On Wilf equivalence for alternating permutations

Sherry H. F. Yan
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, P.R. China
huifangyan@hotmail.com

Submitted: Mar 27, 2013; Accepted: Sep 21, 2013; Published: Oct 7, 2013
Mathematics Subject Classifications: 05A05, 05C30

Abstract

In this paper, we obtain several new classes of Wilf-equivalent patterns for alternating permutations. In particular, we prove that for any nonempty pattern τ, the patterns $12\dots k \oplus \tau$ and $k \dots 21 \oplus \tau$ are Wilf-equivalent for alternating permutations, paralleling a result of Backelin, West, and Xin for Wilf-equivalence for permutations.

Keywords: alternating permutation; pattern avoiding; Wilf-equivalent; alternating Young diagram.

1 Introduction

A permutation $\pi = \pi_1\pi_2\dots\pi_n$ of length n on $[n] = \{1, 2, \dots, n\}$ is said to be an *alternating* permutation if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Similarly, π is said to be a *reverse* alternating permutation if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. We denote by $\mathcal{A}_n$ and $\mathcal{A}'_n$ the set of alternating and reverse alternating permutations of length n, respectively.

Denote by $\mathcal{S}_n$ the set of all permutations on $[n]$. Given a permutation $\pi = \pi_1\pi_2\dots\pi_n \in \mathcal{S}_n$ and a permutation $\tau = \tau_1\tau_2\dots\tau_k \in \mathcal{S}_k$, we say that π contains the *pattern* τ if there exists a subsequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ of π that is order-isomorphic to τ. Otherwise, π is said to *avoid* the pattern τ or be τ-*avoiding*.

Pattern avoiding permutations have been extensively studied over last decade. For a thorough summary of the current status of research, see Bóna's book [4] and Kitaev's book [8]. Analogous to the ordinary permutations, Mansour [11] initiated the study of alternating permutations avoiding a given pattern. For any pattern of length 3, the number of alternating permutations of a given length avoiding that pattern is given by

Catalan numbers, see [11, 13]. Recently, Lewis [9] considered the enumeration of alternating permutations avoiding a given pattern of length 4. Let $\mathcal{A}_n(\tau)$ and $\mathcal{A}'_n(\tau)$ be the set of τ-avoiding alternating and reverse alternating permutations of length n, respectively. Lewis [9] showed that there is a bijection between $\mathcal{A}_{2n}(1234)$ and the set of standard Young tableaux of shape (n, n, n), and between the set $\mathcal{A}_{2n+1}(1234)$ and the set of standard Young tableaux of shape $(n + 1, n, n - 1)$. Using the hook length formula for standard Young tableaux [12], he deduced that $|\mathcal{A}_{2n}(1234)| = \frac{2(3n)!}{n!(n+1)!(n+2)!}$ and $|\mathcal{A}_{2n+1}(1234)| = \frac{16(3n)!}{(n-1)!(n+1)!(n+3)!}$. Later, Lewis [10] showed that the generating trees for 2143-avoiding alternating permutations of even length and odd length are isomorphic to the generating trees for standard Young tableaux of shape (n, n, n) and shifted standard Young tableaux of shape $(n+2, n+1, n)$, respectively. In his paper [10], Lewis posed several conjectures on the enumeration of alternating permutations avoiding a given pattern of length 4 and 5. Some of these conjectures were proved by Bóna [5], Chen et al. [6] and Xu et al. [14].

The *reverse* of a permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ is given by $\pi^r = \pi_n\pi_{n-1} \ldots \pi_1$ and the *complement* by $\pi^c = (n + 1 - \pi_1)(n + 1 - \pi_2) \ldots (n + 1 - \pi_n)$. Recall that two patterns α and β are said to be Wilf-equivalent if $|\mathcal{S}_n(\alpha)| = |\mathcal{S}_n(\beta)|$ for all natural numbers n. We denote this by $\alpha \sim \beta$. It is clear that $\sigma \sim \sigma^c \sim \sigma^r \sim \sigma^{rc}$. It is easily seen that the reverse and complement operation of an alternating permutation of even length gives another alternating permutations, and the reverse of an alternating permutation of odd length gives another alternating permutation. Thus, given a pattern σ, we have $|\mathcal{A}_{2n}(\sigma)| = |\mathcal{A}_{2n}(\sigma^{rc})|$ and $|\mathcal{A}_{2n+1}(\sigma)| = |\mathcal{A}_{2n+1}(\sigma^r)|$ for all $n \geqslant 0$. In this context, σ and σ^{rc} are said to be *trivially* Wilf-equivalent for alternating permutations of even length. Similarly, σ and σ^r are said to be *trivially* Wilf-equivalent for alternating permutations of odd length.

Definition 1.1. *The direct sum of two permutations* $\alpha = \alpha_1\alpha_2 \ldots \alpha_k \in \mathcal{S}_k$ *and* $\beta = \beta_1\beta_2 \ldots \beta_l \in \mathcal{S}_l$*, denoted by* $\alpha \oplus \beta$*, is the permutation* $\alpha_1\alpha_2 \ldots \alpha_k(\beta_1+k)(\beta_2+k) \ldots (\beta_l+k)$*.*

Definition 1.2. *The skew sum of two permutations* $\alpha = \alpha_1\alpha_2 \ldots \alpha_k \in \mathcal{S}_k$ *and* $\beta = \beta_1\beta_2 \ldots \beta_l \in \mathcal{S}_l$*, denoted by* $\alpha \ominus \beta$*, is the permutation* $(\alpha_1 + l)(\alpha_2 + l) \ldots (\alpha_k + l)\beta_1\beta_2 \ldots \beta_l$*.*

In this paper, we are mainly concerned with the Wilf-equivalent classes of patterns for alternating permutations, which are the analogue of a result for permutations proved by Backelin, West, Xin [2] and for involutions proved by Bousquet-Mélou and Steingrímsson [3]. We obtain the following non-trivial Wilf equivalence for alternating permutations.

Theorem 1.3. *Let* $n \geqslant 1$ *and* $k \geqslant 2$*. The equality* $|\mathcal{A}_n(12 \ldots k \oplus \tau)| = |\mathcal{A}_n(k \ldots 21 \oplus \tau)|$ *holds for any nonempty pattern* τ*.*

Theorem 1.4. *Let* $n \geqslant 1$ *and* $k \geqslant 2$*. The equality* $|\mathcal{A}_n(k-1 \ldots 21k \oplus \tau)| = |\mathcal{A}_n(k \ldots 21 \oplus \tau)|$ *holds for any nonempty pattern* τ*.*

Theorem 1.5. *Let* $n \geqslant 1$ *and* $k \geqslant 3$*. The equality* $|\mathcal{A}_{2n+1}(23 \ldots k1 \ominus \tau)| = |\mathcal{A}_{2n+1}(12 \ldots k \ominus \tau)|$ *holds for any nonempty pattern* τ*.*

Theorem 1.6. *Let $n \geqslant 1$ and $k \geqslant 3$. The equality $|\mathcal{A}_{2n}(23\dots k1 \ominus \tau)| = |\mathcal{A}_{2n}(12\dots k \ominus \tau)|$ holds for any (possibly empty) pattern τ.*

Note that Theorems 1.3 through 1.6 were proved by Gowravaram and Jagadeesan [7] for $k = 2, 3$.

2 Alternating-shape-Wilf-equivalence for alternating Young diagrams

In this section, we follow the approach given in [1], [2], [3] and [7]. We study pattern avoidance for slightly more general objects than alternating permutation, namely, *transversals of alternating Young diagrams.* Let us begin with some necessary definitions and notations. We draw Young diagrams in English notation and use matrix coordinates, and for example $(1, 2)$ is the second square in the first row of a Young diagram.

Definition 2.1. *Let λ be a Young diagram with k columns and D be a subset of nonconsecutive positive integers of $[k]$. If for any $i \in D$ column i and column $i+1$ have the same length, then we call the pair (λ, D) an alternating Young diagram. An alternating Young diagram (λ, D) is said to be a strict alternating Young diagram if $D \subseteq [k-1]$.*

A *transversal* of a Young diagram λ is a filling of the squares of λ with $1's$ and $0's$ such that every row and column contains exactly one 1. Denote by $T = \{(t_i, i)\}_{i=1}^{k}$ the transversal in which the square (t_i, i) is filled with a 1 for all $i \leqslant k$. For example the transversal $T = \{(1,5),(2,4),(3,2),(4,3),(5,1)\}$ are illustrated as Figure 1.

0	0	0	0	1
0	0	0	1	
0	1	0		
0	0	1		
1				

Figure 1: The transversal $T = \{(1,5),(2,4),(3,2),(4,3),(5,1)\}$.

The notion of pattern avoidance is extended to transversals of a Young diagram in [1] and [2]. In this section, we will consider permutations as permutation matrices. Given a permutation $\pi = \pi_1\pi_2\dots\pi_n \in \mathcal{S}_n$, its corresponding permutation matrix is a n by n matrix M in which the entry on column i and row π_i is 1 and all the other entries are zero.

Given a permutation α of $[m]$, let M be its permutation matrix. A transversal L of a Young diagram λ will be said to contain α if there exists two subsets of the index set $[n]$, $R = \{r_1 < r_2 < \ldots < r_m\}$ and $C = \{c_1 < c_2 < \ldots < c_m\}$, such that the matrix on R and C is a copy of M and each of the squares (r_j, c_j) falls within the Young diagram. In this context, the permutation α is called a *pattern*. Denote by $\mathcal{S}_\lambda(\alpha)$ the set of all transversals of Young diagram λ that avoid α. Two patterns α and β are said to be *shape-Wilf-equivalent*, denoted by $\alpha \sim_s \beta$, if for all Young diagram λ, we have $|\mathcal{S}_\lambda(\alpha)| = |\mathcal{S}_\lambda(\beta)|$.

