Repeated columns and an old chestnut

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Abstract

Let $t \geqslant 1$ be a given integer. Let $\mathcal{F}$ be a family of subsets of $[m] = \{1, 2, \ldots, m\}$. Assume that for every pair of disjoint sets $S, T \subset [m]$ with $|S| = |T| = k$, there do not exist $2t$ sets in $\mathcal{F}$ where t subsets of $\mathcal{F}$ contain S and are disjoint from T and t subsets of $\mathcal{F}$ contain T and are disjoint from S. We show that $|\mathcal{F}|$ is $O(m^k)$.

Our main new ingredient is allowing, during the inductive proof, multisets of subsets of $[m]$ where the multiplicity of a given set is bounded by $t-1$. We use a strong stability result of Anstee and Keevash. This is further evidence for a conjecture of Anstee and Sali. These problems can be stated in the language of matrices. Let $t \cdot M$ denote t copies of the matrix M concatenated together. We have established the conjecture for those configurations $t \cdot F$ for any $k \times 2$ (0,1)-matrix F.

Keywords: extremal set theory, extremal hypergraphs, (0,1)-matrices, multiset, forbidden configurations, trace, subhypergraph.

1 Introduction

We will be considering a problem in extremal hypergraphs that can be phrased as how many edges a hypergraph on m vertices can have when there is a forbidden subhypergraph. There are a variety of ways to define this problem (we could, but do not, restrict to (simple) k-uniform hypergraphs). We can encode a hypergraph on m vertices as an m-rowed (0,1)-matrix where the ith column is the incidence vector of the ith hyperedge.

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A hypergraph is *simple* if there are no repeated edges. We define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. We will use the language of matrices in this paper.

Let M be an m-rowed (0,1)-matrix. Some notation about repeated columns is needed. For an $m \times 1$ (0,1)-column α, we define $\mu(\alpha, M)$ as the multiplicity of column α in a matrix M. We consider matrices of bounded column multiplicity. We define a matrix A to be t-*simple* if it is a (0,1)-matrix and every column α of A has $\mu(\alpha, A) \leqslant t$. Simple matrices are 1-*simple*. For a given matrix M, let $\text{supp}(M)$ denote the maximal simple m-rowed submatrix of M, so that if $\mu(\alpha, M) \geqslant 1$ then $\mu(\alpha, \text{supp}(M)) = 1$. The matrices below are a 3-simple matrix M and its support $\text{supp}(M)$.

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \qquad \text{supp}(M) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

For two $(0,1)$-matrices F and A, we say that F is a *configuration* in A, and write $F \prec A$ if there is a row and column permutation of F which is a submatrix of A. Let $\mathcal{F}$ denote a finite set of (0,1)-matrices. Let $\text{Avoid}(m, \mathcal{F}, t)$ denote all m-rowed t-simple matrices A for which $F \nprec A$ for all $F \in \mathcal{F}$. We are most interested in cases with $|\mathcal{F}| = 1$ [5]. We do not require any $F \in \mathcal{F}$ to be simple which is quite different from usual forbidden subhypergraph problems. Let $\|A\|$ denote the number of columns of A. Our extremal function of interest is

$$\text{forb}(m, \mathcal{F}) = \max_A \{\|A\| \,:\, A \in \text{Avoid}(m, \mathcal{F}, 1)\}.$$

We find it helpful to also define

$$\text{forb}(m, \mathcal{F}, t) = \max_A \{\|A\| \,:\, A \in \text{Avoid}(m, \mathcal{F}, t)\}.$$

If $A \in \text{Avoid}(m, \mathcal{F}, t)$ then $\text{supp}(A) \in \text{Avoid}(m, \mathcal{F}, 1)$ and $\|A\| \leqslant t \cdot \|\text{supp}(A)\|$. We obtain

$$\text{forb}(m, \mathcal{F}) \leqslant \text{forb}(m, \mathcal{F}, t) \leqslant t \cdot \text{forb}(m, \mathcal{F}), \tag{1}$$

so that the asymptotic growth of $\text{forb}(m, \mathcal{F})$ is the same as that of $\text{forb}(m, \mathcal{F}, t)$ for fixed t.

We have an important conjecture about $\text{forb}(m, F)$. We use the notation $[M \,|\, N]$ to denote the matrix obtained from concatenating the two matrices M and N. We use the notation $k \cdot M$ to denote the matrix $[M|M|\cdots|M]$ consisting of k copies of M concatenated together. Let I_k denote the $k \times k$ *identity* matrix and let I_k^c denote the (0,1)-complement of I_k. Let T_k denote the $k \times k$ *triangular* (0,1)-matrix with the (i, j) entry being 1 if and only if $i \leqslant j$. For an $m_1 \times n_1$ matrix X and an $m_2 \times n_2$ matrix Y, we define the 2-fold product $X \times Y$ as the $(m_1 + m_2) \times n_1 n_2$ matrix where each column consisting of a column of X placed on a column of Y and this is done in all possible ways. This extends to p-fold products.

