The Ramsey Number $R(3, K_{10} - e)$ and Computational Bounds for $R(3, G)$

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Abstract

Using computer algorithms we establish that the Ramsey number $R(3, K_{10} - e)$ is equal to 37, which solves the smallest open case for Ramsey numbers of this type. We also obtain new upper bounds for the cases of $R(3, K_k - e)$ for $11 \leqslant k \leqslant 16$, and show by construction a new lower bound $55 \leqslant R(3, K_{13} - e)$.

The new upper bounds on $R(3, K_k - e)$ are obtained by using the values and lower bounds on $e(3, K_l - e, n)$ for $l \leqslant k$, where $e(3, K_k - e, n)$ is the minimum number of edges in any triangle-free graph on n vertices without $K_k - e$ in the complement. We complete the computation of the exact values of $e(3, K_k - e, n)$ for all n with $k \leqslant 10$ and for $n \leqslant 34$ with $k = 11$, and establish many new lower bounds on $e(3, K_k - e, n)$ for higher values of k.

Using the maximum triangle-free graph generation method, we determine two other previously unknown Ramsey numbers, namely $R(3, K_{10} - K_3 - e) = 31$ and $R(3, K_{10} - P_3 - e) = 31$. For graphs G on 10 vertices, besides $G = K_{10}$, this leaves 6 open cases of the form $R(3, G)$. The hardest among them appears to be $G = K_{10} - 2K_2$, for which we establish the bounds $31 \leqslant R(3, K_{10} - 2K_2) \leqslant 33$.

Keywords: Ramsey number; triangle-free graphs; almost-complete graphs; computation

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1 Notation and Preliminaries

Notation, definitions and tools of this work are analogous to those in our recent study of the classical two-color Ramsey numbers $R(3,k)$ [10], where $R(3,k)$ is defined as the smallest m such that no m-vertex triangle-free graph with independence number less than k exists. Using coloring terminology, such graphs can be seen as 2-colorings of the edges of K_m, which have no triangles in the first color and no monochromatic K_k in the second color. In this paper we study mainly the case when $K_k - e$, the complete graph of order k with one missing edge, is avoided in the second color.

Let J_k denote the graph $K_k - e$, and hence the Ramsey number $R(K_3, J_k)$ is the smallest m such that every triangle-free graph on m vertices contains J_k in the complement. Similarly as in [10], a graph F will be called a $(G, H; n, e)$-graph, if $|V(F)| = n, |E(F)| = e$, F does not contain G, and $\overline{F}$ (i.e. the complement of F) does not contain H. By $\mathcal{R}(G, H; n, e)$ we denote the set of all $(G, H; n, e)$-graphs. We will often omit the parameter e, or both e and n, or give some range to either of these parameters, when referring to special $(G, H; n, e)$-graphs or sets $\mathcal{R}(G, H; n, e)$. If G or H is the complete graph K_k, we will often write k instead of G. For example, a $(3, J_k; n)$-graph F is a $(K_3, K_k - e; n, e)$-graph for $e = |E(F)|$.

In the remainder of this paper we will study only triangle-free graphs, and mainly K_k or J_k will be avoided in the complement. Note that for any $G \in \mathcal{R}(3, J_k)$ we have $\Delta(G) < k$, since all neighborhoods of vertices in G are independent sets. $e(3, J_k, n)$ $(= e(K_3, J_k, n))$ is defined as the smallest number of edges in any $(3, J_k; n)$-graph. The sum of the degrees of all neighbors of a vertex v in G will be denoted by $Z_G(v)$. Similarly as in [17, 18, 23], one can easily generalize the tools used in analysis of the classical case $R(3,k)$ [10, 11, 12, 20, 21], as described in the sequel.

Let G be a $(3, J_k; n, e)$-graph. For any vertex $v \in V(G)$, we will denote by G_v the graph induced in G by the set $V(G) \setminus (N_G(v) \cup \{v\})$. Note that if $d = \deg_G(v)$, then G_v is a $(3, J_{k-1}; n - d - 1, e - Z_G(v))$-graph. This also implies that

$$\gamma(v) = \gamma(v, k, G) = e - Z_G(v) - e(3, J_{k-1}, n - d - 1) \geqslant 0, \tag{1}$$

where $\gamma(v)$ is the so called *deficiency* of vertex v (as in [11]). Finally, the deficiency of the graph G is defined as

$$\gamma(G) = \sum_{v \in V(G)} \gamma(v, k, G) \geqslant 0. \tag{2}$$

The condition that $\gamma(G) \geqslant 0$ is often sufficient to derive good lower bounds on $e(3, J_k, n)$, though a stronger condition that all summands $\gamma(v, k, G)$ of (2) are non-negative sometimes implies better bounds. It is easy to compute $\gamma(G)$ just from the degree sequence of G [11, 12, 17]. If n_i is the number of vertices of degree i in a $(3, J_k; n, e)$-graph G, then

$$\gamma(G) = ne - \sum_i n_i\big(i^2 + e(3, J_{k-1}, n-i-1)\big) \geqslant 0, \tag{3}$$

where $n = \sum_{i=0}^{k-1} n_i$ and $2e = \sum_{i=0}^{k-1} i n_i$.

We obtain a number of improvements on lower bounds for $e(3, J_k, n)$ and upper bounds for $R(3, J_k)$, summarized at the end of the next section. The main computational result of this paper solves the smallest open case for the Ramsey numbers of the type $R(3, J_k)$, namely we establish that $R(3, J_{10}) = 37$ by improving the previous upper bound $R(3, J_{10}) \leqslant 38$ [17] by one.

Section 3 describes how the algorithms of this work differed from those used by us in the classical case [10], and how we determined two other previously unknown Ramsey numbers, namely $R(3, K_{10}-K_3-e) = 31$ and $R(3, K_{10}-P_3-e) = 31$, using the maximum triangle-free graph generation method. Section 4 presents progress on $e(3, J_k, n)$ and $R(3, J_k)$ for $k \leqslant 11$, and Section 5 for $k \geqslant 12$.