Definition 2.2. *Let λ be a Young diagram with k columns. Given a transversal $T = \{(t_i, i)\}_{i=1}^k$ of an alternating Young diagram (λ, D), let $Peak(T) = \{i | t_{i-1} < t_i > t_{i+1}\}$ with the assumption $t_0 = t_{k+1} = 0$. Then T is said to be a valid transversal of (λ, D) if we have $D \subseteq Peak(T)$.*

Denote by $\mathcal{T}_\lambda^D$ the set of all valid transversals of alternating Young diagram (λ, D). Similarly, denote by $\mathcal{T}_\lambda^D(\alpha)$ the set of all transversals of alternating Young diagram (λ, D) that avoid α.

Definition 2.3. *Two patterns α and β are called alternating-shape-Wilf-equivalent if $|\mathcal{T}_\lambda^D(\alpha)| = |\mathcal{T}_\lambda^D(\beta)|$ for all alternating Young diagrams (λ, D). We denote this by $\alpha \sim_{as} \beta$. Similarly, two patterns α and β are called strict-alternating-shape-Wilf-equivalent if $|\mathcal{T}_\lambda^D(\alpha)| = |\mathcal{T}_\lambda^D(\beta)|$ for all strict alternating Young diagrams (λ, D). We denote this by $\alpha \sim_{sas} \beta$.*

Backelin, West, Xin [2] proved the following shape-Wilf equivalences for transversals of Young diagrams. Let $I_k = 12\ldots k$, $J_k = k\ldots 21$ and $F_k = (k-1)\ldots 21k$.

Theorem 2.4. *([2], Proposition 2.3) For any patterns α, β and σ, if $\alpha \sim_s \beta$, then $\alpha \oplus \sigma \sim_s \beta \oplus \sigma$.*

Theorem 2.5. *([2], Proposition 3.1) For all $k \geqslant 2$, $F_k \sim_s J_k$.*

Theorem 2.6. *([2], Proposition 2.2) For all $k \geqslant 2$, $I_k \sim_s J_k$.*

In this section, we will adapt their proof of Theorem 2.4 to obtain the following theorem.

Theorem 2.7. *For any nonempty patterns α, β and σ, if $\alpha \sim_{sas} \beta$, then $\alpha \oplus \sigma \sim_{as} \beta \oplus \sigma$.*

Proof. For any Young diagram λ with k columns and a subset D of non-consecutive integers of $[k-1]$, let $f_\lambda^D : \mathcal{T}_\lambda^D(\alpha) \to \mathcal{T}_\lambda^D(\beta)$ implied by the hypothesis. Now fix Young diagram λ and a subset D of non-consecutive integers of $[k-1]$. We proceed to construct a bijection $g_\lambda^D : \mathcal{T}_\lambda^D(\alpha \oplus \sigma) \to \mathcal{T}_\lambda^D(\beta \oplus \sigma)$. Given a transversal $T \in \mathcal{T}_\lambda^D(\alpha \oplus \sigma)$, we color the cell (r, c) of λ by white if the board of λ lying above and to the right of it contains σ, or gray otherwise. Then find the $1's$ coloured by gray, and colour the corresponding rows and columns gray. Denote the white board by λ'.

The white board λ' is a Young diagram since if a cell is colored white, then each cell to the left and above it. Suppose that $c \in D$. It is easily seen that if $c \in D$ and the cell (r, c) is filled with a 1 in T, then if the cell (r, c) is coloured by white, then all the cell $(r', c+1)$

is also colored by white for all $r' \leqslant r$. This implies that the white board form a strict alternating Young diagram. Denote this by (λ', D'). Denote by T' the the transversal T restricted to the strict alternating Young diagram (λ', D'). Next we aim to show that T' is a valid transversal of (λ', D'). Suppose that $c \in D$, the squares $(r_1, c-1)$, (r_2, c) and $(r_3, c+1)$ are filled with $1's$ in T, and the square (r_2, c) is coloured by white. Then the squares $(r_1, c-1)$ and $(r_3, c+1)$ are also coloured by white. This implies that T' is a valid transversal of (λ', D'). Since T avoids $\alpha \oplus \sigma$, we have $T' \in \mathcal{T}_{\lambda'}^{D'}(\alpha)$. Applying the map $f_{\lambda'}^{D'}$ to T', we get a valid transversal in $\mathcal{T}_{\lambda'}^{D'}(\beta)$. Restoring the gray cells of λ and their contents, we obtain a transversal L of the alternating Young diagram (λ, D) avoiding the pattern $\beta \oplus \sigma$. It is easy to check that L is a valid transversal of the alternating Young diagram (λ, D).

In order to show that the map g_λ^D is a bijection, we show that the above procedure is invertible. It is obvious that the map g_λ^D only changes the white cells and leaves the gray cells fixed. Hence when applying the inverse map $(g_\lambda^D)^{-1}$, the coloring of L will result in the same semi-standard Young diagram (λ', D') on which to apply the inverse transformation $(f_{\lambda'}^{D'})^{-1}$. This completes the proof. □

In the next section, we will give a bijective proof of the following analogous result of Theorem 2.5 given in [2].

Theorem 2.8. *For all $k \geqslant 3$, $F_k \sim_{sas} J_k$.*

In order to prove Theorem 1.4, we also need the following Wilf equivalence for alternating permutations, which was proved by Gowravaram and Jagadeesan [7].

Theorem 2.9. *([7], Theorem 4.4) Fix $n \geqslant 1$. For any nonempty patterns σ, we have $|\mathcal{A}_n(12 \oplus \sigma)| = |\mathcal{A}_n(21 \oplus \sigma)|$.*

Note that Theorem 2.9 can also be proved by similar reasoning as in the proof of Theorem 2.7.

The proofs of Theorems 1.3 through 1.6. Combining Theorems 2.7 and 2.8, we deduce that $F_k \oplus \sigma \sim_{as} J_k \oplus \sigma$ for any nonempty pattern σ and $k \geqslant 3$. Note that the permutation matrix of an alternating (resp. reverse alternating) permutation of length n is a valid transversal of an alternating Young diagram (λ, D), where λ is a n by n square diagram and $D = \{2, 4, \ldots, \lfloor \frac{n}{2} \rfloor\}$ (resp. $D = \{1, 3, \ldots, \lceil \frac{n}{2} \rceil\}$). Hence for $n \geqslant 0$ and $k \geqslant 3$, the equalities

$$|\mathcal{A}_n(k-1\ldots21k \oplus \tau)| = |\mathcal{A}_n(k\ldots21 \oplus \tau)| \tag{2.1}$$

and

$$|\mathcal{A}'_n(k-1\ldots21k \oplus \tau)| = |\mathcal{A}'_n(k\ldots21 \oplus \tau)| \tag{2.2}$$

hold for any for any nonempty pattern τ.

When λ is a $2n$ by $2n$ square diagram and $D = \{1, 3, \ldots, 2n-1\}$, a valid transversal of (λ, D) is the permutation matrix of an reverse alternating permutations of even length. Therefore for $n \geqslant 1$ and $k \geqslant 3$, Theorem 2.8 leads to the equality

$$|\mathcal{A}'_{2n}(k-1\ldots21k)| = |\mathcal{A}'_{2n}(k\ldots21)|. \tag{2.3}$$

Since the complement map is a bijection between the set $\mathcal{A}_n$ and the set $\mathcal{A'}_n$, we obtain Theorems 1.5 and 1.6 by combining Formulae (2.1), (2.2) and (2.3). Combining Theorem 2.9 and Formula 2.1, we get Theorem 1.4. From Theorem 1.4, we immediately get Theorem 1.3. This completes the proof. □

3 The bijection

In this section, we will prove Theorem 2.8 by establishing a bijection between $\mathcal{T}_\lambda^D(F_k)$ and $\mathcal{T}_\lambda^D(J_k)$ for every strict alternating Young diagram (λ, D). Let us first describe two transformations defined by Backelin, West and Xin [2]

The transformation ϕ Given a transversal L of a Young diagram λ, if L contains no J_k, we simply define $\phi(L) = L$. Otherwise, find the highest square (a_1, b_1) containing a 1, such that there is a J_k in L in which (a_1, b_1) is the leftmost 1. Then, find the leftmost (a_2, b_2) containing a 1, such that there is a J_k in L in which (a_1, b_1) and (a_2, b_2) are the leftmost two 1's. Finally, find $(a_3, b_3), (a_4, b_4), \ldots, (a_k, b_k)$ one by one as (a_2, b_2). Let $\phi(L)$ be a transversal of λ such that the squares $(a_2, b_1), (a_3, b_2), \ldots, (a_k, b_{k-1}), (a_1, b_k)$ are filled with $1's$ and the other rows and columns are the same as L.

The transformation ψ Given a transversal L of a Young diagram λ, if L contains no F_k, we simply define $\psi(L) = L$. Otherwise, find the lowest square (a_1, b_1) containing a 1, such that there is a F_k in L in which (a_1, b_1) is the rightmost 1. Then, find the lowest (a_2, b_2) containing a 1, such that there is a F_k in L in which (a_1, b_1) and (a_2, b_2) are the rightmost two 1's. Finally, find $(a_3, b_3), (a_4, b_4), \ldots, (a_k, b_k)$ one by one as (a_2, b_2). Let $\psi(L)$ be a transversal of λ such that the squares $(a_1, b_k), (a_k, b_{k-1}), \ldots, (a_2, b_1)$ are filled with $1's$ and the other rows and columns are the same as L.