Definition 1. Let $X(F)$ be the smallest p so that $F \prec A_1 \times A_2 \times \cdots \times A_p$ for every choice of A_i as either $I_{m/p}$, $I^c_{m/p}$ or $T_{m/p}$ for sufficiently large m.

Alternatively, assuming $F \nprec I$ or $F \nprec I^c$ or $F \nprec T$, then $X(F) - 1$ is the largest choice of p so that $F \nprec A_1 \times A_2 \times \cdots \times A_p$ for some choices of A_i as either $I_{m/p}$, $I^c_{m/p}$ or $T_{m/p}$. We note that if $A_1 \times A_2 \times \cdots \times A_p \in \text{Avoid}(m, F)$, then $\text{forb}(m, F)$ is $\Omega(m^p)$.

Details are in [5]. We are assuming m is large and divisible by p, in particular that $m \geqslant (k+1)(k\ell+1)$ so that $m/p \geqslant k\ell+1$. Divisibility by p does not affect the asymptotic growth, thus $\text{forb}(m, F)$ is $\Omega(m^{X(F)-1})$ using an appropriate $(X(F) - 1)$-fold product.

Conjecture 2. [4] Let F be given. Then $\text{forb}(m, F) = \Theta(m^{X(F)-1})$. $\square$

The conjecture was known to be true for all 3-rowed F [4] and all $k \times 2$ F [3]. Section 3 shows how Theorem 3 establishes the conjecture for matrices $t \cdot F$ when F is a $k \times 2$ matrix. It is of interest to generalize Conjecture 2 to $\text{forb}(m, \mathcal{F})$ where $|\mathcal{F}| > 1$ but we know example of $\mathcal{F}$ where the modified form of the conjecture fails (see [5]).

We define $F_{e,f,g,h}$ as the $(e + f + g + h) \times 2$ matrix consisting of e rows $[1\,1]$, f rows $[1\,0]$, g rows $[0\,1]$ and h rows $[0\,0]$. Let $\mathbf{1}_e\mathbf{0}_f$ denote the $(e + f) \times 1$ vector of e 1's on top of f 0's so that $F_{e,f,g,h} = [\mathbf{1}_{e+f}\mathbf{0}_{g+h} \,|\, \mathbf{1}_e\mathbf{0}_f\mathbf{1}_g\mathbf{0}_h]$. We let $\mathbf{1}_e$ denote the $e \times 1$ vector of e 1's and $\mathbf{0}_f$ denote the $f \times 1$ vector of f 0's. Our main result is the following which had foiled many previous attempts.

Theorem 3. *Let $t \geqslant 2$ be given. Then forb$(m, t \cdot F_{0,k,k,0})$ is $\Theta(m^k)$.*

The forbidden configuration $t \cdot F_{0,k,k,0}$ in the language of sets, consists of two disjoint k-sets S, T, and a family of t sets containing S but disjoint from T, and the other family of another t sets containing T but disjoint from S. This theorem echoes our statement in the abstract.

The result for $t = 2$ and $k = 2$ was proven in [1] and many details worked out for $t = 2$ and $k > 2$ by the first author and Peter Keevash. The extension for $t > 2$, $k = 2$ had been open since then [5]. The proof for $t > 2$, $k = 2$ is in Section 2. The proof for $t > 2$, $k > 2$ is in Section 3. Matrices $F_6(t), F_7(t)$ were given in [5] as 4-rowed forbidden configurations (with some columns of multiplicity t) for which Conjecture 2 predicts $\text{forb}(m, F_6(t))$ and $\text{forb}(m, F_7(t))$ are $O(m^2)$. Note that $t \cdot F_{0,2,2,0} \prec F_6(t)$ and $t \cdot F_{0,2,2,0} \prec F_7(t)$ and so Theorem 3 is a step towards these bounds which would establish Conjecture 2 for all 4-rowed F. Our proof use a new induction given in Section 2 that considers t-simple matrices as well as a strong stability result Lemma 10. We offer some additional applications in Section 4.