2 Summary of Prior and New Results

In 1995, Kim [13] obtained a breakthrough result using probabilistic methods by establishing the exact asymptotics for the classical case, namely $R(3, k) = \Theta(n^2/\log n)$. The asymptotic behaviour of $R(3, J_k)$ is clearly the same, since $K_{k-1} \subset J_k \subset K_k$. The monotonicity of $e(3, G, n)$ and Ramsey numbers $R(3, G)$ implies that for all n and k we have

$$e(3, k, n) = e(K_3, K_k, n) \leqslant e(K_3, J_k, n) \leqslant e(K_3, K_{k-1}, n), \tag{4}$$

$$R(3, k-1) = R(K_3, K_{k-1}) \leqslant R(K_3, J_k) \leqslant R(K_3, K_k). \tag{5}$$

For the small cases of $R(3, J_k)$ much of the progress was obtained by deriving and using good lower bounds on $e(3, J_k, n)$. Explicit formulas for $e(3, J_{k+2}, n)$ are known for all $n \leqslant 13k/4 - 1$, and for $n = 13k/4$ when $k = 0 \bmod 4$, as follows:

Theorem 1 ([23, 20]) *For all $n, k \geqslant 1$, for which $e(3, J_{k+2}, n)$ is finite, we have*

$$e(3, J_{k+2}, n) = \begin{cases} 0 & \text{if } n \leqslant k+1, \\ n-k & \text{if } k+2 \leqslant n \leqslant 2k \text{ and } k \geqslant 1, \\ 3n-5k & \text{if } 2k < n \leqslant 5k/2 \text{ and } k \geqslant 3, \\ 5n-10k & \text{if } 5k/2 < n \leqslant 3k \text{ and } k \geqslant 6, \\ 6n-13k & \text{if } 3k < n \leqslant 13k/4-1 \text{ and } k \geqslant 6. \end{cases} \tag{6}$$

Furthermore, $e(3, J_{k+2}, n) = 6n - 13k$ for $k = 4t$ and $n = 13t$, and the inequality $e(3, J_{k+2}, n) \geqslant 6n - 13k$ holds for all n and $k \geqslant 6$. All critical graphs have been characterized whenever the equality in the theorem holds for $n \leqslant 3k$.

Our main focus in this direction is to obtain new exact values and bounds on $e(3, J_{k+2}, n)$ for $n \geqslant 13k/4$. This in turn will permit us to prove the new upper bounds on $R(3, J_k)$, for $10 \leqslant k \leqslant 16$.

The general method we use is first to compute, if feasible, the exact value of $e(3, J_k, n)$, or to derive a lower bound using a combination of equalities (3) and (6), and computations. Better lower bounds on $e(3, J_{k-1}, m), m < n$, often lead to better lower bounds on $e(3, J_k, n)$. If we show that $e(3, J_k, n) = \infty$, then we obtain an upper bound $R(3, J_k) \leqslant n$.

Full enumeration of the sets $\mathcal{R}(3, J_k)$ for $k \leqslant 6$ was completed in [18], all such graphs for $k = 7$ were uploaded by Fidytek at a website [8], and they were confirmed in this work. Radziszowski computed the values of $e(3, J_7, n)$ and $e(3, J_8, n)$ in [18]. Some of the values and bounds for $k = 9$ and $k = 10$, beyond those given by Theorem 1, were obtained by McKay, Piwakowski and Radziszowski in [17]. In this paper we complete this census for all cases of n with $k \leqslant 10$, and give new lower bounds for some higher parameters.

A $(3, J_k; n)$-graph is called *critical* for a Ramsey number $R(3, J_k)$ if $n = R(3, J_k) - 1$. In [17], McKay et al. determined that there are at least 6 *critical* triangle Ramsey graphs for J_9. Using the maximum triangle-free method (see Section 3), we find one more such graph and thus determine that there are exactly 7 critical graphs for $R(3, J_9)$. They can be downloaded from the *House of Graphs* [1] by searching for the keywords "critical ramsey graph for R(3,K9-e)".

There is an obvious similarity between Theorem 1 and the results for $e(3, k, n)$ obtained in [22] as summarized in Theorem 2 in [10], though also note that there are some differences. In particular, various cases are now restricted to $k > c$. The graphs showing that these restrictions are necessary are listed in [23].

Our new results on $R(3, J_k)$ are marked in bold in Table 1, which presents the values and best known bounds on the Ramsey numbers $R(3, J_k)$ and $R(3, K_k)$ for $k \leqslant 16$. The new upper bounds for J_{10} and J_{11} improve on the bounds given in [17] by 1 and 2, respectively. Other upper bounds in bold are recorded for the first time.

k	$R(3, J_k)$	$R(3, K_k)$	k	$R(3, J_k)$	$R(3, K_k)$
3	5	6	10	**37**	40–42
4	7	9	11	42–**45**	47–50
5	11	14	12	47–**53**	52–59
6	17	18	13	**55**–**62**	59–68
7	21	23	14	59–**71**	66–77
8	25	28	15	69–**80**	73–87
9	31	36	16	73–**91**	82–98

Table 1: Ramsey numbers $R(3, J_k)$ and $R(3, K_k)$, for $k \leqslant 16$, $J_k = K_k - e$.

The results $R(3, 11) \geqslant 47$ [6] and $R(3, 16) \geqslant 82$ [7] were recently obtained by Exoo. Our recent work [10], after 25 years of no progress, improved the upper bound on $R(3, 10)$

from 43 [21] to 42, similarly as all other upper bounds for $R(3,k)$ in the last column of Table 1. The references for all other bounds and values, and the previous bounds, are listed in [19, 10].

In a related cumulative work, Brinkmann, Goedgebeur and Schlage-Puchta [3] completed the computation of all Ramsey numbers of the form $R(3,G)$ for graphs G on up to 10 vertices, except 10 cases. The exceptions included K_{10}, J_{10}, and 8 other graphs close to K_{10}. The complements of these 10 graphs are depicted in Figure 1. In fact, the authors of [3] showed in their article that the Ramsey number for all of these remaining cases is at least 31. In addition to J_{10}, two of these cases are solved in this work, namely $R(3, K_{10} - K_3 - e) = R(3, K_{10} - P_3 - e) = 31$. Hence for graphs G on 10 vertices, besides $G = K_{10}$, this leaves 6 other open cases of the form $R(3,G)$. The hardest among them appears to be $G = K_{10} - 2K_2$, for which we establish the bounds $31 \leqslant R(3, K_{10} - 2K_2) \leqslant 33$.