Example 3.1. *Let $k = 3$. Given a transversal $L = \{(1,2), (2,4), (3,3), (4,1)\}$ of 4 by 4 square diagram λ, by applying the transformation ϕ, we get a transversal $L' = \{(1,2), (2,3), (3,1), (4,4)\}$ as shown in Figure 2, where the selected J_k is illustrated in bold. Conversely, given a transversal $L' = \{(1,2), (2,3), (3,1), (4,4)\}$ of a square diagram λ we can recover the transversal L by applying the transformation ψ as shown in figure 2, where the selected F_k is illustrated in bold.*

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{1} \\ \hline 0 & 0 & \mathbf{1} & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 \\ \hline \end{array} \quad \underset{\psi}{\overset{\phi}{\rightleftarrows}} \quad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \mathbf{1} & 0 \\ \hline \mathbf{1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{1} \\ \hline \end{array}$$

Figure 2: An example of the transformations ϕ and ψ.

Backelin, West and Xin [2] proved the following properties of ϕ and ψ, which were essential in the construction of their bijection between $\mathcal{S}_\lambda(F_k)$ and $\mathcal{S}_\lambda(J_k)$. In the following lemmas, for all $1 \leqslant i \leqslant k$, let (a_i, b_i) be the square selected in the application of the transformations ϕ and ψ.

Lemma 3.2. *([2], Lemma 4.1) There is no J_k strictly above row a_1 in $\phi(L)$.*

Lemma 3.3. *([2], Lemma 4.2) If L contains no F_k with at least one square below row a_1, then $\phi(L)$ contains no such F_k.*

Lemma 3.4. *([2], Lemma 4.3) There is no J_t above a_1, below row a_{t+1} and to the left of column b_{t+1} in $\phi(L)$.*

Lemma 3.5. *([2], Lemma 4.4) If L contains no F_k with at least one square below row a_1, then $\psi(L)$ contains no such F_k.*

Lemma 3.6. *([2], Lemma 4.5) If L contains no J_k that is above row a_1, then $\psi(L)$ contains no such J_k.*

Lemma 3.7. *([2], Lemma 4.6) If L contains no J_k that is above row a_1, the board that is above and to the right of (a_t, b_{t-1}) cannot contain a J_{k-t} in $\psi(L)$ such that the lowest 1 of this J_{k-t} is to the left of (a_{t+1}, b_t), and this J_{t-k} , combining with the $1'$s positioned at squares (a_1, b_1), (a_2, b_2), $\ldots$, (a_t, b_{t-1}) forms a J_k in $\psi(L)$.*

Combining Lemmas 3.3 and 3.4 yields the following result.

Lemma 3.8. *If L is a transversal containing no F_k with at least one square below row a_1, then we have $\psi(\phi(L)) = L$.*

Lemmas 3.6 and 3.7 imply the following result.

Lemma 3.9. *If L is a transversal containing no J_k above row a_1, then we have $\phi(\psi(L)) = L$.*

It is obvious that the transformation ϕ (resp. ψ) changes every occurrence of J_k (resp. F_k) to an occurrence of F_k(resp. J_k). Lemmas 3.2 and 3.5 imply that after finitely many iterations of ϕ (resp. ψ), there will be no occurrence of J_k (resp. F_k) in L. Denote by ϕ^* (resp. ψ^*) the iterated transformation, that recursively transforms every occurrence of J_k (resp. F_k) into F_k (resp. J_k). Backelin, West and Xin [2] proved the following theorem.

Theorem 3.10. *([2], Proposition 3.1) For every Young diagram λ, the transformations ϕ^* and ψ^* induce a bijection between $\mathcal{S}_\lambda(F_k)$ and $\mathcal{S}_\lambda(J_k)$.*

In Example 3.1, it is easily seen that L is a valid transversal of the strict alternating Young diagram (λ, D) with $D = \{1, 3\}$, while the transversal L' is not a valid transversal of (λ, D). Hence, in order to establish a bijection between $\mathcal{T}_\lambda^D(F_k)$ and $\mathcal{T}_\lambda^D(J_k)$ for any strict alternating Young diagram (λ, D), we need to define two transformations Φ and Ψ on valid transversals of the strict alternating Young diagram (λ, D) by slightly modifying the transformations ϕ and ψ.

The transformation Φ Given a valid transversal L of a strict alternating Young diagram (λ, D), if L contains no J_k, we simply define $\Phi(L) = L$. Otherwise, applying the map ϕ to L, we get a transversal $\phi(L)$. Suppose that when we apply the map ϕ to L, the selected $1's$ are positioned at $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$, where $b_1 < b_2 < \ldots < b_k$. Suppose that the squares $(a', b_k - 1)$ and $(a'', b_k + 1)$ are filled with $1's$ in $\phi(L)$. Define $\Phi(L)$ as follows:

(i) if $b_k - 1, b_k + 1 \notin D$, then let $\Phi(L) = \phi(L)$;

(ii) if $b_k - 1 \in D$ and $b_k + 1 \notin D$, then let $\Phi(L)$ be a transversal in which $(a_1, b_k - 1)$ and (a', b_k) are filled with $1's$, and all other rows and columns are the same as $\phi(L)$;

(iii) if $b_k - 1 \in D$ and $b_k + 1 \in D$, then let $\Phi(L)$ be a transversal in which $(a_1, b_k - 1)$ and (a', b_k) are filled with $1's$, and all other rows and columns are the same as $\phi(L)$ when $a' < a''$, and let $\Phi(L)$ be a transversal in which $(a_1, b_k + 1)$ and (a'', b_k) are filled with $1's$, and all other rows and columns are the same as $\phi(L)$ when $a' > a''$;

(iv) if $b_k - 1 \notin D$ and $b_k + 1 \in D$, then let $\Phi(L)$ be a transversal in which $(a_1, b_k + 1)$ and (a'', b_k) are filled with $1's$, and all other rows and columns are the same as $\phi(L)$;

The transformation Ψ Given a valid transversal L of a strict alternating Young diagram (λ, D), if L contains no F_k, we simply define $\Psi(L) = L$. Otherwise, find the lowest square (a_1, b_1) containing a 1, such that there is a F_k in L in which (a_1, b_1) is the rightmost 1. Then, find the lowest (a_2, b_2) containing a 1, such that there is a F_k in L in which (a_1, b_1) and (a_2, b_2) are the rightmost two 1's. Finally, find $(a_3, b_3), (a_4, b_4), \ldots, (a_k, b_k)$ one by one as (a_2, b_2). We define L' as follows:

(i') if $b_1 \notin D$, then let $L' = L$;

(ii') if $b_1 \in D$, then suppose that the squares $(a', b_1 - 1)$ and $(a'', b_1 + 1)$ are filled with $1's$ in L. If $a' > a''$, then let L' be a transversal in which the squares $(a_1, b_1 - 1)$ and (a', b_1) are filled with $1's$, and all other rows and columns are the same as L; If $a' < a''$, then let L' be a transversal in which the squares $(a_1, b_1 + 1)$ and $(a'', b_1)$are filled with $1's$, and all other rows and columns are the same as L;

Set $\Psi(L) = \psi(L')$.

Now we proceed to verify that Φ and Ψ have the following analogous properties of ϕ and ψ.

Lemma 3.11. *Fix $k \geqslant 3$. There is no J_k strictly above row a_1 in $\Phi(L)$. Moreover, if $\Phi(L)$ contains a J_k whose leftmost 1 is positioned at the square (a_1, b_1'), then we have $b_1' > b_1$.*

Proof. By Lemma 3.2, there is no J_k above row a_1 in $\phi(L)$. From the definition of Φ, it is easily seen that there is no J_k above row a_1 in $\Phi(L)$. Also, it is easy to check that for $k \geqslant 3$, if $\Phi(L)$ contains a J_k whose leftmost 1 is positioned at the square (a_1, b_1'), then we have $b_1' > b_1$. This completes the proof. $\square$

Lemma 3.12. *Fix $k \geqslant 3$. If L contains no F_k with at least one square below row a_1, then $\Phi(L)$ contains no such F_k.*

Proof. Suppose that H is a copy of F_k with at least one square below row a_1 in $\Phi(L)$. By Lemma 3.3, there is no F_k with at least one square below row a_1 in $\phi(L)$. Hence the 1 positioned at (a', b_k) or (a'', b_k) must fall in H. Moreover, we have $a' < a_1$ and $a'' < a_1$. Thus, neither (a', b_k) nor (a'', b_k) is the rightmost square of H. Suppose that the rightmost 1 of H is positioned at (r, c) in $\Phi(L)$. Then the $1's$ positioned at squares $(a_1, b_1), (a_2, , b_2), \ldots, (a_{k-1}, b_{k-1}), (r, c)$ will form a F_k in L. This contradicts the fact that L contains no F_k with at least one square below a_1. Thus $\Phi(L)$ contains no F_k with at least one row below a_1. This completes the proof. □

Lemma 3.13. *Fix $k \geqslant 3$. If L contains no F_k with at least one square below row a_1 and no J_k strictly above the row a_1 , then $\Psi(L)$ contains no such F_k. Moreover, if $\Psi(L)$ contains a copy of F_k whose rightmost 1 is positioned at (a_1, b'_1), then we have $b'_1 < b_1$.*

Proof. First we need to show that there is no F_k with at least one square below row a_1 in L'. If $b_1 \notin D$, then we have $L' = L$. Thus it is obvious that L' contains no F_k with at least one square below row a_1 and contains a F_k whose rightmost 1 is positioned at the square (a_1, b_1). If $b_1 \in D$, then we have the following two cases: $a' < a''$ or $a' > a''$ since the squares $(a', b_1 - 1)$ and $(a'', b_1 + 1)$ are filled with $1's$ in L. According to the construction of L', it is easily seen that in L' the squares $(a_1, b_1 + 1)$ and (a'', b_1) are filled with $1's$ in the former case, while the squares $(a_1, b_1 - 1)$ and (a', b_1) are filled with $1's$ in the latter case, and the other rows and columns are the same as L.