2 New Induction

We consider a new form of the standard induction for forbidden configurations [5]. Let F be a matrix with maximum column multiplicity t. Thus $F \prec t \cdot \text{supp}(F)$. Let

$A \in \text{Avoid}(m, F, t-1)$. Assume $\|A\| = \text{forb}(m, \mathcal{F}, t-1)$. Given a row r we permute rows and columns of A to obtain

$$A = \begin{array}{c} \text{row } r \rightarrow \\ \\ \end{array} \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ G & H \end{bmatrix}. \tag{2}$$

Now $\mu(\alpha, G) \leqslant t-1$ and $\mu(\alpha, H) \leqslant t-1$. For those α for which $\mu(\alpha, [G\,H]) \geqslant t$, let C be formed with $\mu(\alpha, C) = \min\{\mu(\alpha, G), \mu(\alpha, H)\}$. We rewrite our decomposition of A as follows:

$$A = \begin{array}{c} \text{row } r \rightarrow \\ \\ \end{array} \begin{bmatrix} 0\,0 \cdots 0 & 1\,1 \cdots 1 \\ B \quad C & C \quad D \end{bmatrix}. \tag{3}$$

Then we deduce that $[BCD]$ and C are both $(t-1)$-simple. The former follows from $\mu(\alpha, [B\,C\,D]) = \mu(\alpha, G) + \mu(\alpha, H) - \min\{\mu(\alpha, G), \mu(\alpha, H)\} \leqslant t-1$. We have that $F \nprec [B\,C\,D]$ for $F \in \mathcal{F}$. Also for any $F' \prec C$ then $[0\,1] \times F' \prec A$ so we define

$$\mathcal{G} = \{F' \,:\, \text{for } F \in \mathcal{F},\; F \prec [0\,1] \times F' \text{ and } F \nprec [0\,1] \times F'' \text{ for all } F'' \prec F',\; F'' \neq F'\}. \tag{4}$$

Basically, $\mathcal{G}$ is the family after removing redundancy from all configurations F' that are obtained by removing one row from any F in $\mathcal{F}$.

Also since each column α of C has $\mu(\alpha, [G\,H]) \geqslant t$, we deduce that $\text{supp}(F) \nprec C$ for each $F \in \mathcal{F}$. Our induction on m becomes:

$$\begin{aligned} \text{forb}(m, \mathcal{F}, t-1) &= \|A\| = \|[BCD]\| + \|C\| \\ &\leqslant \text{forb}(m-1, \mathcal{F}, t-1) + (t-1) \cdot \text{forb}(m-1, \mathcal{G} \cup \{\text{supp}(F) \,:\, F \in \mathcal{F}\}). \end{aligned} \tag{5}$$

Lemma 4. *Let H be a given simple matrix satisfying forb(m, H) is $O(m^{\ell})$. Then forb$(m, t \cdot H)$ is $O(m^{\ell+1})$.*

Proof. We use the induction (5) where $F = t \cdot H$ and $H = \text{supp}(F)$. Induction on m yields the desired bound. □

Proof of Theorem 3 for $k = 2$: We will use induction on m to show $\text{forb}(m, t \cdot F_{0,2,2,0}, t)$ is $O(m^2)$. The maximum multiplicity of a column in $t \cdot F_{0,2,2,0}$ is t and $F_{0,2,2,0} = \text{supp}(t \cdot F_{0,2,2,0})$. Also $t \cdot F_{0,2,2,0} \prec [0\,1] \times (t \cdot F_{0,2,1,0})$. Let $A \in \text{Avoid}(m, t \cdot F_{0,2,2,0}, t-1)$ with $\|A\| = \text{forb}(m, t \cdot F_{0,2,2,0}, t-1)$. Apply (5). We have

$$\begin{aligned} \text{forb}(m, t \cdot F_{0,2,2,0}, t-1) &= \|A\| = \|[BCD]\| + \|C\| \\ &\leqslant \text{forb}(m-1, t \cdot F_{0,2,2,0}, t-1) + (t-1) \cdot \text{forb}(m-1, \{F_{0,2,2,0}, t \cdot F_{0,2,1,0}\}). \end{aligned}$$

We apply Lemma 5 with induction on m to deduce that $\text{forb}(m, t \cdot F_{0,2,2,0}, t-1)$ is $O(m^2)$. Then by (1), $\text{forb}(m, t \cdot F_{0,2,2,0})$ is also $O(m^2)$. □

Theorem 3 was proven for $t = k = 2$ in [1] using induction in the spirit of (5) ($(t-1)$-simple matrices are simple) and Lemma 5 for $t = 2$.

Lemma 5. *We have that forb*$(m,\{F_{0,2,2,0},t\cdot F_{0,2,1,0}\})$ *is* $O(m)$.