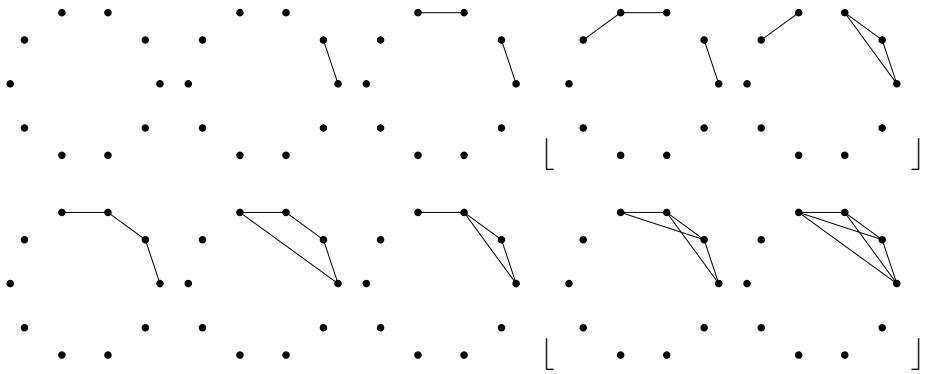

Figure 1: The complements $\overline{G}$ of the 10 remaining graphs of order 10 which have $R(3,G) \geqslant 31$, for which Brinkmann et al. [3] were unable to determine the Ramsey number. Graphs which must have the same Ramsey number are grouped by $\lfloor$ and $\rfloor$ (see [3, 9]).

We note that in three cases of Table 1, namely for $k = 12, 14$ and 16, the best known lower bounds for $R(3,J_k)$ are the same as for $R(3,K_{k-1})$, and thus likely they can be improved. Our new bounds on $R(3,J_k)$ are summarized in the following theorem.

Theorem 2 (a) $R(3,J_{10}) = 37$, (b) $R(3,J_{11}) \leqslant 45$, $R(3,J_{12}) \leqslant 53$, $R(3,J_{13}) \leqslant 62$, $R(3,J_{14}) \leqslant 71$, $R(3,J_{15}) \leqslant 80$, $R(3,J_{16}) \leqslant 91$, *and* (c) $R(3,J_{13}) \geqslant 55$.

Proof. The lower bound $R(3,J_{10}) \geqslant 37$ was established in [17]. The remaining sections describe our computational methods, and intermediate values and bounds on $e(3,J_k,n)$. These imply the upper bound for (a) in Section 4, and the bounds for (b) in Section 5. Result (c) follows from a circular $(3,J_{13};54)$-graph which we constructed. It has arc distances $\{2,3,9,16,20,24\}$. Note that all of these distances have a nontrivial `gcd` with 54, and thus this graph cannot be transformed to an isomorphic circulant by a modular multiplier, so that 1 appears as one of the distances. □

Problem (Erdős-Sós, 1980 [4, 5]) Let $\Delta_k = R(3,k) - R(3,k-1)$. Is it true that

$$\Delta_k \stackrel{k}{\to} \infty \ ? \qquad \Delta_k/k \stackrel{k}{\to} 0 \ ?$$

Only easy bounds $3 \leqslant \Delta_k \leqslant k$ are known. The results of this paper don't give any general improvement on the bounds for Δ_k, however we note that better understanding of the behavior of $R(3,J_k)$ relative to $R(3,K_k)$ may lead to such improvements since

$$\Delta_k = \big(R(3,K_k) - R(3,J_k)\big) + \big(R(3,J_k) - R(3,K_{k-1})\big). \tag{7}$$

The new results on $R(3,G)$ for some of the open cases listed in Figure 1 are as follows.

Theorem 3 (a) $31 \leqslant R(3, K_{10} - 2K_2) \leqslant 33$,
(b) $R(3, K_{10} - K_3 - e) = R(3, K_{10} - P_3 - e) = 31$.

Proof. The lower bound of 31 for each of the three cases was established in [3].

The upper bound of 33 for (a) was obtained using essentially the same method as the one used in the main case of this paper for J_{10} (see Section 3 for more details). This improves over the trivial bound of 37 implied by Theorem 2(a). Table 14 in Appendix 1 contains information about data used to derive the new bound: the values of $e(3, K_9 - 2K_2; n)$ and the counts of corresponding graphs. Using only degree sequence analysis and the values of $e(3, K_9 - 2K_2; n)$ one obtains the bound $R(3, K_{10} - 2K_2) \leqslant 35$. Further computations using the neighbour gluing algorithm were required to obtain the upper bound 33.

The computations applying the maximum triangle-free method from [3] were enhanced as described in Section 3, and gave the upper bounds needed for (b). □

Interestingly, contrary to our initial intuition, the case of $K_{10} - 2K_2$ appears to be significantly more difficult than J_{10}. The computational effort which was required to prove $R(3, K_{10} - 2K_2) \leqslant 33$ was similar to the computational effort to prove $R(3, J_{10}) \leqslant 37$, but it looks like it is computationally infeasible to improve the upper bound for $R(3, K_{10} - 2K_2)$ any further by our current algorithms. Our numerous attempts to improve the lower bound failed, and consequently we conjecture that $R(3, K_{10} - 2K_2) = 31$. If true, this would imply that for each of the 6 remaining open cases of G on 10 vertices (except K_{10}), we have $R(3,G) = 31$.

Finally, we would like to note that we performed exhaustive searches for circulant graphs on up to 61 vertices in an attempt to improve lower bounds for $R(3,k)$, $R(3,J_k)$, and for $R(3,G)$ for the remaining graphs of order 10. If any of these lower bounds can still be improved, it must be by using graphs which are not circulant.

3 Algorithms

Similarly as in [10], we use two independent techniques to determine triangle Ramsey numbers: the maximum triangle-free method and the neighborhood gluing extension method. These methods are outlined in the following subsections.

Maximum Triangle-Free Method

Generating maximal triangle-free graphs

A maximal triangle-free graph (in short, an *mtf* graph) is a triangle-free graph such that the insertion of any new edge forms a triangle. It is easy to see that there exists a $(3, J_k; n)$-graph if and only if there is an mtf $(3, J_k; n)$-graph. Brinkmann, Goedgebeur and Schlage-Puchta [3] developed an algorithm to exhaustively generate mtf graphs and mtf Ramsey graphs efficiently. They implemented their algorithm in a program called *triangleramsey* [2]. We refer the reader to [3, 9] for more details about the algorithm. Using this program they determined the triangle Ramsey number $R(3, G)$ of nearly all graphs G of order 10. The complements of the 10 graphs for which they were unable to determine the Ramsey number are shown in Figure 1 in Section 2.