We claim that there is no F_k with at least one row below a_1 in L'. If not, suppose that G is such a F_k. Then at least one of the squares (a'', b_1) and $(a_1, b_1 + 1)$ must fall in G when $a' < a''$ and at least one of the squares $(a_1, b_1 - 1)$ and (a', b_1) must fall in G when $a' > a''$. Since $b_1 \in D$ and L is valid, we have $a' < a_1$ and $a'' < a_1$. Thus the $1's$ positioned at rows $b_1 - 1$, b_1 and $b_1 + 1$ cannot be the rightmost 1 of G. Suppose that the rightmost 1 of G is positioned at the square (r, c). Then we have $c \geqslant b_1 + 2$. Then the $1's$ positioned at the squares $(r, c), (a_2, b_2), \ldots, (a_k, b_k)$ will form a F_k in L and the square (r, c) is below row a_1. This contradicts the fact that L contains no F_k with at least one square below row a_1. Thus we conclude that L' contains no F_k with at least one square below row a_1.

Next we aim to show that when $b_1 \in D$ the transversal L' contains a F_k whose rightmost 1 is at row a_1. Clearly, the statement is true for the case when $a' < a''$ and $b_1 \notin D$. It remains to consider the case when $a' > a''$. Since L contains no J_k above row a_1, we have $b_2 \neq b_1 - 1$. Thus the $1's$ positioned at squares $(a_1, b_1 - 1)$, (a_2, b_2), $\cdots$, (a_k, b_k) form a F_k in L'. Recall that we have shown that there is no F_k whose rightmost 1 is strictly above row a_1. Therefore, we select a copy of F_k whose rightmost 1 is at row a_1 in the application of ψ to L'. From Lemma 3.5, it follows that $\Psi(L)$ contains no F_k with at least one square below row a_1.

Suppose that $\Psi(L)$ contains a F_k whose rightmost 1 is positioned at the square (a_1, b'_1). Recall that when we apply the map ψ to the transversal L', the rightmost 1 of the selected

F_k is positioned at the square (a_1, b_1) when $b_1 \notin D$. Moreover, when $b_1 \in D$, the rightmost 1 of the selected F_k is positioned at the $(a_1, b_1 + 1)$ (resp. $(a_1, b_1 - 1)$) if $a' < a''$ (resp. $a' > a''$). Since we move the 1 positioned at row a_1 to the left in the application of ψ to L', we conclude that $b'_1 < b_1$. This completes the proof. □

Lemma 3.14. *Fix $k \geqslant 3$. If L contains no J_k above row a_1, then $\Psi(L)$ contains no such J_k.*

Proof. It is easy to check that L' contains no J_k above row a_1. Recall that in the proof of Lemma 3.13, we have shown that when we apply ψ to L', we select a copy of F_k whose rightmost 1 is at row a_1. By Lemma 3.6, we deduce that $\Psi(L)$ contains no J_k above row a_1. This completes the proof. □

Lemma 3.11 states that the square (a_1, b_1) we find in the transformation Φ can only go down or slide right. Similarly, Lemmas 3.13 and 3.14 tells us that the square (a_1, b_1) selected in the application of the transformation Ψ can only go up or slide left. Hence, there will no J_k (resp. F_k) in the resulting transversal after finitely many iterations of Φ (resp. Ψ). Denote by $\Phi^*(L)$ (resp. $\Psi^*(L)$) the resulting transversal.

The key to our paper is the following theorem.

Theorem 3.15. *Fix $k \geqslant 3$. For any strict alternating Young diagram (λ, D), the transformations Φ^* and Ψ^* induce a bijection between $\mathcal{T}_\lambda^D(F_k)$ and $\mathcal{T}_\lambda^D(J_k)$.*

4 The proof

In this section, we will give a proof of Theorem 3.15. First we need to show that the transformations Φ and Ψ transform a valid transversal into a valid transversal. Denote by $\Phi^0(L) = L$ and $\Psi^0(L) = L$ for any transversal L.

Theorem 4.1. *Fix $n \geqslant 0$ and $k \geqslant 3$. For any strict alternating Young diagram (λ, D) and any transversal $L \in \mathcal{T}_\lambda^D(F_k)$, the transversal $\Phi^n(L)$ is a valid transversal of (λ, D).*

Proof. We proceed by induction on n. Clearly, the theorem holds for $n = 0$. For $n \geqslant 1$, assume that $\Phi^{n-1}(L)$ is a valid transversal of (λ, D). Now we aim to show that $\Phi^n(L)$ is also a valid transversal of (λ, D). Let $c \in D$. Suppose that the squares $(r_1, c - 1)$, (r_2, c) and $(r_3, c + 1)$ are filled with $1's$ in $\Phi^n(L)$. In order to show that $\Phi^n(L)$ is a valid transversal, it suffices to show that $r_1 < r_2 > r_3$.

Suppose that when we apply Φ to $\Phi^{n-1}(L)$, we select a copy of J_k whose $1's$ are positioned at squares (a_1, b_1), (a_2, b_2), $\ldots$, (a_k, b_k). First we need to show that $b_k \notin D$. If not, then suppose that $b_k \in D$ and the square $(r, b_k + 1)$ is filled with a 1 in $\Phi^{n-1}(L)$. Since $\Phi^{n-1}(L)$ is valid, we have $a_k > r$. Thus the $1's$ positioned at squares (a_2, b_2), (a_3, b_3), $\ldots$, (a_k, b_k) and $(r, b_k + 1)$ will form a J_k in $\Phi^{n-1}(L)$, which contradicts the selection of (a_1, b_1). Thus we conclude that $b_k \notin D$.

Next we proceed to prove $r_1 < r_2 > r_3$ by considering the following cases.

Case 1. The square (a_1, b_k) is filled with a 1 in $\Phi^n(L)$: According to the definition of Φ, we have $b_k - 1, b_k + 1 \notin D$. Moreover, the squares (a_2, b_1), (a_3, b_2), …, (a_k, b_{k-1}) and (a_1, b_k) are filled with $1's$ in $\Phi^n(L)$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c-1, c, c+1$ belongs to the set $\{b_1, b_2, \ldots, b_k\}$.

Subcase 1.1. $c = b_t$ for some integer $t \leqslant k-1$: According to the definition of Φ, we have $r_2 = a_{t+1}$. Suppose that the square $(r_1', c-1)$ is filled with a 1 in $\Phi^{n-1}(L)$. Since $\Phi^{n-1}(L)$ is valid and $b_t \in D$, we have $r_1' < a_t$. This implies that $c-1 \neq b_{t-1}$. Thus $r_1 = r_1'$. We claim that $r_1 < a_{t+1} = r_2$. If not, suppose that $r_1 > a_{t+1}$. Then the $1's$ positioned at squares $(a_1, b_1), (a_2, b_2)$, …, (a_{t-1}, b_{t-1}), $(r_1, b_t - 1), (a_{t+1}, b_{t+1}), \ldots, (a_k, b_k)$ will form a J_k in $\Phi^{n-1}(L)$, which contradicts the selection of (a_t, b_t). It remains to show that $a_{t+1} > r_3$. We have two cases. If $c + 1 = b_{t+1}$, then we have $r_3 = a_{t+2}$ according to the definition of Φ. In this case, we have $r_2 = a_{t+1} > a_{t+2} = r_3$. If $c + 1 \neq b_{t+1}$, then the square $(r_3, c+1)$ is also filled with a 1 in $\Phi^{n-1}(L)$. Since $\Phi^{n-1}(L)$ is valid, we have $a_t > r_3$. We claim that $r_2 = a_{t+1} > r_3$. If not, then suppose that $a_{t+1} < r_3$. Then the $1's$ positioned at squares (a_2, b_2), (a_3, b_3), …, (a_t, b_t) $(r_3, b_t + 1)$, (a_{t+1}, b_{t+1}), …, (a_k, b_k) will form a J_k in L, which contradicts the selection of (a_1, b_1). Thus we conclude that $r_1 < r_2 > r_3$.

Subcase 1.2. $c + 1 = b_t$ and $c \neq b_{t-1}$ for some integer $t \leqslant k-1$: According to the definition of Φ, we have $r_3 = a_{t+1}$ and the square (r_2, c) is also filled with a 1 in $\Phi^{n-1}(L)$. Since $\Phi^{n-1}(L)$ is valid, we have $r_2 > a_t$. If $c - 1 \neq b_{t-1}$, then the square $(r_1, c-1)$ is also filled with a 1 in $\Phi^{n-1}(L)$. Thus we have $r_1 < r_2 > a_t > a_{t+1} = r_3$. If $c - 1 = b_{t-1}$, then we have $r_1 = a_t$. Since $r_2 > a_t$, we have $r_1 < r_2 > a_t > a_{t+1} = r_3$.

Subcase 1.3. $c - 1 = b_t$ and $c + 1 \neq b_{t+1}$ for some integer $t \leqslant k-1$: According to the definition of Φ, we have $r_1 = a_{t+1}$. Moreover, the squares (r_2, c) and $(r_3, c+1)$ are also filled with $1's$ in $\Phi^{n-1}(L)$. Thus we have $r_2 > r_3$ and $r_2 > a_t$. This implies that $r_1 = a_{t+1} < a_t < r_2 > r_3$.

Case 2. The square $(a_1, b_k - 1)$ is filled with a 1 in $\Phi^n(L)$ and $b_{k-1} = b_k - 1$: According to the definition of Φ, we have $b_k - 1 \in D$. Moreover, the squares (a_2, b_1), …, (a_{k-1}, b_{k-2}), (a_1, b_{k-1}) are filled with $1's$ in $\Phi^n(L)$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c-1, c, c+1$ belongs to the set $\{b_1, b_2, \ldots, b_{k-1}\}$. Here we only consider the case when $c = b_{k-1}$. The other cases can be verified by similar arguments as in the proofs of Case 1. It is obvious that when $c = b_{k-1}$ the squares $(r_1, c-1)$ and $(r_3, c+1)$ are also filled with $1's$ in $\Phi^{n-1}(L)$. By the induction hypothesis, we have $r_1 < a_{k-1} > a_k = r_3$. Since $r_2 = a_1$ and $a_1 > a_{k-1} > a_k$, we have $r_1 < r_2 > r_3$.