Proof. Let $A\in\text{Avoid}(m,\{F_{0,2,2,0},t\cdot F_{0,2,1,0}\})$. Avoiding $F_{0,2,2,0}$ creates structure: Let X_i denote the columns of A of column sum i. Let $J_{a\times b}$ denote the $a\times b$ matrix of 1's and let $0_{a\times b}$ denote the $a\times b$ matrix of 0's. Now $F_{0,2,2,0}\nprec X_i$ and so for $\|X_i\|\geqslant 3$, we may deduce that there is a partition of the rows $[m]$ into $A_i\cup B_i\cup C_i$. Let $x_i=|X_i|$. After suitable row and column permutations, we have X_i as follows:

$$\text{type 1: } X_i=\begin{array}{r}A_i\{\\ B_i\{\\ C_i\{\end{array}\left[\begin{array}{l}I_{x_i}\\ J_{(i-1)\times x_i}\\ 0_{(m-x_i-i+1)\times x_i}\end{array}\right]\text{ or type 2: } X_i=\begin{array}{r}A_i\{\\ B_i\{\\ C_i\{\end{array}\left[\begin{array}{l}I^c_{x_i}\\ J_{(i-x_i+1)\times x_i}\\ 0_{(m-i-1)\times x_i}\end{array}\right].$$

We will say i *is of type* j ($j=1$ or $j=2$) if the columns of sum i are of type j. These are the *sunflowers* (for type 1) and *inverse sunflowers* (type 2) of [7] where for type 1 the petals are A_i with center B_i.

Let $T(1)=\{i : i \text{ is of type 1 and } \|X_i\|\geqslant t+2\}$. We wish to show for that $B_i\subset B_j$ for $i,j\in T(1)$ and $i<j$. Assume $p\in B_i\backslash B_j$. Given that $|B_i|<|B_j|$, there are two rows $r,s\in B_j\backslash B_i$. Then we find a copy of $t\cdot F_{0,2,1,0}$ in rows p,r,s of $[X_i\,X_j]$ (we would not choose the possible column of X_i that has a 1 in row r and the column of X_i that has a 1 in row s), a contradiction showing no such p exists and hence $B_i\subset B_j$.

We form a matrix Y_1 from those X_i with $i\in T(1)$. We have $\|Y_1\|=\sum_{i\in T(1)}\|X_i\|=\sum_{i\in T(1)}|A_i|$. Assume $\sum_{i\in T(1)}|A_i|>(t+1)m$. Then there is some row p and $(t+2)$-set $\{s(1),s(2),\dots,s(t+2)\}$ with $p\in A_i$ for all $i\in\{s(1),s(2),\dots,s(t+2)\}$. Assume $s(1)<s(2)<\cdots<s(t+2)$. We have $B_{s(1)}\subset B_{s(2)}\subset\cdots\subset B_{s(t+2)}$. We may choose $r,s\in B_{s(t+2)}\backslash B_{s(t)}$ so that $r,s\in A_{s(i)}\cup C_{s(i)}$ for $i=1,2,\dots,t$. We find a copy of $t\cdot F_{0,2,1,0}$ in rows p,r,s as follows. We take one column from each $X_{s(j)}$ for $j=1,2,\dots,t$ and t columns from the $X_{s(t+2)}$. We conclude that $\|Y_1\|\leqslant(t+1)m$. Similarly the matrix Y_2 formed from those X_i such that i is of type 2 and $\|X_i\|\geqslant t+2$ has $\|Y_2\|\leqslant(t+1)m$. Now Y_1 and Y_2 represent all columns of A with the exception of columns of sum i with $\|X_i\|\leqslant t+1$ and so we conclude $\|A\|\leqslant\|Y_1\|+\|Y_2\|+(t+1)(m-1)+2$. Thus $\|A\|$ is $O(m)$. □

3 More evidence for the Conjecture

This section first explores the Conjecture 2 for $t\cdot F$ when F is $k\times 2$. The section concludes with the proof of Theorem 3 for $k>2$. The following verifies Conjecture 2 for all $k\times 2$ F. Note that any $k\times 2$ matrix F can be written as $F_{a,b,c,d}$ ($b\geqslant c$) under proper row and column permutations. Since forb(m,F) is invariant under taking $(0,1)$-complement, we can further assume $a\geqslant d$. The case of $t=1$ was solved in [3] by the following theorem.

Theorem 6. *[3] Suppose* $a\geqslant d$ *and* $b\geqslant c$. *Then forb*$(m,F_{a,b,c,d})$ *is* $\Theta(m^{a+b-1})$ *if either* $b>c$ *or* $a,b\geqslant 1$. *Also forb*$(m,F_{a,0,0,d})$ *is* $\Theta(m^a)$ *and forb*$(m,F_{0,b,b,0})$ *is* $\Theta(m^b)$. □

Note that Conjecture 2 is verified if there is a product construction avoiding F yielding the same asymptotic growth as an upper bound on $\mathrm{forb}(m, F)$. The k-fold product $I_{m/k} \times I_{m/k} \times \cdots \times I_{m/k} \in \mathrm{Avoid}(m, t \cdot F_{0,k,k,0})$ has $\Theta(m^k)$ columns. Thus Theorem 3 verifies the conjecture for $t \cdot F_{0,k,k,0}$. The following results verify the conjecture for $t \cdot F$ for the remaining $k \times 2$ F.

