Proof of Sun's conjecture on the divisibility of certain binomial sums

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Abstract

In this paper, we prove the following result conjectured by Z.-W. Sun:

$$(2n-1)\binom{3n}{n}\,\Bigg|\,\sum_{k=0}^{n}\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}$$

by showing that the left-hand side divides each summand on the right-hand side.

Keywords: congruences, binomial coefficients, super Catalan numbers, Stirling's formula

1 Introduction

In [4], Z.-W. Sun proved some new series for $1/\pi$ as well as related congruences on sums of binomial coefficients, such as

$$\sum_{n=0}^{\infty}\frac{n}{864^n}\sum_{k=0}^{n}\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}=\frac{1}{\pi},$$

and, for any prime $p>3$,

$$\sum_{n=0}^{p-1}\frac{n}{864^n}\sum_{k=0}^{n}\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}\equiv 0 \pmod{p^2}.$$

Sun [4] also proposed many interesting related conjectures, one of which is

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Conjecture 1.1 [4, Conjecture 4.2] *For $n = 0, 1, 2, \ldots,$ define*

$$s_n := \frac{1}{(2n-1)\binom{3n}{n}} \sum_{k=0}^{n} \binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}.$$

Then $s_n \in \mathbb{Z}$ for all n. Also,

$$\lim_{n\to\infty} \sqrt[n]{s_n} = 64. \tag{1.1}$$

Sun himself has proved that $s_n \equiv 0 \pmod 8$ for $n \geqslant 1$ and $s_{p-1} \equiv \lfloor (p-1)/6 \rfloor \pmod p$ for any prime p. In this paper, we shall prove that Conjecture 1.1 is true by establishing the following two theorems.

Theorem 1.2 *For $0 \leqslant k \leqslant n$, we have*

$$\frac{1}{(2n-1)\binom{3n}{n}} \binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k} \in \mathbb{Z}.$$

Note that, in [5,6], Sun proved many similar results on the divisibility of binomial coefficients.

Theorem 1.3 *For $n \geqslant 1$ and $0 \leqslant k < n/2$, we have*

$$\begin{aligned}
&\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k} \\
&\quad \geqslant \binom{6k+6}{3k+3}\binom{3k+3}{k+1}\binom{6(n-k-1)}{3(n-k-1)}\binom{3(n-k-1)}{n-k-1},
\end{aligned}$$

and hence

$$\frac{2}{2n-1}\binom{6n}{3n} \leqslant s_n \leqslant \frac{n+1}{2n-1}\binom{6n}{3n}. \tag{1.2}$$

It is easy to see that (1.1) follows from (1.2) and Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

2 Proof of Theorem 1.2

We need the following two lemmas.

Lemma 2.1 *Let $m, n \geqslant 1$ and $0 \leqslant k \leqslant n$. Then*

$$\binom{mn}{n} \,\Big|\, \binom{2mk}{mk}\binom{mk}{k}\binom{2m(n-k)}{m(n-k)}\binom{m(n-k)}{n-k}.$$

Proof. Observe that

$$\begin{aligned}
&\binom{2mk}{mk}\binom{mk}{k}\binom{2m(n-k)}{m(n-k)}\binom{m(n-k)}{n-k}\Big/\binom{mn}{n}\\
&=\frac{(2mk)!(2mn-2mk)!}{(mk)!(mn-mk)!(mn)!}\binom{(m-1)n}{(m-1)k}\binom{n}{k}. \qquad (2.1)
\end{aligned}$$

The proof then follows from the fact that numbers of the form

$$\frac{(2a)!(2b)!}{a!b!(a+b)!},$$

called the *super Catalan numbers*, are integers (see [1, 2, 7]). □

Lemma 2.2 *Let* $0 \leqslant k \leqslant n$ *be integers. Then*

$$(2n-1)\,\bigg|\,\frac{(6k)!(6n-6k)!(2n)!}{(3k)!(3n-3k)!(3n)!(2k)!(2n-2k)!},$$

or equivalently,

$$\frac{(6k)!(6n-6k)!(2n)!(2n-2)!}{(3k)!(3n-3k)!(3n)!(2k)!(2n-2k)!(2n-1)!} \in \mathbb{Z}.$$

Corollary 2.3 *Let* $n \geqslant 1$*. Then* $(2n-1)$ *divides* $\binom{6n}{3n}$*.*

For the p-adic order of $n!$, there is a known formula

$$\operatorname{ord}_p n! = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor, \qquad (2.2)$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to a real number x. In order to prove Lemma 2.2, we first establish the following result.