Case 3. The square $(a_1, b_k - 1)$ is filled with a 1 in $\Phi^n(L)$ and $b_{k-1} \neq b_k - 1$: Recall that $\phi(\Phi^{n-1}(L))$ is a transversal in which squares (a_2, b_1), (a_3, b_2), …, (a_k, b_{k-1}), (a_1, b_k) are filled with $1's$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Suppose that the squares $(a', b_k - 1)$ and $(a'', b_k + 1)$ are filled with $1's$ in $\phi(\Phi^{n-1}(L))$. Since $b_{k-1} \neq b_k - 1$, the squares $(a', b_k - 1)$ and $(a'', b_k + 1)$ are also filled with $1's$ in $\Phi^{n-1}(L)$. According to the definition of Φ, we have $b_k - 1 \in D$. Moreover, the squares (a_2, b_1), …, (a_k, b_{k-1})

$(a_1, b_k - 1)$ and (a', b_k) are filled with $1's$ in $\Phi^n(L)$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c-1, c, c+1$ belongs to the set $\{b_1, b_2, \ldots, b_{k-1}, b_k - 1, b_k\}$. Here we only consider the case when $c = b_k - 1$ or $c - 1 = b_k$. The other cases can be treated in the same manner as Case 1.

Subcase 3.1. $c = b_k - 1$" According to the definition of Φ, we have $r_2 = a_1$ and $r_3 = a'$. By Lemma 3.12, there is no F_k with at least one square below row a_1 in $\Phi^{n-1}(L)$. This implies that $a' < a_1$. Thus we have $r_2 > r_3 = a'$. It remains to show that $r_1 < r_2$. We consider two cases. If $c - 1 = b_{k-1}$, then $r_1 = a_k$. Thus $r_1 = a_k < a_1 = r_2$. If $c - 1 \neq b_{k-1}$, then the square $(r_1, c-1)$ is also filled with a 1 in $\Phi^{n-1}(L)$. Recall that the squares $(r_1, c-1)$ and (a', c) are filled with $1's$ in L. By the induction hypothesis, it follows that $r_1 < a'$. Thus we have $r_1 < a' < a_1 = r_2$.

Subcase 3.2. $c - 1 = b_k$: According to the definition of Φ, we have $a' < a''$, $r_1 = a'$ and $r_2 = a''$. Moreover, the squares $(r_3, c+1)$ and (r_2, c) are also filled with $1's$ in $\Phi^{n-1}(L)$. By the induction hypothesis, it follows that $r_2 > r_3$. Thus we have $r_1 < r_2 > r_3$.

Case 4. The square $(a_1, b_k + 1)$ is filled with a 1 in $\Phi^n(L)$: Recall that $\phi(\Phi^{n-1}(L))$ is a transversal in which squares (a_2, b_1), (a_3, b_2), …, (a_k, b_{k-1}), (a_1, b_k) are filled with $1's$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Suppose that the squares $(a', b_k - 1)$ and $(a'', b_k + 1)$ are filled with $1's$ in $\phi(\Phi^{n-1}(L))$. Since the map ϕ only changes columns $b_1, b_2, \ldots, b_k$, the square $(a'', b_k + 1)$ is also filled with a 1 in $\Phi^{n-1}(L)$. According to the definition of Φ, we have $b_k + 1 \in D$. Moreover, the squares (a_2, b_1), …, (a_k, b_{k-1}) $(a_1, b_k + 1)$ and (a'', b_k) are filled with $1's$ in $\Phi^n(L)$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$.

We claim that $b_k - 1 \neq b_{k-1}$. If not, suppose that $b_k - 1 = b_{k-1}$. Then we have $a' = a_k$. Recall that the squares (a_k, b_k) and $(a'', b_k + 1)$ are filled with $1's$ in $\Phi^{n-1}(L)$. By the induction hypothesis, since $b_k + 1 \in D$, we have $a_k > a''$. Thus the $1's$ positioned at squares (a_2, b_2), (a_3, b_3), …, (a_k, b_k), $(a'', b_k + 1)$ will form a J_k in $\Phi^{n-1}(L)$, which contradicts the selection of (a_1, b_1). Thus the square $(a', b_k - 1)$ is also filled with a 1 in $\Phi^n(L)$ and $\Phi^{n-1}(L)$.

In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c-1, c, c+1$ belongs to the set $\{b_1, b_2, \ldots, b_k, b_k + 1\}$. Here we only consider the case when $c = b_k + 1$ or $c + 1 = b_k$. The other cases can be treated in the same manner as Case 1.

Subcase 4.1. $c = b_k + 1$: Thus we have $r_1 = a''$ and $r_2 = a_1$. By Lemma 3.12, there is no F_k with at least one square below row a_1 in $\Phi^{n-1}(L)$. Recall that the square $(a'', b_k + 1)$ is filled with a 1 in $\Phi^{n-1}(L)$. Thus we have $a'' < a_1$. This implies that $r_1 < r_2$. It remains to show that $r_2 > r_3$. According to the definition of Φ, the squares $(r_3, b_k + 2)$ and $(a'', b_k + 1)$ are also filled with $1's$ in $\Phi^{n-1}(L)$. Since $b_k + 1 \in D$, by the induction hypothesis we have $a'' > r_3$. Thus we have $r_3 < a'' < a_1 = r_2$.

Subcase 4.2. $c + 1 = b_k$: In this case, we have $r_3 = a''$. Recall that the square $(r_2, b_k - 1)$ is also filled with a 1 in $\Phi^{n-1}(L)$ and $\phi(\Phi^{n-1}(L))$. Thus we have $r_2 = a'$.

Note that $b_k - 1, b_k + 1 \in D$ and the squares $(a', b_k - 1)$ and $(a'', b_k + 1)$ are filled with $1's$ in $\phi(\Phi^{n-1}(L))$. According to the definition of Φ, we have $a' > a''$. Thus we have $r_2 = a' > a'' = r_3$. Now it remains to show that $r_1 < r_2$. If $c - 1 \neq b_{k-1}$, then the square $(r_1, c-1)$ is also filled with 1 in $\Phi^{n-1}(L)$. By the induction hypothesis, it follows that $r_1 < r_2$. If $c - 1 = b_{k-1}$, then we have $r_1 = a_k$. Recall that the squares $(a', b_k - 1)$ and (a_k, b_k) are filled with $1's$ in $\Phi^{n-1}(L)$ and $b_k - 1 \in D$. By the induction hypothesis, we have $a' > a_k$. Thus we deduce that $r_1 < r_2$.

This completes the proof. □

Theorem 4.2. *Fix $n \geqslant 0$ and $k \geqslant 3$. For any strict alternating Young diagram (λ, D) and any transversal $L \in \mathcal{T}_\lambda^D(F_k)$, the transversal $\Psi^n(L)$ is a valid transversal of (λ, D).*

Proof. We proceed by induction on n. Clearly, the theorem holds for $n = 0$. For $n \geqslant 1$, assume that $\Psi^{n-1}(L)$ is a valid transversal of (λ, D). Now we aim to show that $\Psi^n(L)$ is also a valid transversal of (λ, D). Suppose that when we apply Ψ to $\Psi^{n-1}(L)$, we select a copy of F_k whose $1's$ are positioned at squares $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$, where $b_1 > b_2 > \ldots > b_k$. Assume that the squares $(a', b_1 - 1)$ and $(a'', b_1 + 1)$ are filled with $1's$ in $\Psi^{n-1}(L)$.

Let $c \in D$. Suppose that the squares $(r_1, c-1)$, (r_2, c) and $(r_3, c+1)$ are filled with $1's$ in $\Psi^n(L)$. In order to show that $\Psi^n(L)$ is a valid transversal, it suffices to show that $r_1 < r_2 > r_3$. We consider the following cases.

Case 1. $b_1 \notin D$: According to the selection of (a_1, b_1), we have $b_1 - 1, b_1 + 1 \notin D$. Otherwise, either the $1's$ positioned at squares $(a', b_1 - 1), (a_2, b_2), \cdots, (a_k, b_k)$ or those positioned at squares $(a', b_1 - 1), (a_2, b_2), \cdots, (a_k, b_k)$ will form a F_k in $\Psi^{n-1}(L)$, hence contradicting the selection of (a_1, b_1). From the definition of Ψ, it follows that $\Psi^n(L)$ is a transversal in which squares (a_1, b_k), (a_k, b_{k-1}), $\ldots$, (a_2, b_1) are filled with $1's$ in and all the other rows and columns are the same as $\Psi^{n-1}(L)$. In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c - 1, c, c + 1$ belongs to the set $\{b_1, b_2, \ldots, b_k\}$.

Subcase 1.1. $c = b_t$ for some integer $1 < t \leqslant k$: It is easily seen that the square $(r_1, c-1)$ is also filled with a 1 in $\Psi^{n-1}(L)$. Moreover, we have $r_2 = a_{t+1}$ for $t < 1$ and $r_2 = a_1$ otherwise. Recall that the square (a_t, b_t) is filled with a 1 in $\Psi^{n-1}(L)$. Since $b_t \in D$, by the induction hypothesis we have $r_1 < a_t$. Thus we have $r_1 < a_t < a_{t+1} < a_1$. Thus we deduce that $r_1 < r_2$. It remains to show that $r_2 > r_3$. We have two cases. If $c + 1 = b_{t-1}$, then we have $r_3 = a_t$, which implies that $r_3 < r_2$. If $c + 1 \neq b_{t-1}$, then the square $(r_3, c+1)$ is also filled with a 1 in $\Psi^{n-1}(L)$. Since $b_t \in D$, by the induction hypothesis we have $a_t > r_3$. Thus we have $r_2 = a_{t+1} > a_t > r_3$.