It is computationally infeasible to use *triangleramsey* do determine all mtf $(3, J_{10})$-graphs. However, we executed *triangleramsey* on a large computer cluster and were able to determine all mtf Ramsey graphs for $K_{10}-P_3-e$ up to 31 vertices (where P_k is the path with k vertices). This is one of the remaining graphs whose Ramsey number could not be determined by Brinkmann et al. The new computations took approximately 20 CPU years and the result is that there are 4 mtf Ramsey graphs with 30 vertices for $K_{10} - P_3 - e$ and no mtf Ramsey graphs with 31 vertices. Thus, this proves that $R(3, K_{10} - P_3 - e) = 31$. By monotonicity of Ramsey numbers we have $R(3, K_{10} - K_3 - e) \leqslant R(3, K_{10} - P_3 - e)$, and thus the lower bound of 31 for both cases [3, 9] implies that $R(3, K_{10} - K_3 - e) = R(3, K_{10} - P_3 - e) = 31$.

We also performed sample runs with *triangleramsey* for the other remaining graphs, but it looks like it will be computationally infeasible to complete this task by this method. E.g., we estimate that approximately 144 CPU years would be required to determine $R(3, K_{10} - P_4)$ by running *triangleramsey*, and sample tests for $R(3, K_{10} - K_4)$ indicate that this case will take much longer than 200 CPU years.

Generating complete sets of triangle Ramsey graphs

In order to determine $e(3, J_k, n)$ also non-maximal triangle-free Ramsey graphs are required. Given all mtf $(3, J_k; n)$-graphs, we can obtain all $(3, J_k; n)$-graphs by recursively removing edges in all possible ways and testing if the obtained graphs are still $(3, J_k; n)$-graphs. We used *nauty* [15, 16] to make sure no isomorphic copies are output. We generated, amongst others, the full sets $\mathcal{R}(3, J_9; 28), \mathcal{R}(3, J_9; 29)$ and $\mathcal{R}(3, J_9; 30)$ (see Appendix 1 for detailed results) using this method.

This mtf method is too slow for generating all $(3, J_{10}; n)$-graphs for n which were needed in this work. Nevertheless, we used this method to verify the correctness of our other programs for smaller parameters. The results agreed in all cases in which more than one method was used (see Appendix 2 for more details).

Neighborhood Gluing Extension Method

The main method we used to improve upper bounds for $R(3, J_k)$ is the neighborhood gluing extension method. In this method our extension algorithm takes a $(3, J_k; m)$-graph H as input and produces all $(3, J_{k+1}; n, e)$-graphs G, often with some specific restrictions on n and e, such that for some vertex $v \in V(G)$ graph H is isomorphic to G_v. The program also gets an expansion degree $d = n - m - 1$ as input. Thus it connects, or *glues*, the d neighbors of a vertex v to H in all possible ways. Note that each neighbor of v is glued to an independent set, otherwise the extended graph would contain triangles. Similarly as in [10], various optimizations and bounding criteria are used to speed up the algorithm.

For example, suppose that we are aiming to construct $(3, J_{k+1})$-graphs. Note that the complement of a graph G contains J_k if and only if G contains a spanning subgraph of $\overline{J_k}$ as an induced subgraph. If two neighbors u_1 and u_2 of v have already been connected to independent sets S_1 and S_2 in H and $H[V(H) \setminus (S_1 \cup S_2)]$ contains a spanning subgraph of $\overline{J_{k-1}}$ as induced subgraph, we can abort the recursion, since this cannot yield any $(3, J_{k+1})$-graphs.

There is, however, also one optimization which is specific to J_k. Namely, we do not have to connect the neighbors of v to independent sets S for which $H[V(H) \setminus S]$ induces an independent set of order $k - 1$, since otherwise this graph would contain $\overline{J_{k+1}}$ as induced subgraph (an independent set of order $k - 1$ together with the disjoint edge $\{u, v\}$).

For more details about the general gluing algorithm, we refer the reader to [10, 9].

Most values and new bounds for $e(3, J_k, n)$, which are listed in Section 4, were obtained by the gluing extension method. In Appendix 2 we describe how we tested the correctness of our implementation.

The strategy we used to determine if the parameters of the input graphs to which our extender program was applied are sufficient, i.e. that it is guaranteed that all $(3, J_{k+1}; n, e)$-graphs are generated, is the same as in [10] and is outlined in the next subsection.

Degree Sequence Feasibility

This method is based on the same principles as in the classical case [10, 21, 14]. Suppose we know the values or lower bounds on $e(3, J_k, m)$ for some fixed k and we wish to know all feasible degree sequences of $(3, J_{k+1}; n, e)$-graphs. We construct the system of integer constraints consisting of $n = \sum_{i=0}^{k} n_i$, $2e = \sum_{i=0}^{k} i n_i$, and the inequality (3). If it has no solutions then we can conclude that no such graphs exist. Otherwise, we obtain solutions for n_i's which include all potential degree sequences.

4 Progress on Computing Small $e(3, J_k, n)$

vertices	k													
n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	2													
4	4	2												
5	∞	4	2											
6		6	3	2										
7		∞	6	3	2									
8			8	4	3	2								
9			12	7	4	3	2							
10			15	10	5	4	3	2						
11			∞	14	8	5	4	3	2					
12				18	11	6	5	4	3	2				
13				24	15	9	6	5	4	3	2			
14				30	19	12	7	6	5	4	3	2		
15				35	24	15	10	7	6	5	4	3	2	
16				40	30	20	13	8	7	6	5	4	3	2
17				∞	37	25	16	11	8	7	6	5	4	3
18					43	30	20	14	9	8	7	6	5	4
19					54	37	25	17	12	9	8	7	6	5
20					60	44	30	20	15	10	9	8	7	6
21					∞	51	35	25	18	13	10	9	8	7
22						59	**42**	30	21	16	11	10	9	8
23						70	**49**	35	25	19	14	11	10	9
24						80	**56**	40	30	22	17	12	11	10
25						∞	**65**	46	35	25	20	15	12	11
26							**73**	52	40	30	23	18	13	12
27							**81**	**61**	45	35	26	21	16	13
28							**95**	**68**	51	40	30	24	19	14
29							**106**	**77**	**58**	45	35	27	22	17
30							**117**	**86**	**66**	50	40	30	25	20
31							∞	**95**	**73**	56	45	35	28	23

Table 2: Exact values of $e(3, J_k, n)$, for $3 \leqslant k \leqslant 16$, $3 \leqslant n \leqslant 31$.

Most of the values of $e(3, J_k, n)$ collected in Table 2 are implied by Theorem 1, others were obtained in [18, 17, 23], and those in bold are the result of this work. The bottom-left blank part covers cases where the graphs with corresponding parameters do not exist, while all entries in the top-right blank area indicate 0 edges. By Theorem 1 and [22], for fixed k, $e(3, J_k, n)$ is equal to $e(3, K_{k-1}, n)$ for most small n, and from the data presented in the following it looks like that this equality persists further as n grows. Only sporadic

counterexamples to such behavior for n not much larger than $13k/4$ are known: seven such cases are listed in [23] for $k \leqslant 7$, and another one can be noted in Table 4 for $k = 11, n = 32$. In other words, the second inequality of (4) seems to be much closer to equality than the first, and the opposite seems to hold in (5). If true, we can expect that the first part of Δ_k in (7) is significantly larger than the second part.