Theorem 7. *For $b > c$ or $a, b \geqslant 1$ then $\mathrm{forb}(m, t \cdot F_{a,b,c,d})$ is $\Theta(m^{a+b})$.*

Proof. The upper bound follows from $\mathrm{forb}(m, F_{a,b,c,d})$ being $\Theta(m^{a+b-1})$ and then applying Lemma 4. The lower bound follows from $2 \cdot \mathbf{1}_{a+b} \prec t \cdot F_{a,b,c,d}$ so that the $(a+b)$-fold product $I_{m/(a+b)} \times I_{m/(a+b)} \times \cdots \times I_{m/(a+b)} \in \mathrm{Avoid}(m, F_{a,b,c,d})$ and hence $\mathrm{forb}(m, t \cdot F_{a,b,c,d})$ is $\Omega(m^{a+b})$. □

Theorem 8. *Let $a \geqslant d$ be given. Then $\mathrm{forb}(m, t \cdot F_{a,0,0,d})$ is $\Theta(m^a)$.*

Proof. This follows using Lemma 9 repeatedly and also $\mathrm{forb}(m, t \cdot F_{a,0,0,0})$ is $O(m^a)$ using Theorem 14. The a-fold product $I_{m/a} \times I_{m/a} \times \cdots \times I_{m/a} \in \mathrm{Avoid}(m, t \cdot F_{a,0,0,d})$. □

The following result can be found in the survey on forbidden configurations [5]

Lemma 9. *Assume $\mathrm{forb}(m, F)$ is $O(m^\ell)$. Then $\mathrm{forb}(m, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \times F)$ is $O(m^{\ell+1})$.*

Here is the summary of results on $\mathrm{forb}(m, t \cdot F_{a,b,c,d})$ ($a \geqslant d$ and $b \geqslant c$), which verify Conjecture 2 for all $k \times 2$ F.

t	Configuration	result	reference	Lower bound construction
$t = 1$	$F_{a,b,c,d}$ ($b > c$ or $a, b \geqslant 1$)	$\Theta(m^{a+b-1})$	[3]	$\overbrace{I \times I \times \cdots I \times I}^{a+b-1}$
	$F_{a,0,0,d}$	$\Theta(m^a)$	[3]	$\overbrace{I \times I \times \cdots I \times I}^{a}$
	$F_{0,b,b,0}$	$\Theta(m^b)$	[3]	$\overbrace{I \times I \times \cdots I \times I}^{b-1} \times T$
$t \geqslant 2$	$t \cdot F_{a,b,c,d}$ ($b > c$ or $a, b \geqslant 1$)	$\Theta(m^{a+b})$	Lemma 4	$\overbrace{I \times I \times \cdots I \times I}^{a+b}$
	$t \cdot F_{a,0,0,d}$	$\Theta(m^a)$	Lemma 4	$\overbrace{I \times I \times \cdots I \times I}^{a}$
	$t \cdot F_{0,b,b,0}$	$\Theta(m^b)$	Theorem 3	$\overbrace{I \times I \times \cdots I \times I}^{b-1} \times T$

Table 1: All cases of $\mathrm{forb}(m, t \cdot F_{a,b,c,d})$ with $a \geqslant d$ and $b \geqslant c$.

We note that the bound for $\mathrm{forb}(m, t \cdot F_{a,0,0,d})$ can be readily established by a pigeonhole argument. We return to Theorem 3 and first obtain some useful lemmas. Let

$X_i \in \text{Avoid}(m, F_{0,k,k,0})$ with all column sums i. We define X_i to be of *type* (a,b) if $a,b \geqslant 0$ are integers with $a+b=k-1$ and there is a partition $C_i \cup D_i = [m]$ with $|D_i|+a-b=i$ such that any column α of X_i has exactly a 1's in rows C_i and exactly b 0's in rows D_i. We are able to use this structure in view of the following 'strong stability' result:

Lemma 10. *[3] Let $Y_i \in \text{Avoid}(m, F_{0,k,k,0})$ with all column sums i. Assume $\|Y_i\| \geqslant (6(k-1))^{5k+2}m^{k-2}$. Then there is an m-rowed submatrix X_i of Y_i and a pair of integers $a,b \geqslant 0$ with $a+b=k-1$ such that X_i is of type (a,b) and where $\|Y_i\|-\|X_i\| \leqslant m^{k-3}$.*