Subcase 1.2. $c + 1 = b_t$ and $c \neq b_{t+1}$ for some integer $1 < t < k$: It is easily seen that $r_3 = a_{t+1}$ and the square (r_2, c) is also filled with a 1 in $\Psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $r_2 > a_t$. We claim that $r_2 > a_{t+1}$. If not, then suppose that $r_2 < a_{t+1}$. Then the $1's$ positioned at squares $(a_1, b_1), \ldots, (a_{t-1}, b_{t-1})$, $(r_2, b_t - 1)$, $(a_{t+1,b_{t+1}}), \ldots, (a_k, b_k)$ will form a F_k in $\Psi^{n-1}(L)$, which contradicts the selection of (a_t, b_t).

Thus we have $r_2 > r_3$. It remains to show that $r_1 < r_2$. We have three cases. (i) If $c-1 \neq b_{t+1}$, then the square $(r_1, c-1)$ is also filled with a 1 in $\Psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $r_1 < r_2$. (ii) If $c-1 = b_{t+1}$ and $t < k-1$, then $r_1 = a_{t+2}$. We claim that $r_1 < r_2$. If not, suppose that $a_{t+2} > r_2$. Since $r_2 > a_{t+1}$, we will get a contradiction with the selection of (a_{t+1}, b_{t+1}). (iii) If $c-1 = b_k$, then $r_1 = a_1$. We claim that $a_1 < r_2$. If not, suppose that $a_1 > r_2$. Recall that the squares (r_2, c) and $(a_k, c-1)$ are filled with $1's$ in $\Psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $r_2 > a_k$. Thus the $1's$ positioned at squares $(a_1, b_1), (a_3, b_3), \ldots, (a_k, b_k)(r_2, c)$ will form a F_k in $\psi^{n-1}(L)$, which contradicts the selection of (a_2, b_2). Thus we have $r_1 = a_1 < r_2$.

Subcase 1.3. $c+1 = b_k$: It is easy to check that $r_3 = a_1$ and the squares $(r_1, c-1)$ and (r_2, c) are also filled with $1's$ in $\Psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $r_1 < r_2$ and $r_2 > a_k$. We claim that $r_2 > r_3$. If not, then suppose that $r_2 < a_1$. Then the $1's$ positioned at squares $(a_1, b_1), (a_3, b_3), \ldots, (a_k, b_k)(r_2, b_k - 1)$ will form a F_k in L, which contradicts the selection of (a_2, b_2). Thus we have $r_2 > a_1 = r_3$.

Subcase 1.4. $c-1 = b_t$ and $c+1 \neq b_{t-1}$ for some integer $1 < t \leqslant k$: It is easy to check that the squares (r_2, c) and $(r_3, c+1)$ are also filled with $1's$ in $\psi^{n-1}(L)$. Moreover, we have $r_1 = a_{t+1}$ when $t < k$ and $r_1 = a_1$ otherwise. Since $c \in D$, by the induction hypothesis, we have $a_t < r_2 > r_3$. We claim that $r_2 > r_1$. If not, then suppose that $r_2 < a_{t+1} = r_1$ when $t < k$ and $r_2 < a_1$ otherwise. Then the $1's$ positioned at squares $(a_1, b_1), (a_2, b_2), \ldots, (a_{t-1}, b_{t-1}), (r_2, b_t + 1), (a_{t+1}, b_{t+1}), \ldots, (a_k, b_k)$ form a F_k in $\Psi^{n-1}(L)$ in the former case, while those positioned at squares $(a_1, b_1), (a_2, b_2), \ldots, (a_{k-1}, b_{k-1})(r_2, c)$ form a F_k in $\Psi^{n-1}(L)$ in the latter case. Both of them contradict the selection of (a_t, b_t). Thus we conclude that $r_1 < r_2$.

Case 2. $b_1 \in D$ and $a' < a''$: Let us first recall the procedure of constructing $\Psi^n(L)$ from $\Psi^{n-1}(L)$. First we get a transversal L' in which (a'', b_1) and $(a_1, b_1 + 1)$ are filled with $1's$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. Then, we apply ψ to L' to get $\Psi^n(L)$. By Lemma 3.14, there is no J_k above the row a_1 in $\Psi^{n-1}(L)$. Thus we have $a'' > a' \geqslant a_2$.

Subcase 2.1. $a'' < a_3$: We claim that when we apply ψ to L' the $1's$ selected are positioned at squares $(a_1, b_1 + 1)$, (a'', b_1), (a_3, b_3), $\ldots$, (a_k, b_k). If not, suppose that G is a copy of F_k selected in the application of ψ to L' such that at least one of the squares $(a_1, b_1 + 1)$, (a'', b_1), (a_2, b_2), (a_3, b_3), $\ldots$, (a_k, b_k) does not fall in G. We label the $1's$ of G from right to left by $g_1, g_2, \ldots, g_k$. Obviously, g_1 and g_2 are positioned at the squares $(a_1, b_1 + 1)$ and (a'', b_1), respectively. When $a' = a_2$, let p be the least integer such that g_p is not positioned at (a_p, b_p) for $p \geqslant 3$. Clearly, g_p is below row a_p. Then the $1's$ positioned at the squares (a_1, b_1), (a_2, b_2) , $\ldots$, (a_{p-1}, b_{p-1}) combining with $g_p, \ldots, g_k$, form a F_k in $\Psi^{n-1}(L)$. This contradicts the selection of (a_p, b_p). When $a' > a_2$, then the $1's$ positioned at the squares (a_1, b_1) and $(a', b_1 - 1)$, combining $g_3, \ldots, g_k$, form a F_k in $\Psi^{n-1}(L)$, hence contradicting the selection of (a_2, b_2).

By the definition of Ψ, the transversal $\Psi^n(L)$ is a transversal in which squares (a_1, b_k), (a_k, b_{k-1}), $\ldots$, (a_4, b_3), (a_3, b_1) are filled with $1's$ in $\Psi^n(L)$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c-1, c, c+1$ belongs to the set $\{b_1, b_3, \ldots, b_k\}$. Here we only consider the case when $c = b_1$. The other cases can be treated similarly as Case 1. Let $c = b_1$. It is easy to check that the square $(a', b_1 - 1)$ is also filled with a 1 in $\Psi(L)$. Thus we have $r_1 = a'$. Note that $r_2 = a_3$ and $r_3 = a''$. Since $a' < a'' > a_3$, we have $r_1 < r_2 > r_3$.

Subcase 2.2. $a'' > a_3$: Recall that the squares (a'', b_1) and $(a_1, b_1 + 1)$ are filled with $1's$ in L' and all the other rows and columns are the same as $\psi^{n-1}(L)$.

We claim that the selected $1's$ of a copy of F_k are positioned at squares $(a_1, b_1 + 1)$, (a_2, b_2), (a_3, b_3), ..., (a_k, b_k) in the procedure of applying ψ to L'. If not, suppose that G is a copy of F_k when we apply the map ψ to L' such that at least one of the squares $(a_1, b_1 + 1)$, (a_2, b_2), (a_3, b_3), ..., (a_k, b_k) does not fall in G. We label the $1's$ of G from right to left by $g_1, g_2, \ldots, g_k$. Thus g_1 and g_2 must be positioned at the squares (a_1, b_1+1) and (a'', b_1). Since $a'' > a_3$, thus g_3 must be below row a_3. Since there is no J_k above row a_1 in $\Psi^{n-1}(L)$, we have $a' \geqslant a_2$. When $a' = a_2$, then the $1's$ positioned at the squares (a_1, b_1) and (a_2, b_2), combining with $g_3, \ldots, g_k$, form a F_k in $\Psi^{n-1}(L)$. This contradicts the selection of (a_3, b_3). When $a' > a_2$, then the $1's$ positioned at the squares (a_1, b_1) and $(a', b_1 - 1)$, combining $g_3, \ldots, g_k$, form a F_k in $\Psi^{n-1}(L)$. This also contradicts the selection of (a_2, b_2).

By the definition of Ψ, the squares (a_1, b_k), (a_k, b_{k-1}), ..., $(a_2, b_1 + 1)$, (a'', b_1) are filled with $1's$ in $\Psi^n(L)$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c-1, c, c+1$ belongs to the set $\{b_1 + 1, b_1, b_2, \ldots, b_k\}$. Here we only consider the case when $c = b_1$ or $c - 1 = b_1 + 1$. The discussion for other cases is the same as Case 1.

Suppose that $c = b_1$. It is easy to check that $r_2 = a''$ and $r_3 = a_2$. Since $a'' > a_3 > a_2$, we have $r_2 > r_3$. Next we aim to show that $r_1 < r_2$. We have two cases. (i) If $b_1 - 1 = b_2$, then we have $r_1 = a_3$, which implies that $r_1 < r_2$. (ii) If $b_1 - 1 \neq b_2$, then the square $(r_1, c-1)$ is also filled with a 1 in $\Psi^{n-1}(L)$. Thus we have $r_1 = a'$. Since $a' < a''$, we deduce that $r_1 < r_2$.

Suppose that $c-1 = b_1+1$. Obviously, the squares (r_2, c) and $(r_3, c+1)$ are also filled with $1's$ in $\psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $r_2 > r_3$. We claim that $a_2 < r_2$. Otherwise, the squares (a_k, b_k), $(a_{k-1}, b_{k-1}), \ldots, (a_2, b_2), (r_2, c)$ will form a J_k above row a_1. This contradicts the fact that there is no J_k above row a_1 in $\Psi^{n-1}(L)$. Thus we have $r_1 = a_2 < r_2$.