Exact values of $e(3, J_9, n)$

The values of $e(3, J_9, \leqslant 21)$ are determined by Theorem 1. The values of $e(3, J_9, n)$ for $22 \leqslant n \leqslant 30$ were obtained by computations, mostly by the gluing extender algorithm which is outlined in Section 3, and they are presented in Table 2. These values improve over previously reported lower bounds [23, 17]. We note that $e(3, J_9, n) = e(3, K_8, n)$ for all $9 \leqslant n \leqslant 26$.

Exact values of $e(3, J_{10}, n)$

The values of $e(3, J_{10}, \leqslant 26)$ are determined by Theorem 1. The values for $27 \leqslant n \leqslant 37$ were obtained by the gluing extender algorithm of Section 3, and they are presented in Table 3. These values improve over previously reported lower bounds [23, 17]. We note that $e(3, J_{10}, n) = e(3, K_9, n)$ for all $10 \leqslant n \leqslant 35$ (see [10]).

n	$e(3, J_{10}, n)$	previous bound/comments
26	52	Theorem 1
27	61	58
28	68	65
29	77	72
30	86	81
31	95	90
32	104	99
33	118	110
34	129	121
35	140	133
36	156	146, maximum 162
37	∞	hence $R(3, J_{10}) \leqslant 37$, Theorem 2(a)

Table 3: Values of $e(3, J_{10}, n)$, for $n \geqslant 26$.

Values and lower bounds on $e(3, J_{11}, n)$

Table 4 presents what we know about $e(3, J_{11}, n)$ beyond Theorem 1, which determines the values of $e(3, J_{11}, \leqslant 28)$. The values and bounds for $29 \leqslant n \leqslant 41$ were obtained by the gluing extender algorithm outlined in Section 3. The lower bounds on $e(3, J_{11}, \geqslant 42)$ are based on solving integer constraints (2) and (3), using the exact values of $e(3, J_{10}, n)$ listed in Table 3. We note that $e(3, J_{11}, n) = e(3, K_{10}, n)$ for all $11 \leqslant n \leqslant 34$, except

for $n = 32$ (see [10]). Four seemingly exceptional $(3, J_{11}; 32, 80)$-graphs can be obtained from the *House of Graphs* [1] by searching for the keywords "exceptional minimal ramsey graph". One of them is the Wells graph, also called the Armanios-Wells graph. It is a double cover of the complement of the Clebsch graph. One of the other special graphs is formed by two disjoint copies of the Clebsch graph itself. This works, since the Clebsch graph is the unique $(3, J_6; 16, 40)$-graph [18] (denoted G_4 in the latter paper).

n	$e(3, J_{11}, n) \geqslant$	comments
28	51	exact, Theorem 1
29	58	exact
30	66	exact
31	73	exact
32	80	exact, $e(3, 10, 32) = 81$
33	90	exact
34	99	exact
35	107	extender
36	117	extender
37	128	extender
38	139	extender
39	151	extender
40	161	extender
41	172	extender
42	185	$e(3, 10, 42) = \infty$
43	201	
44	217	maximum 220
45	∞	hence $R(3, J_{11}) \leqslant 45$, Theorem 2(b)

Table 4: Values and lower bounds on $e(3, J_{11}, n)$, for $n \geqslant 28$.

5 Progress on $e(3, J_k, n)$ and $R(3, J_k)$ for Higher k

The results in Tables 3 and 4 required computations of our gluing extender algorithm. We did not perform any such computations in an attempt to improve the lower bounds on $e(3, J_k, n)$ for $k \geqslant 12$, because such computations would be hardly feasible. The results presented in this section depend only on the degree sequence analysis described in Section 3, using constraints (2), (3) and the results for $k \leqslant 11$ from the previous section.

Lower bounds on $e(3, J_{12}, n)$
Beyond the range of equality in Theorem 1 (for $n \geqslant 32$), the lower bounds we obtained for $e(3, J_{12}, n)$ are the same as for $e(3, K_{11}, n)$, for all $33 \leqslant n \leqslant 49$, except for $n = 38$ (see

[10]), and they are presented in Table 5. They were obtained by using constraints (2) and (3) as described in Section 3. In one case, for $n = 39$, we can improve the lower bound by one as in the following lemma.

Lemma 4 $e(3, J_{12}, 39) \geqslant 117$.

Proof. Suppose that G is a $(3, J_{12}; 39, e)$-graph with $e \leqslant 116$. Using (2) and (3) with the bounds of Table 4 gives no solutions for $e < 116$ and two feasible degree sequences n_i for $e = 116$: $n_4 = 1, n_6 = 38$ and $n_5 = 2, n_6 = 37$. If $\deg(v) = 4$ then $Z_G(v) = 24$, and if $\deg(v) = 5$ then $Z_G(v) \geqslant 29$. In both cases this contradicts inequality (1), and thus we have $e(3, J_{12}, 39) \geqslant 117$. □

n	$e(3, J_{12}, n) \geqslant$	comments
31	56	exact, Theorem 1
32	62	Theorem 1
33	68	Theorem 1
34	75	
35	83	
36	92	
37	100	
38	108	$e(3, 11, 32) \geqslant 109$
39	117	improvement by Lemma 4
40	128	
41	138	
42	149	
43	159	
44	170	
45	182	
46	195	
47	209	
48	222	unique solution $n_7 = 36, n_8 = 12$
49	237	
50	252	$e(3, 11, 50) = \infty$
51	266	
52	280	maximum 286
53	∞	hence $R(3, J_{12}) \leqslant 53$, Theorem 2(b)

Table 5: Lower bounds on $e(3, J_{12}, n)$, for $n \geqslant 31$.