Lemma 11. *Let $X_i \in \text{Avoid}(m, F_{0,k,k,0})$ have all columns of sum i and assume X_i is of type (a,b) with $a,b \geqslant 1$ with $a+b=k-1$. Let $C_i \cup D_i = [m]$ be the associated partition of the rows. We form a bipartite graph $G_i = (V_i, E_i)$ with $V_i = \binom{C_i}{a} \cup \binom{D_i}{b}$ where we have $(C,D) \in E_i$ if there is a column of X_i with a 1's in rows C and $D_i \backslash D$ and b 0's in rows D and $C_i \backslash C$. Assume $|E_i| \geqslant 2km^{k-2}$. Then there is subgraph $G'_i = (V'_i, E'_i)$ of G_i with $|E'| \geqslant \frac{1}{2}|E_i|$ such that for every pair $C \in \binom{C_i}{a}$ and $D \in \binom{D_i}{b}$ with $(C,D) \in E'$ we have*

$$d_{G'_i}(C) \geqslant (b+1/2)m^{b-1}, \qquad d_{G'_i}(D) \geqslant (a+1/2)m^{a-1}. \tag{6}$$

Proof. Simply delete vertices $C \in \binom{C_i}{a}$ with $d_G(C) < (b+1/2)m^{b-1}$ and vertices $D \in \binom{D_i}{b}$ with $d_G(D) < (a+1/2)m^{a-1}$ and continue deleting vertices until conditions (6) are satisfied for any remaining vertices of G'. This will delete a maximum of $(b+1/2)m^{b-1}\binom{|C_i|}{a} + (a+1/2)m^{a-1}\binom{|D_i|}{b} < km^{k-2}$ edges which deletes less than half the edges of G. $\square$

Lemma 12. *Let k be given. Then forb$(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\})$ is $O(m^{k-1})$.*

Proof. Let $A \in \text{Avoid}(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\})$. Let Y_i denote the columns of A of column sum i. For all i for which $|Y_i| < (6(k-1))^{5k+2}m^{k-2}$, delete the columns of Y_i from A. This may delete $(6(k-1))^{5k+2}m^{k-1}$ columns. For i with $|Y_i| \geqslant (6(k-1))^{5k+2}m^{k-2}$, apply Lemma 10 and obtain X_i with $|X_i| \geqslant (6(k-1))^{5k+2}m^{k-2} - m^{k-3}$.

We consider a choice a,b with $a+b=k-1$. Let $T(a,b) = \{i : X_i \text{ is of type } (a,b)\}$. We will show that $\sum_{i \in T(a,b)} |X_i| \leqslant (tk)m^{k-1}$.

Case 1. $a,b \geqslant 1$.

Create G_i as described in Lemma 11 to obtain G'_i for each $i \in T(a,b)$. Now if $\sum_{i \in T(a,b)} |E'_i| > (t+1)m^{a+b}$, then there will be some edge $(C,D) \in E'_i$ for at least $t+2$ choices $i \in T(a,b)$. Let those choices be $s(1), s(2), \ldots, s(t+2)$ where $s(1) < s(2) < \cdots < s(t+2)$. We wish to show that $X_{s(i)}$ has $t \cdot F_{0,k-1,0,0}$ on rows $C \cup D$.

$$\begin{array}{rc}
 & \begin{array}{ccc} & \overbrace{}^{t} & \overbrace{}^{t} \end{array} \\
\text{rows } C \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{ccc} 1 & 11\cdots1 & 00\cdots0 \\ 1 & 11\cdots1 & 00\cdots0 \\ 1 & 11\cdots1 & 00\cdots0 \end{array} \\
\text{rows } D \left\{ \begin{array}{c} \\ \\ \end{array} \right. & \begin{array}{ccc} 0 & 11\cdots1 & 00\cdots0 \\ 0 & 11\cdots1 & 00\cdots0 \end{array}
\end{array}$$

For a given set $D \in \binom{D_{s(i)}}{b}$, we compute $|\{H \in \binom{D_{s(i)}}{b} : H \cap D \neq \emptyset\}| \leqslant \sum_{j=1}^{b} \binom{b}{j}\binom{D_{s(i)}\backslash D}{b-j} < bm^{b-1}$.

Now if $d_{G'}(C) \geqslant (b+1/2)m^{b-1}$ and $(C, D) \in E'_{s(i)}$ then there are at least t edges $(C, H) \in E'_{s(i)}$ with $H \cap D = \emptyset$. We are using $(b+1/2)m^{b-1} > bm^{b-1} + t + 2$ which is true for m large enough and so asymptotics are unaffected. Thus we have t columns of $X_{s(1)}$ with $\mathbf{1}_{k-1}$ on rows $C \cup D$ and, because these columns have a 1's on rows $C \subseteq C_{s(1)}$, these columns are 0's on the remaining rows of $C_{s(1)}\backslash C$.