Case 3. $b_1 \in D$ and $a' > a''$: Let us first recall the procedure of constructing $\Psi^n(L)$ from $\Psi^{n-1}(L)$. First we get a transversal L' in which $(a_1, b_1 - 1)$ and (a', b_1) are filled with $1's$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. Then, we apply ψ to L' to get $\Psi^n(L)$. By Lemma 3.14, there is no J_k above the row a_1 in $\Psi^{n-1}(L)$. Thus we have $a' \neq a_2$.

We claim that when we apply ψ to L', the selected $1's$ are positioned at squares $(a_1, b_1 - 1)$, (a_2, b_2), (a_3, b_3), ..., (a_k, b_k). If not, suppose that G is a copy of F_k when we apply the map ψ to L' such that at least one of the squares $(a_1, b_1 - 1)$, (a_2, b_2), (a_3, b_3), ..., (a_k, b_k) does not fall in G. We label the $1's$ of G from right to left by $g_1, g_2, \ldots, g_k$.

According to the choice of (a_1, b_1), we have that g_1 must be positioned at $(a_1, b_1 - 1)$. Let p be the least integer such that g_p is not positioned at (a_p, b_p). It follows that g_p is below row a_p. Thus the $1's$ positioned at the squares (a_1, b_1), (a_2, b_2), …, (a_{p-1}, b_{p-1}), combining $g_p, \ldots, g_k$ will form a F_k in $\Psi^{n-1}(L)$, hence contradicting the selection of (a_p, b_p).

By the definition of Ψ, the squares (a_1, b_k), (a_k, b_{k-1}), …, $(a_2, b_1 - 1)$, (a', b_1) are filled with $1's$ in $\Psi^n(L)$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. In order to show that $r_1 < r_2 > r_3$, it suffices to consider the case when at least one of $c-1, c, c+1$ belongs to the set $\{b_1, b_1 - 1, b_2, \ldots, b_k\}$. Here we only consider the case when $c = b_1$ or $c + 1 = b_1 - 1$. The other cases can be treated similarly as Case 1.

Suppose that $c = b_1$. It is easy to check that $r_2 = a'$ and $r_3 = a''$ and $r_1 = a_2$. Since $a' > a''$, we have $r_2 > r_3$. By Lemma 3.14, there is no no J_k above row a_1 in $\Psi^{n-1}(L)$. Thus we have $a' > a_2$. This implies that $r_1 < r_2$.

Suppose that $c + 1 = b_1 - 1$. It is easy to check that $r_3 = a_2$. By Lemma 3.14, there is no J_k above row a_1 in $\Psi^{n-1}(L)$. Thus we have $b_2 \notin D$ and $a_2 < a'$. Thus, the square (r_2, c) is also filled with a 1 in $\psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $r_2 > a'$. Thus we have $r_2 > a' > a_2 = r_3$. Next we proceed to show that $r_1 < r_2$. We have two cases. If $c-1 \neq b_2$, then the square $(r_1, c-1)$ is also filled with a 1 in $\Psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $r_1 < r_2$. If $c - 1 = b_2$, then we have $r_1 = a_3$. Recall that the squares $(a_2, c-1)$ and (r_2, c) are filled with $1's$ in $\Psi^{n-1}(L)$. Since $c \in D$, by the induction hypothesis we have $a_2 < r_2$. We claim that $r_1 < r_2$. If not, then the $1's$ positioned at squares $(a_1, b_1), (r_2, c), (a_3, b_3), \ldots, (a_k, b_k)$ will form a F_k in $\Psi^{n-1}(L)$, which contradicts the selection of (a_2, b_2). Thus we conclude that $r_1 < r_2$.

This completes the proof. □

Proof of Theorem 3.15. By Theorems 4.1 and 4.2, it suffices to show that Φ^* and Ψ^* are inverses of each other. First, we aim to show $\Psi(\Phi^n(L)) = \Phi^{n-1}(L)$ for any $L \in \mathcal{T}_\lambda^D(F_k)$. Assume that $\Phi^n(L) \neq \Phi^{n-1}(L)$.

Suppose that at the nth application of Φ to $\Phi^{n-1}(L)$ we select a copy of J_t in $\Phi^{n-1}(L)$. Assume that the selected $1's$ are positioned at the squares (a_1, b_1), $(a_2, b_2), \ldots, (a_k, b_k)$, where $b_1 < b_2 < \ldots < b_k$.

By Lemmas 3.8 and 3.12, we have $\psi(\phi(\Phi^{n-1}(L))) = \Phi^{n-1}(L)$. In order to show that $\Psi(\Phi^n(L)) = \Phi^{n-1}(L)$, it suffices to show that $\Psi(\Phi^n(L)) = \psi(\phi(\Phi^{n-1}(L)))$. There are four cases to consider.

Case 1. The square (a_1, b_k) is filled with a 1 in $\Phi^n(L)$: In this case, we have $\Phi(\Phi^{n-1}(L)) = \phi(\Phi^{n-1}(L))$. This implies that the squares $(a_1, b_k), (a_2, b_1), \ldots, (a_k, b_{k-1})$ are filled with $1's$ in $\Phi^n(L)$ and the other rows and columns are the same as $\Phi^{n-1}(L)$. By Lemma 3.12, there is no F_k with at least one square below row a_1 in $\Phi^{n-1}(L)$. Thus, Lemmas 3.3 and 3.4 guarantee that when we apply Ψ to $\Phi^n(L)$, the selected $1's$ of a copy of F_k are positioned at the squares $(a_1, b_k), (a_2, b_1), \ldots, (a_k, b_{k-1})$. We claim that $b_k \notin D$. If not, suppose that $b_k \in D$ and the square $(r, b_k + 1)$ is filled with a 1 in $\Phi^{n-1}(L)$. By Theorem 4.1, the transversal $\Phi^{n-1}(L)$ is valid. This implies that $a_k > r$. Then the squares $(a_2, b_2), \ldots (a_k, b_k), (r, b_k + 1)$ will form a J_k in $\Phi^{n-1}(L)$. This contradicts the selection of (a_1, b_1). Thus we conclude that $b_k \notin D$. According to the definition of Ψ, we have $\Psi(\Phi^n(L)) = \psi(\Phi^n(L)) = \psi(\phi(\Phi^{n-1}(L)))$.

Case 2. The square $(a_1, b_k - 1)$ is filled with a 1 in $\Phi^n(L)$ and $b_{k-1} = b_k - 1$: From the definition of Φ, we have $b_k - 1 \in D$. Moreover, the squares (a_2, b_1), …, (a_{k-1}, b_{k-2}), $(a_1, b_k - 1)$ are filled with $1's$ in $\Phi^n(L)$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Note that the squares (a_2, b_1), …, $(a_{k-1}, b_{k-2}), (a_k, b_k - 1)$, (a_1, b_k) are filled with $1's$ in $\phi(\Phi^{n-1}(L))$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Suppose that the square $(a', b_k - 2)$ is filled with a 1 in $\Phi^n(L)$. Since $b_k - 1 \in D$, we have $b_k - 2 \neq b_{k-2}$. This implies that the square $(a', b_k - 2)$ is also filled with a 1 in both $\Phi^{n-1}(L)$ and $\phi(\Phi^{n-1}(L))$. By Theorem 4.1, the transversal $\Phi^{n-1}(L)$ is valid. Thus we have $a' < a_{k-1}$. Lemmas 3.3, 3.4, 3.11 and 3.12 ensure that when we apply the map Ψ to $\Phi^n(L)$, the selected $1's$ of a copy of F_k are positioned at the squares (a_2, b_1), …, $(a_{k-1}, b_{k-2}), (a', b_k - 2)$, $(a_1, b_k - 1)$. By Lemma 3.11, there is no J_k above row a_1 in $\Phi^n(L)$. This implies that $a' < a_k$. Recall that the squares (a_k, b_k) and $(a', b_k - 2)$ are filled with a $1's$ in $\Phi^n(L)$, we have $\Psi(\Phi^n(L)) = \psi(\phi(\Phi^{n-1}(L)))$.

Case 3. The square $(a_1, b_k - 1)$ is filled with a 1 in $\Phi^n(L)$ and $b_{k-1} \neq b_k - 1$: Suppose that $(a', b_k - 1)$ is filled with a 1 in $\Phi^{n-1}(L)$. From the definition of Φ, we have $b_k - 1 \in D$. Moreover, the squares (a_2, b_1), …, (a_{k-1}, b_{k-2}), (a_k, b_{k-1}), $(a_1, b_k - 1)$ and (a', b_k) are filled with $1's$ in $\Phi^n(L)$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Note that the squares (a_2, b_1), …, $(a_{k-1}, b_{k-2}), (a_k, b_{k-1})$, (a_1, b_k) are filled with $1's$ in $\phi(\Phi^{n-1}(L))$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Lemmas 3.3, 3.4 and 3.12 guarantee that when we apply the map Ψ to $\Phi^n(L)$, the selected $1's$ of a copy of F_k are positioned at the squares (a_2, b_1), …, (a_{k-1}, b_{k-2}), (a_k, b_{k-1}), $(a_1, b_k - 1)$.