Lower bounds on $e(3, J_{13}, n)$

Beyond the range of equality in Theorem 1 (for $n \geqslant 35$), the lower bounds we obtained for $e(3, J_{13}, n)$ are the same as for $e(3, K_{12}, n)$, for all $35 \leqslant n \leqslant 58$, except for $n = 45, 46$ (see [10]), and they are presented in Table 6. They were obtained by using constraints (2) and (3) as described in Section 3.

n	$e(3, J_{13}, n) \geqslant$	comments
34	61	exact, Theorem 1
35	67	Theorem 1
36	73	Theorem 1
37	79	Theorem 1
38	86	
39	93	
40	100	unique solution $n_5 = 40$
41	109	unique solution $n_5 = 28, n_6 = 13$
42	119	
43	128	
44	138	
45	147	$e(3, 12, 45) \geqslant 148$
46	157	$e(3, 12, 46) \geqslant 158$
47	167	
48	179	
49	191	
50	203	
51	216	
52	229	
53	241	
54	255	
55	269	
56	283	unique solution $n_{10} = 50, n_{11} = 6$
57	299	
58	316	
59	333	$e(3, 12, 59) = \infty$
60	350	maximum 360
61	366	must be regular $n_{12} = 61$
62	∞	hence $R(3, J_{13}) \leqslant 62$, Theorem 2(b)

Table 6: Lower bounds on $e(3, J_{13}, n)$, for $n \geqslant 34$.

Lower bounds on $e(3, J_{14}, n)$

Beyond the range of equality in Theorem 1 (for $n \geqslant 40$), the lower bounds we obtained for $e(3, J_{14}, n)$ are the same as for $e(3, K_{13}, n)$, for all $41 \leqslant n \leqslant 67$, except for $n = 54, 55$ (see [10]), and they are presented in Table 7. They were obtained by using constraints (2) and (3) as described in Section 3.

n	$e(3, J_{14}, n) \geqslant$	comments
39	78	exact, Theorem 1
40	84	Theorem 1
41	91	
42	97	
43	104	
44	112	
45	120	
46	128	
47	136	
48	146	
49	157	
50	167	
51	177	
52	189	
53	200	
54	210	$e(3, 13, 54) \geqslant 212$
55	222	$e(3, 13, 55) \geqslant 223$
56	234	
57	247	
58	260	
59	275	
60	289	
61	303	
62	319	
63	334	
64	350	
65	365	
66	381	
67	398	
68	416	$e(3, 13, 68) = \infty$
69	434	
70	451	maximum 455
71	∞	hence $R(3, J_{14}) \leqslant 71$, Theorem 2(b)

Table 7: Lower bounds on $e(3, J_{14}, n)$, for $n \geqslant 39$.

Lower bounds on $e(3, J_{15}, n)$
The lower bounds we obtained for $e(3, J_{15}, n)$ are the same as for $e(3, K_{14}, n)$ for all $71 \leqslant n \leqslant 76$ (see [10]), and they are presented in Table 8. They were obtained by using constraints (2) and (3) as described in Section 3.

n	$e(3, J_{15}, n) \geqslant$	comments
71	398	
72	415	
73	432	
74	449	451 needed for $R(3, J_{16}) \leqslant 90$
75	468	
76	486	473 sufficient for $R(3, J_{16}) \leqslant 91$
77	505	$e(3, 14, 77) = \infty$
78	524	
79	543	maximum 553
80	∞	hence $R(3, J_{15}) \leqslant 80$, Theorem 2(b)

Table 8: Lower bounds on $e(3, J_{15}, n)$, for $n \geqslant 71$.

A 15-regular $(3, J_{16}; 90, 675)$-graph G is feasible when for every vertex v its G_v is a $(3, J_{15}; 74, 450)$-graph. Constraints (2) and (3) have no feasible solution for $(3, J_{16}; 91)$-graphs, and thus $R(3, J_{16}) \leqslant 91$.

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Appendix 1: Graph Counts

Tables 9–13 below contain all known exact counts of $(3, J_k; n, e)$-graphs for specified n, for $k = 7, 8, 9, 10$ and 11, respectively. All graph counts were obtained by the algorithms described in Section 3. Empty entries indicate 0. In all cases, the maximum number of edges is bounded by $\Delta(G)n/2 \leqslant (k-1)n/2$. All $(3, J_k; n, e(3, J_k, n))$-graphs, $k \leqslant 10$, which were constructed by our programs can be obtained from the *House of Graphs* [1] by searching for the keywords "minimal ramsey graph * Kk-e".

All $(3, J_7)$-graphs were previously determined by Fidytek [8]. We include the counts of $(3, J_7)$-graphs in Table 9 for completeness and more uniform presentation, since Fidytek provided statistics for J_7-free graphs whose complement does not contain a K_3, while we list triangle-free graphs whose complement does not contain a J_7. The latter is similar to the tables with data about $(3, J_k)$-graphs for $k \leqslant 6$ in [18].

Table 14 contains all known exact counts of $(3, K_9 - 2K_2; n, e)$-graphs. These graph counts were also obtained by the algorithms described in Section 3. All edge minimal $(3, K_9 - 2K_2; n, e(3, K_9 - 2K_2, n))$-graphs which were constructed by our programs can be obtained from the *House of Graphs* [1] by searching for the keywords "minimal ramsey graph * K9-2K2".

edges	number of vertices n												
e	8	9	10	11	12	13	14	15	16	17	18	19	20
3	1												
4	6	1											
5	14	2	1										
6	31	14	1										
7	51	41	5										
8	69	108	27	1									
9	76	195	102	3									
10	66	291	327	29									
11	41	329	771	131	1								
12	22	302	1355	643	8								
13	9	204	1778	2158	47								
14	3	117	1808	5239	398								
15	2	53	1439	8961	2434	1							
16	1	25	918	11450	9872	16							
17		9	492	11072	26586	241							
18		4	231	8505	49752	2665							
19		1	99	5260	67226	16313	1						
20		1	44	2794	68351	60891	13						
21			19	1294	54124	145452	300						
22			7	578	34707	238525	3997						
23			3	233	18757	280341	28889						
24			2	101	8976	247162	117123	2					
25			1	41	3942	169011	291706	14					
26				18	1669	93503	477533	305					
27				6	693	43149	543408	4521					
28				3	289	17392	451296	32828					
29				1	115	6217	286635	121140					
30				1	52	2073	146341	256923	3				
31					21	626	63112	338238	22				
32					10	190	24207	296128	361				
33					4	50	8505	181637	3251				
34					2	14	2841	83169	14968				
35					1	3	884	30257	35296				
36					1	1	275	9648	45855				
37							75	2865	34944	1			
38							22	883	16583	54			
39							5	273	5269	349			
40							2	94	1334	1070			
41								32	350	1501			
42								11	134	1174			
43								4	50	522	2		
44								1	25	147	8		
45								1	8	26	38		
46									4	6	61		
47									1	1	58		
48									1	1	36		
49										1	17		
50											4		
51											1		
52-53													
54												1	
55-59													
60													1
$\lvert\mathcal{R}(3, J_7; n)\rvert$	392	1697	9430	58522	348038	1323836	2447170	1358974	158459	4853	225	1	1