Similarly, because $d_{G'_i}(D) \geqslant (a+1/2)m^{a-1}$ there will be $t+2$ edges $(K, D) \in E_{s(i)}$ with $K \cap C = \emptyset$ and so there are t columns of $X_{s(t+2)}$ with $\mathbf{0}_{k-1}$ on rows $C \cup D$ and, because these columns have 0's on rows D, these columns are 1's on rows of $D_{s(t+2)}\backslash D$.

We choose k rows in $Z = D_{s(t+2)}\backslash D_{s(1)}$ so that $Z \subseteq C_{s(1)}$. We deduce that in the chosen t columns of $X_{s(1)}$ we have $\mathbf{0}_k$ in rows Z since $Z \subseteq C_{s(1)}\backslash C$ and the columns have $\mathbf{1}_{k-1}$ in rows $C \cup D$. In the chosen t columns of $X_{s(t+2)}$ we have $\mathbf{1}_k$ in rows Z since $Z \subset D_{s(t+2)}\backslash D$ and the columns have $\mathbf{0}_{k-1}$ in rows $C \cup D$. This yields $t \cdot F_{0,k,k-1,0}$, a contradiction. Thus $\sum_{i \in Type(a,b)} |E'_i| \leqslant (t+1)m^{k-1}$. This concludes Case 1.

Case 2. $a = k-1, b = 0$ or $a = 0, b = k-1$.

We proceed similarly. We need only consider $a = k-1, b = 0$ since the case $a = 0, b = k-1$ is just the (0,1)-complement. For $i \in T(k-1, 0)$, X_i has partition $C_i \cup D_i = [m]$ and columns of X_i have 1's on exactly $k-1$ rows of C_i and all 1's on rows D_i. Assume $\sum_{i \in T(k-1,0)} |X_i| \geqslant (tk)m^{k-1}$. Then there are tk choices $s(1), s(2), \ldots, s(tk) \in T(k-1, 0)$ where $s(1) < s(2) < \cdots < s(tk)$ such that, for some $C \in \binom{C_{s(i)}}{k-1}$, each $X_{s(i)}$ has a column with 1's in rows $C \cup D_{s(i)}$ and 0's in rows $C_{s(i)}\backslash C$. We wish to find $t \cdot F_{0,k-1,0,0}$ in A in rows C as follows using one column from each of $X_{s(i)}$ for $i = 1, 2, \ldots, t$ and t columns from $X_{s(tk)}$.

$$
\begin{array}{r c c c c c c}
 & & & & & & \overbrace{}^{t} \\
\text{rows } C \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} & \ldots & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} & \begin{array}{c} 0\,0\cdots0 \\ 0\,0\cdots0 \\ 0\,0\cdots0 \end{array} \\
 & X_{s(1)} & X_{s(2)} & & X_{s(t)} & X_{s(tk)} & X_{s(tk)}
\end{array}
$$

Given our choice $C \in \binom{C_{s(tk)}}{k-1}$, we compute that $|\{K \in \binom{C_{s(kt)}}{k-1} : K \cap C \neq \emptyset\}| < km^{k-2}$. Thus with $|X_{s(kt)}| \geqslant km^{k-2}$, there will be t choices $K_1, K_2, \ldots, K_t$ disjoint from C and hence one column of $X_{s(kt)}$ for each $i = 1, 2, \ldots, t$ with $\mathbf{1}_{k-1}$ on rows of $K_i \subseteq C_{s(kt)}\backslash C$ and 0's on $C_{s(kt)}\backslash K_i$ and hence $\mathbf{0}_{k-1}$ on rows C.

We will show below that we can choose $D \subset D_{s(kt)} \backslash \cup_{i=1}^{t} D_{s(i)}$ with $|D| = k$. Then we can find $t \cdot F_{0,k,k-1,0}$ as follows. We have one column in $X_{s(i)}$ for each $i = 1, 2, \ldots, t$ which is $\mathbf{1}_{k-1}$ on rows C and $\mathbf{0}_k$ on rows D (since $D \subset C_{s(i)}\backslash C$ for each $i = 1, 2, \ldots, t$). The t columns of $X_{s(tk)}$ we have selected have $\mathbf{0}_{k-1}$ on rows C and 1's on $D_{s(kt)}$ where $D \subseteq D_{s(kt)}$ and hence $\mathbf{1}_k$ on rows D. These $2t$ columns yield $t \cdot F_{0,k,k-1,0}$ in $[X_{s(1)} \,|\, X_{s(2)} \cdots |\, X_{s(t)} \,|\, X_{s(kt)}]$.

