_{s=1}^{n}\sum_{t=1}^{d_k} v_{st} \sum_{s'=1}^{n}\sum_{t'\in T_{ss'}}\sum_{r=1}^{d_k} \varrho^{(k)}_{rt}(t')\bar{\varrho}^{(k),s}_{rj} \quad \text{(by Corollary 5)} \\
&= \sum_{s=1}^{n}\sum_{r=1}^{d_k} \left[\sum_{t=1}^{d_k}\sum_{s'=1}^{n}\sum_{t'\in T_{ss'}} v_{st}\varrho^{(k)}_{rt}(t')\right]\bar{\varrho}^{(k),s}_{rj} \\
&= \sum_{s=1}^{n}\sum_{r=1}^{d_k} \lambda v_{sr}\bar{\varrho}^{(k),s}_{rj} \\
&= \lambda\sum_{s=1}^{n}\sum_{r=1}^{d_k} v_{sr}\bar{\varrho}^{(k),s}_{rj} \\
&= \lambda v^j_{(k)}
\end{aligned}
$$

as desired. Since $\left\{\bar{\varrho}^{(k),s}_{ij} \mid 1 \leqslant k \leqslant m, 1 \leqslant s \leqslant n, 1 \leqslant i,j \leqslant d_k\right\}$ is an orthogonal basis of V (the corresponding vector space of the adjacency matrix A), $v^j_{(k)}$'s are distinct and linearly independent. □

An eigenvector of the adjacency matrix of a graph Γ is said to be *main eigenvector* if it is not orthogonal to the all ones vector $\mathbf{j}$. An eigenvalue of a graph Γ is said to be a *main eigenvalue* if it has a main eigenvector. By Perron-Frobenius Theorem, the largest eigenvalue of a graph is a main eigenvalue. It is also well known that a graph is regular if and only if it has exactly one main eigenvalue. So for every Cayley graph $\Gamma = \mathrm{Cay}(G,S)$, $|S|$ is the only main eigenvalue of Γ. Since n-Cayley graphs, for $n \geqslant 2$ are not necessarily regular, determining the main eigenvalues of these graphs seems to be important. In the following corollary we determine the main eigenvalues of n-Cayley graphs.

Corollary 12. *Let $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leqslant i,j \leqslant n)$ be an n-Cayley graph over a finite group G and $n \geqslant 2$. The main eigenvalues of Γ is equal to main eigenvalues of the matrix $A_1 = [|T_{ji}|]_{1\leqslant i,j\leqslant n}$.*

Proof. Using the notations of Lemma 11, we have

$$
\begin{aligned}
v^j_{(k)} \cdot \mathbf{j} &= \left(\sum_{s=1}^{n}\sum_{g\in G}\left[v_s \cdot \bar{\varrho}^{(k)}_j(g)\right] e^s_g\right)\cdot \mathbf{j} \\
&= \sum_{s=1}^{n}\sum_{g\in G}\left[v_s \cdot \bar{\varrho}^{(k)}_j(g)\right] e^s_g \cdot \mathbf{j} \\
&= \sum_{s=1}^{n}\sum_{g\in G}\left[v_s \cdot \bar{\varrho}^{(k)}_j(g)\right]
\end{aligned}
$$

$$= \sum_{s=1}^{n} v_s \cdot \left[\sum_{g \in G} \bar{\varrho}_j^{(k)}(g) \right].$$

Also $\sum_{g\in G} \bar{\varrho}_j^{(k)}(g)$ is the complex conjugate of jth column of $\sum_{g\in G} \varrho^{(k)}(g)$. On the other hand, by Schur's Lemma $\sum_{g\in G} \varrho^{(1)}(g) = |G|$ and for all $k \neq 1$, $\sum_{g\in G} \varrho^{(k)}(g) = 0_{d_k}$, where 0_{d_k} is the all zeros matrix of order d_k. This implies that

$$v_{(k)}^{j} \cdot \mathbf{j} = \begin{cases} 0 & k \neq 1 \\ |G| \sum_{s=1}^{n} v_s & k = 1. \end{cases}$$

Since $v_{(1)}^{j} \cdot \mathbf{j} = |G| \sum_{s=1}^{n} v_s = |G|(v_{(1)} \cdot \mathbf{j}')$, where $\mathbf{j}'$ is the all ones vector $1 \times n$, the result is clear. □

Corollary 13. *Let* $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ *be a 2-Cayley graph over a group* G*. Then* Γ *has exactly two main eigenvalues if and only if* $|T_{11}| \neq |T_{22}|$*.*

Proof. Let Γ has exactly two main eigenvalues. If $|T_{11}| = |T_{22}|$, then Γ is regular and by [6, Proposition 1.4], Γ must have exactly one main eigenvalue which is a contradiction. Conversely, let $|T_{11}| \neq |T_{22}|$. Then Γ is not regular. Also by Corollary 12, Γ has at most two main eigenvalues and by Perron-Frobenius Theorem, the largest eigenvalue of Γ is a main eigenvalue. So by [6, Proposition 1.4], Γ has exactly two main eigenvalues. □

Corollary 14. *Let* $\Gamma = \mathrm{Cay}(G, T_{ij} \mid 1 \leqslant i, j \leqslant 2)$ *be a 2-Cayley graph over a group* G*. If* $|T_{11}| \neq |T_{22}|$ *then* Γ *has exactly two orbits on* $V(\Gamma)$ *which are the same orbits of* R_G*.*

Proof. Let $|T_{11}| \neq |T_{22}|$. By Corollary 13, Γ has exactly two main eigenvalues. On the other hand, $A := \mathrm{Aut}(\Gamma)$ has at least two orbits on $V(\Gamma)$, say α^A and β^A. Let $O_1 = \alpha^{R_G}$ and $O_2 = \beta^{R_G}$ be two orbits of R_G then $O_1 \subseteq \alpha^A$ and $O_2 \subseteq \beta^A$. Hence $O_1 \cap O_2 = \emptyset$ and so $O_1 \cup O_2 = V = \alpha^A \cup \beta^A$. This shows that Γ has exactly two distinct orbits which are the same orbits of R_G. □

Note that the converse of the above corollary is not true. To see this, consider the generalized Peterson graph $\Gamma = P(h, t)$, where $t^2 \neq 1 \pmod h$. Then $P(h, t)$ is not vertex-transitive (see [2, pp. 104, 105]), and so as we proved in the above corollary, Γ has two orbits on $V(\Gamma)$ and is a 2-Cayley graph over a cyclic group $\langle a \rangle$ of order h, where $T_{11} = \{a, a^{-1}\}$, $T_{22} = \{a^t, a^{-t}\}$ and $T_{12} = T_{21} = \{1\}$.

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