Table 9: Number of $(3, J_7; n, e)$-graphs, for $n \geqslant 8$.

edges	number of vertices n									
e	15	16	17	18	19	20	21	22	23	24
15	1									
16	2									
17	18									
18	188									
19	?									
20	?	2								
21	?	17								
22	?	358								
23	?	10659								
24	?	?								
25	?	?	2							
26	?	?	44							
27	?	?	2576							
28	?	?	117474							
29	?	?	?							
30	?	?	?	2						
31	?	?	?	22						
32	?	?	?	1175						
33	?	?	?	79025						
34-36	?	?	?	?						
37	?	?	?	?	20					
38	?	?	?	?	2031					
39	?	?	?	?	130297					
40	?	?	?	?	3939009					
41-43	?	?	?	?	?					
44	?	?	?	?	?	169				
45	?	?	?	?	?	8231				
46	?	?	?	?	?	310400				
47	?	?	?	?	?	5839714				
48-50	?	?	?	?	?	?				
51	?	?	?	?	?	?	7			
52	?	?	?	?	?	?	375			
53		?	?	?	?	?	14141			
54		?	?	?	?	?	255635			
55		?	?	?	?	?	2262269			
56-58		?	?	?	?	?	?			
59			?	?	?	?	?	2		
60				?	?	?	?	13		
61				?	?	?	?	162		
62				?	?	?	?	1630		
63				?	?	?	?	9101		
64					?	?	?	26611		
65					?	?	?	42700		
66					?	?	?	41455		
67						?	?	26459		
68						?	?	11716		
69						?	?	3657		
70						?	?	957	1	
71							?	208	2	
72							?	42	8	
73							?	10	6	
74								2	4	
75									1	
76									1	
77									13	
78-79										
80										1
81-83										0
84										8
$\lvert\mathcal{R}(3, J_8; n)\rvert$	?	?	?	?	?	?	?	164725	36	9

Table 10: Number of $(3, J_8; n, e)$-graphs, for $n \geqslant 15$.

edges	number of vertices n										
e	20	21	22	23	24	25	26	27	28	29	30
30	5										
31	64										
32	2073										
33-34	?										
35	?	1									
36	?	20									
37	?	951									
38	?	39657									
39-41	?	?									
42	?	?	21								
43	?	?	1592								
44	?	?	86833								
45	?	?	3963053								
46-48	?	?	?								
49	?	?	?	103							
50	?	?	?	9102							
51	?	?	?	514099							
52-55	?	?	?	?							
56	?	?	?	?	54						
57	?	?	?	?	3639						
58	?	?	?	?	173608						
59-64	?	?	?	?	?						
65	?	?	?	?	?	547					
66	?	?	?	?	?	48964					
67	?	?	?	?	?	2538589					
68-72	?	?	?	?	?	?					
73	?	?	?	?	?	?	62				
74	?	?	?	?	?	?	1857				
75	?	?	?	?	?	?	36799				
76	?	?	?	?	?	?	755052				
77-80	?	?	?	?	?	?	?				
81		?	?	?	?	?	?	4			
82		?	?	?	?	?	?	24			
83		?	?	?	?	?	?	197			
84		?	?	?	?	?	?	1126			
85			?	?	?	?	?	6206			
86			?	?	?	?	?	42468			
87			?	?	?	?	?	384398			
88			?	?	?	?	?	2843005			
89-94				?	?	?	?	?			
95					?	?	?	?	1		
96					?	?	?	?	14		
97						?	?	?	107		
98						?	?	?	1062		
99						?	?	?	5182		
100						?	?	?	16588		
101							?	?	34077		
102							?	?	50241		
103							?	?	51686		
104							?	?	39702		
105								?	21621		
106								?	9379	1	
107								?	2864	0	
108								?	843	0	
109									158	2	
110									49	6	
111									7	9	
112									91	6	
113-115										0	
116										1	
117											1
118											1
119											1
120											4
$\lvert\mathcal{R}(3, J_9; n)\rvert$	?	?	?	?	?	?	?	?	233672	25	7

Table 11: Number of $(3, J_9; n, e)$-graphs, for $n \geqslant 20$.

edges	number of vertices n												
e	24	25	26	27	28	29	30	31	32	33	34	35	36
40	2												
41	32												
42	2089												
43-45	?												
46	?	1											
47	?	39											
48	?	4113											
49-51	?	?											
52	?	?	1										
53	?	?	1										
54	?	?	444										
55	?	?	58550										
56-60	?	?	?										
61	?	?	?	700									
62	?	?	?	95185									
63	?	?	?	6531339									
64-67	?	?	?	?									
68	?	?	?	?	126								
69	?	?	?	?	17223								
70	?	?	?	?	1204171								
71-76	?	?	?	?	?								
77	?	?	?	?	?	1342							
78	?	?	?	?	?	156982							
79-85	?	?	?	?	?	?							
86	?	?	?	?	?	?	1800						
87	?	?	?	?	?	?	147407						
88-94	?	?	?	?	?	?	?						
95	?	?	?	?	?	?	?	560					
96	?	?	?	?	?	?	?	35154					
97-103	?	?	?	?	?	?	?	?					
104	?	?	?	?	?	?	?	?	39				
105	?	?	?	?	?	?	?	?	952				
106-117	?	?	?	?	?	?	?	?	?				
118				?	?	?	?	?	?	$\geqslant 5$			
119				?	?	?	?	?	?	$\geqslant 86$			
120				?	?	?	?	?	?	$\geqslant 1411$			
121-128				?	?	?	?	?	?	?			
129						?	?	?	?	?	$\geqslant 1$		
130						?	?	?	?	?	$\geqslant 4$		
131							?	?	?	?	$\geqslant 7$		
132-139							?	?	?	?	?		
140									?	?	?	1	
141									?	?	?	0	
142-146									?	?	?	?	
147										?	?	$\geqslant 10$	
148										?	?	$\geqslant 28$	
149											?	$\geqslant 39$	
150											?	$\geqslant 27$	
151											?	$\geqslant 19$	
152											?	$\geqslant 11$	
153											?	$\geqslant 8$	
154-155												?	
156												?	5
157												?	$\geqslant 6$
158													$\geqslant 10$
159													$\geqslant 6$
160													$\geqslant 5$
161													$\geqslant 2$
162													$\geqslant 6$

Table 12: Number of $(3, J_{10}; n, e)$-graphs, for $n \geqslant 24$.

edges	number of vertices n					
e	29	30	31	32	33	34
58	5					
59	1364					
60-65	?					
66	?	5084				
67-72	?	?				
73	?	?	2657			
74-79	?	?	?			
80	?	?	?	4		
81	?	?	?	6601		
82-89	?	?	?	?		
90	?	?	?	?	57099	
91-98	?	?	?	?	?	
99	?	?	?	?	?	$\geqslant 1$
$\geqslant 100$	?	?	?	?	?	?