To show that D can be chosen we first show that $D_{s(i)} \backslash D_{s(j)} \leqslant k-2$ for $s(i) < s(j)$. Assume the contrary, $D_{s(i)} \backslash D_{s(j)} \geqslant k-1$ for $s(i) < s(j)$. We choose $C' \subseteq D_{s(i)} \backslash D_{s(j)}$ with $|C'| = k-1$. Given $s(j) > s(i)$, then $D_{s(j)} \backslash D_{s(i)} \geqslant k$ and so we may choose $D' \subseteq D_{s(j)} \backslash D_{s(i)}$ with $|D'| = k$. Now $C' \subset C_{s(j)}$ and $D' \subset C_{s(i)}$. The number of possible columns of $X_{s(j)}$ with at least one 1 on the rows C' is at most m^{k-2} and with $|X_{s(j)}| \geqslant m^{k-1} + t$, we find t columns of $X_{s(j)}$ with 0's on rows C' and necessarily with 1's on rows D'. The number of possible columns of $X_{s(i)}$ with at least one 1 on the rows of D' is $|D'|m^{k-2} < m^{k-1}$. Given $|X_{s(i)}| \geqslant m^{k-1} + t$, we find t columns of $X_{s(i)}$ with 0's on rows D' and necessarily with 1's on rows C'. This yields $t \cdot F_{0,k,k-1,0}$ in $[X_{s(i)} \,|\, X_{s(j)}]$, a contradiction. Thus $D_{s(i)} \backslash D_{s(j)} \leqslant k-2$ for $s(i) < s(j)$. We may now conclude that $|D_{s(kt} \backslash \cup_{i=1}^{t} D_{s(i)}| \geqslant k$ and so a choice for D exists. We conclude $\sum_{i \in T(k-1,0)} |X_i| \leqslant (tk)m^{k-1}$. This concludes Case 2.

There are $k+1$ choices for type (a,b) and so

$$\sum_{i=0}^{m} |X_i| \leqslant \sum_{j=0}^{k} \left(\sum_{i \in T(j,k-1-j)} |X_i| \right) \leqslant (k+1)(2tk)m^{k-1}$$

and so $\|A\| \leqslant (2tk(k+1))m^{k-1} + (6(k-1))^{5k+2}m^{k-2}$ which is $O(m^{k-1})$. □

Proof of Theorem 3 for $k \geqslant 3$: We use (5) so that

$\text{forb}(m, t \cdot F_{0,k,k,0}, t-1) \leqslant \text{forb}(m-1, t \cdot F_{0,k,k,0}, t-1) + (t-1)\text{forb}(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\})$.

Induction on m and Lemma 12 yields the bound. □

4 Some applications of the Induction

Let K_k denote the $k \times 2^k$ of all possible (0,1)-columns on k rows. The following is the fundamental result about forbidden configurations.

Theorem 13. *[Sauer [10], Perles and Shelah [11], Vapnik and Chervonenkis [12]] We have that*

$$\textit{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Thus forb(m, K_k) is $\Theta(m^{k-1})$.

We can apply this result as follows.

Theorem 14. *[8] Let F be a given $k \times \ell$ (0,1)-matrix. Then forb(m, F) is $O(m^k)$.*

Proof. Let t be the maximum multiplicity of a column in F (of course $t \leqslant \ell$). Then $F \prec t \cdot K_k$ and so $\text{supp}(F) \prec K_k$. Now Lemma 4 combined with Theorem 13 yields the result. □

Interestingly this yields the exact result for forb$(m, 2\cdot K_k)$ [9]. A more precise result of Anstee and Füredi [2] for forb$(m, t\cdot K_k)$ has the leading term being bounded by $\frac{t+k-1}{k+1}\binom{m}{k}$ for $t \geqslant 2$. The following surprising result was obtained by Balogh and Bollobás.

Theorem 15. *[6] Let k be given. There is a constant c_k with forb$(m, \{I_k, I_k^c, T_k\}) = c_k$.*

This yields the following.

Theorem 16. *Let $t, k \geqslant 2$ be given. Then forb$(m, \{t\cdot I_k, t\cdot I_k^c, t\cdot T_k\})$ is $\Theta(m)$.*

Proof. Apply Lemma 4. The matrix $I_m \in$ Avoid$(m, \{t\cdot I_k, t\cdot I_k^c, t\cdot T_k\})$ shows that forb$(m, \{t\cdot I_k, t\cdot I_k^c, t\cdot T_k\})$ is $\Theta(m)$. □

Lemma 4 is interesting for those H for which forb(m, H) is $O(m^\ell)$ and the number of rows in H is bigger than ℓ (see [5] for examples). It is not expected that this will resolve any *boundary cases*, namely those F for which forb$(m, [F\,|\,\alpha])$ is bigger than forb(m, F) by a linear factor (or more) for all choices α which are either not present in F or occur at most once in F. The previously mentioned $F_6(t)$ and $F_7(t)$ have quite complicated structure and the induction (5) does not appear to work directly.

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