Next we aim to show that $\Psi(\Phi^n(L)) = \psi(\phi(\Phi^{n-1}(L)))$. We have two cases. If $b_{k-1} = b_k - 2$, then we have $a_k < a'$ since $b_k - 1 \in D$. Recall that the squares (a', b_k) and $(a_k, b_k - 2)$ are filled with $1's$ in $\Phi^n(L)$. By the definition of Ψ, we have $\Psi(\Phi^n(L)) = \psi(\phi(\Phi^{n-1}(L)))$. If $b_{k-1} \neq b_k - 2$, then suppose that $(a'', b_k - 2)$ is filled with a 1 in $\Phi^n(L)$. Then the square $(a'', b_k - 2)$ is also filled with a 1 in $\Phi^{n-1}(L)$. By Theorem 4.1, the transversal $\Phi^{n-1}(L)$ is valid. Since $b_k - 1 \in D$, we have $a'' < a'$. Recall that the squares $(a'', b_k - 2)$ and (a', b_k) are filled with $1's$ in $\Phi^n(L)$. Thus, by the definition of Ψ, we have $\Psi(\Phi^n(L)) = \psi(\phi(\Phi^{n-1}(L)))$.

Case 4. The square $(a_1, b_k + 1)$ is filled with a 1 in $\Phi^n(L)$: Suppose that $(a', b_k + 1)$ and $(a'', b_k + 2)$ are filled with $1's$ in $\Phi^{n-1}(L)$. From the definition of Φ, we have $b_k + 1 \in D$. Moreover, the squares (a_2, b_1), …, (a_{k-1}, b_{k-2}), (a_k, b_{k-1}), $(a_1, b_k + 1)$ and (a', b_k) are filled with $1's$ in $\Phi^n(L)$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Note that the squares (a_2, b_1), …, $(a_{k-1}, b_{k-2}), (a_k, b_{k-1})$, (a_1, b_k) are filled with $1's$ in $\phi(\Phi^{n-1}(L))$ and all the other rows and columns are the same as $\Phi^{n-1}(L)$. Lemmas 3.3, 3.4 and 3.12 guarantee that when we apply the map Ψ to $\Phi^n(L)$, the selected $1's$ of a copy of F_k are positioned at squares (a_2, b_1), …, (a_{k-1}, b_{k-2}), (a_k, b_{k-1}), $(a_1, b_k + 1)$. By Theorem 4.1, the transversal $\Phi^{n-1}(L)$ is valid. Since $b_k + 1 \in D$, we have $a' > a''$. Recall that the squares (a', b_k) and $(a'', b_k + 2)$ are filled with $1's$ in $\Phi^n(L)$. Thus, by the definition of Ψ, we have $\Psi(\Phi^n(L)) = \psi(\phi(\Phi^{n-1}(L)))$.

Our next goal is to show that $\Phi(\Psi^n(L)) = \Psi^{n-1}(L)$ for any $L \in \mathcal{T}_\lambda^D(J_k)$. Assume that $\Psi^n(L) \neq \Psi^{n-1}(L)$. Suppose that at the nth application of Ψ to $\Psi^{n-1}(L)$, we select a copy of F_k in $\Psi^{n-1}(L)$. Assume that the selected $1's$ of this F_k are positioned at squares (a_1, b_1), (a_2, b_2), $\ldots$, (a_k, b_k), where $b_1 > b_2 \ldots > b_k$. Suppose that the squares $(a', b_1 - 1)$ and $(a'', b_1 + 1)$ are filled with $1's$ in $\Psi^{n-1}(L)$. We consider the following cases.

Case 1. $b_1 \notin D$: According to the selection of (a_1, b_1), we have $a', a'' < a_1$. This implies that $b_1 - 1, b_1 + 1 \notin D$. Since $b_1 \notin D$, we have $\Psi^n(L) = \psi(\Psi^{n-1}(L))$. This implies that the squares (a_1, b_k), (a_k, b_{k-1}), $\ldots$, (a_2, b_1) are filled with $1's$ in $\Psi^n(L)$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. Lemmas 3.6, 3.7 and 3.14 guarantee that when we apply Φ to $\Psi^n(L)$, we will select a copy of J_k whose $1's$ are positioned at squares (a_1, b_k), (a_k, b_{k-1}), $\ldots$, (a_2, b_1). Since $b_1 - 1, b_1 + 1 \notin D$, we have $\Phi(\Psi^n(L)) = \phi(\Psi^n(L)) = \phi(\psi(\Psi^{n-1}(L)))$. By Lemmas 3.9 and 3.14, we have $\phi(\psi(\Psi^{n-1}(L))) = \Psi^{n-1}(L)$.

Case 2. $b_1 \in D$ and $a' < a''$: Let us first recall the procedure of constructing $\Psi^n(L)$ from $\Psi^{n-1}(L)$. First we get a transversal L' in which (a'', b_1) and $(a_1, b_1 + 1)$ are filled with $1's$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. Then, we apply ψ to L' to get $\Psi^n(L)$. Lemma 3.14 ensure there is no J_k above row a_1 in $\Psi^{n-1}(L)$, which implies that $a'' > a' \geqslant a_2$. Recall that in the proof of Theorem 4.2 we have proved that if $a'' < a_3$, the $1's$ selected are positioned at squares (a_1, b_1+1), (a'', b_1), (a_3, b_3), $\ldots$, (a_k, b_k) in the application of ψ to L'. Moreover, if $a'' > a_3$, the selected $1's$ are positioned at squares $(a_1, b_1 + 1)$, (a_2, b_2), (a_3, b_3), $\ldots$, (a_k, b_k) in the procedure of applying ψ to L' .

By the definition of Ψ, we have $\Psi^n(L) = \psi(L')$. Thus, when $a'' < a_3$, $\Psi^n(L)$ is a transversal in which the squares $(a'', b_1 + 1)$, (a_3, b_2), (a_4, b_3), $\ldots$, (a_1, b_k) are filled with $1's$ and all the rows and columns are the same as L'. when $a'' > a_3$, $\Psi^n(L)$ is a transversal in which the squares $(a_2, b_1 + 1)$, (a_3, b_2), (a_4, b_3), $\ldots$, (a_1, b_k) are filled with $1's$ and all the rows and columns are the same as L'. Since there is no J_k above row a_1 in L' and $\Psi^n(L) = \psi(L')$, by Lemmas 3.6 and 3.7, when we apply Φ to $\Psi^n(L)$ we select a copy of J_k which is just created by the application of ψ to L'. Thus we have $\phi(\Psi^n(L)) = \phi(\psi(L')) = L'$. By the definition of Φ, we have $\Phi(\Psi^n(L)) = \Psi^{n-1}(L)$.

Case 3. $b_1 \in D$ and $a' > a''$: Since there is no J_k above row a_1 in $\Psi^{n-1}(L)$ according to Lemma 3.14, we have $b_1 - 1 \neq b_2$. Let us describe the procedure of constructing $\Psi^n(L)$ from $\Psi^{n-1}(L)$. First we get a transversal L' in which the squares $(a_1, b_1 - 1)$ and (a', b_1) are filled with $1's$ and all the other rows and columns are the same as $\Psi^{n-1}(L)$. Recall that we have shown that the $1's$ selected in the application of ψ to L' are positioned at squares $(a_1, b_1 - 1)$, (a_2, b_2), (a_3, b_3), $\ldots$, (a_k, b_k) in the proof of Theorem 4.2. Since there is no J_k above row a_1 in L' and $\Psi^n(L) = \psi(L')$, by Lemmas 3.6 and 3.7, when we apply Φ to $\Psi^n(L)$ we select a copy of J_k which is just created by the application of ψ to L'. That is, these $1's$ are positioned at squares $(a_2, b_1 - 1)$, (a_3, b_2), (a_4, b_3), $\ldots$, (a_1, b_k). Thus we have $\phi(\Psi^n(L)) = \phi(\psi(L')) = L'$. Hence we have $\Phi(\Psi^n(L)) = \Psi^{n-1}(L)$.

This completes the proof. □

Acknowledgments

The author would like to thank the referee for helpful suggestions. This work was supported by the National Natural Science Foundation of China(No. 10901141).

References

[1] E. Babson and J. West, The permutation $123p_4 \dots p_t$ and $321p_4 \dots p_t$ are Wilf-eqivalent. *Graphs Combin.*, 16:373–380, 2001.

[2] J. Backelin, J. West, and G. Xin, Wilf-equivalence for singleton classes. *Adv. Appl. Math.*, 38:133–148, 2007.

[3] M. Bousquet-Mélou, E. Steingrímsson, Decreasing subsequences in permutations and Wilf equivalence for involutions. *J. Algebraic Combin.*, 22:383–409, 2005.

[4] M. Bóna, Combinatorics of Permutations. CRC Press, 2004.

[5] M. Bóna, On a family of conjectures of Joel Lewis on alternating Permutations. *Graphs Combin.*, 2013. doi:10.1007/s00373-013-1291-2.

[6] J. N. Chen, W. Y. C. Chen and R. D. P. Zhou, On pattern avoiding alternating permutations, arXiv:1212.2697v1.

[7] N. Gowravaram and R. Jagadeesan, Beyond alternating permutations: Pattern avoidance in Young diagrams and tableaux, arXiv:1301.6796v1.

[8] S. Kitaev, Patterns in permutations and words. Springer Verlag (EATCS monographs in Theoretical Computer Science book series), 2011.

[9] J. B. Lewis, Pattern avoidance for alternating permutations and Young tableaux. *J. Combin. Theory Ser. A*, 118:1436–1450, 2011.

[10] J. B. Lewis, Generating trees and pattern avoidance in alternating permutations. *Electronic J. Combin.*, 19(1):P21, 2012.

[11] T. Mansour, Restricted 132-alternating permutations and Chebyshev polynomials. *Ann. Combin.*, 7:201–227, 2003.

[12] R. P. Stanley, Enumerative Combinatorics, Volume 2. Cambridge University Press, 2001.

[13] R. P. Stanley, Catalan addendum. http://www-math.mit.edu/~rstan/ec/catadd.pdf, 2012.

[14] Y. Xu and S. H. F. Yan, Alternating permutations with restrictions and standard Young tableaux. *Electronic J. Combin.*, 19(2):P49, 2012.