Table 13: Number of $(3, J_{11}; n, e)$-graphs, for $29 \leqslant n \leqslant 34$.

edges	number of vertices n					
e	21	22	23	24	25	26
45	1					
46	2					
47	61					
48	3743					
49	408410					
50-53	?					
54	?	2				
55	?	299				
56	?	20314				
57	?	985296				
58	?	23618486				
59-60	?	?				
63	?	?	9			
64	?	?	528			
65	?	?	24860			
66	?	?	566836			
67	?	?	5830123			
68-72	?	?	?			
72	?	?	?	104		
73	?	?	?	1068		
74	?	?	?	7913		
75	?	?	?	31134		
76	?	?	?	84634		
77	?	?	?	160815		
78	?	?	?	215365		
79	?	?	?	207752		
80	?	?	?	172746	18	
81	?	?	?	142474	97	
82	?	?	?	121641	333	
83	?	?	?	107869	516	
84	?	?	?	108001	416	
85		?	?	110938	158	
86		?	?	101090	30	
87		?	?	72665	5	
88		?	?	40935	1	
89			?	17722	0	
90			?	6262	0	3
91			?	1779	0	2
92			?	522	5	1
93				129	16	0
94				46	35	0
95				8	34	0
96				5	19	0
97					6	0
98					2	0
99					1	0
100					1	0
101-103						0
104						2
$\lvert\mathcal{R}(3, K_9 - 2K_2; n)\rvert$	?	?	?	1713617	1693	8

Table 14: Number of $(3, K_9 - 2K_2; n, e)$-graphs, for $n \geqslant 21$.

Appendix 2: Testing correctness

Since most results obtained in this paper rely on computations, it is very important that the correctness of our programs has been thoroughly verified. Below we describe how we tested the correctness of our programs.

Correctness

- For every $(3, J_k)$-graph which was output by our programs, we verified that it does not contain a spanning subgraph of $\overline{J_k}$ as induced subgraph by using an independent program.

- Every Ramsey graph for K_k is also a Ramsey graph for J_{k+1}. Therefore, we verified that the complete lists of $(3, J_{k+1}; n, e)$-graphs which were generated by our programs include all known $(3, k; n, e)$-graphs which we had found in [10].

- For every $(3, J_k; n, e(3, J_k, n))$-graph which was generated by our programs, we verified that dropping any edge results in a graph which contains a spanning subgraph of $\overline{J_k}$ as induced subgraph.

- For various $(3, J_k; n, \leqslant e)$-graphs we added up to f edges in all possible ways to obtain $(3, J_k; n, \leqslant e + f)$-graphs. For the cases where we already had the complete set of $(3, J_k; n, \leqslant e+f)$-graphs, we verified that no new $(3, J_k; n, \leqslant e+f)$-graphs were obtained. We used this, amongst other cases, to verify that no new $(3, J_{10}; 26, \leqslant 55)$, $(3, J_{10}; 28, \leqslant 70)$, $(3, J_{10}; 30, \leqslant 87)$ or $(3, J_{11}; 32, \leqslant 81)$-graphs were obtained.

- For various $(3, J_k; n, \leqslant e + f)$-graphs we dropped one edge in all possible ways and verified that no new $(3, J_k; n, \leqslant e + f - 1)$-graphs were obtained. We used this technique, amongst other cases, to verify that no new $(3, J_{10}; 26, \leqslant 54)$, $(3, J_{10}; 28, \leqslant 69)$, $(3, J_{10}; 30, 86)$ or $(3, J_{11}; 32, 80)$-graphs were obtained.

- For various sets of $(3, J_{k+1}; n, \leqslant e)$-graphs we took each member G and constructed from it all G_v's. We then verified that this did not yield any new $(3, J_k; n - deg(v) - 1, \leqslant e - Z(v))$-graphs for the cases where we have all such graphs. We performed this test, amongst other cases, on the sets of $(3, J_9; 28, \leqslant 70)$- and $(3, J_{10}; 32, \leqslant 81)$-graphs.

Various sets of graphs can be obtained by both the maximum triangle-free and neighborhood gluing extension method. Therefore, as a test for the correctness of our implementations, we applied both methods for the generation of several sets of graphs. We also compared our results with known results. In each case, the results were in complete agreement. More details are given below:

- The sets of $(3, J_8; 19, \leqslant 38)$, $(3, J_8; 20, \leqslant 46)$, $(3, J_8; 21, \leqslant 54)$ and $(3, J_9; 27, \leqslant 86)$-graphs were obtained by both the maximum triangle-free method and the neighborhood gluing extension method. The results were in complete agreement.

- The counts of all $(3, J_7)$-graphs are confirmed by [8].
- The counts of all $(3, J_8; 19, 37)$, $(3, J_8; 20, 44)$, $(3, J_8; 21, \leqslant 52)$, $(3, J_8; 22, \leqslant 60)$, $(3, J_8; 23, \leqslant 71)$ and $(3, J_8; 24, \leqslant 81)$-graphs are confirmed by [18].
- The counts of $(3, J_8; 22, \leqslant 65)$, $(3, J_8; 23)$ and $(3, J_8; 24)$-graphs are confirmed by [17].

Since our results are in complete agreement with previous results and since all of our consistency tests passed, we believe that this is strong evidence for the correctness of our implementations and results.

Computation Time

We implemented the extension algorithms described in Section 3 in C. Most computations were performed on a cluster with Intel Xeon L5520 CPU's at 2.27 GHz, on which a computational effort of one CPU year can be usually completed in about 8 elapsed hours. The overall computational effort which was required to improve the upper bounds of $R(3, J_k)$ is estimated to be about 40 CPU years. This includes the time used by a variety of programs. The most CPU-intensive task was the computation to determine all $(3, J_9; \geqslant 28)$-graphs with the maximum triangle-free method. This took approximately 13 CPU years. Also the computations to determine new lower bounds on $e(3, J_{11}, n)$ took relatively long. For example, it took nearly 5 CPU years using the neighborhood gluing method to prove that $e(3, J_{11}, 39) \geqslant 151$.

The CPU time needed to complete the computations of Section 5 was negligible.