Lemma 2.4 *Let* $m \geqslant 2$ *and* $0 \leqslant k \leqslant n$ *be integers. Then*

$$\begin{aligned}
&\left\lfloor \frac{6k}{m} \right\rfloor + \left\lfloor \frac{6n-6k}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + \left\lfloor \frac{2n-2}{m} \right\rfloor\\
&\geqslant \left\lfloor \frac{3k}{m} \right\rfloor + \left\lfloor \frac{3n-3k}{m} \right\rfloor + \left\lfloor \frac{3n}{m} \right\rfloor + \left\lfloor \frac{2k}{m} \right\rfloor + \left\lfloor \frac{2n-2k}{m} \right\rfloor + \left\lfloor \frac{2n-1}{m} \right\rfloor, \qquad (2.3)
\end{aligned}$$

unless $m = 3$, $n \equiv 2 \pmod 3$*, and* $k \not\equiv 1 \pmod 3$.

Proof. For any real numbers x and y, we have (see, for example, [3, Division 8, Problems 8 and 136])

$$\begin{aligned}
\lfloor 2x \rfloor + \lfloor 2y \rfloor &\geqslant \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor,\\
\lfloor x+y \rfloor &\geqslant \lfloor x \rfloor + \lfloor y \rfloor.
\end{aligned}$$

It follows that

$$\left\lfloor\frac{6k}{m}\right\rfloor+\left\lfloor\frac{6n-6k}{m}\right\rfloor+\left\lfloor\frac{2n}{m}\right\rfloor\geqslant\left\lfloor\frac{3k}{m}\right\rfloor+\left\lfloor\frac{3n-3k}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor+\left\lfloor\frac{2k}{m}\right\rfloor+\left\lfloor\frac{2n-2k}{m}\right\rfloor.$$

Now suppose that (2.3) does not hold. Then we must have

$$\left\lfloor\frac{6k}{m}\right\rfloor+\left\lfloor\frac{6n-6k}{m}\right\rfloor=\left\lfloor\frac{3k}{m}\right\rfloor+\left\lfloor\frac{3n-3k}{m}\right\rfloor+\left\lfloor\frac{3n}{m}\right\rfloor, \tag{2.4}$$

$$\left\lfloor\frac{2n}{m}\right\rfloor=\left\lfloor\frac{2k}{m}\right\rfloor+\left\lfloor\frac{2n-2k}{m}\right\rfloor, \tag{2.5}$$

$$\left\lfloor\frac{2n-2}{m}\right\rfloor<\left\lfloor\frac{2n-1}{m}\right\rfloor, \tag{2.6}$$

and so, by (2.6), $m\mid 2n-1$, then by (2.5), $m\mid 2k$ or $m\mid 2n-2k$. If $m\mid 2k$, then $m\mid k$ (since $m\mid 2n-1$ means that m is odd), the identity (2.4) implies that

$$\left\lfloor\frac{6n}{m}\right\rfloor=2\left\lfloor\frac{3n}{m}\right\rfloor. \tag{2.7}$$

Since $m\mid 2n-1$, the identity (2.7) can be written as

$$\frac{2n-1}{m}+\left\lfloor\frac{3}{m}\right\rfloor=2\left\lfloor\frac{n+1}{m}\right\rfloor. \tag{2.8}$$

If $m\geqslant 4$, then the left-hand side of (2.8) equals $(2n-1)/m$, while the right-hand side of (2.8) belongs to

$$\left\{\frac{2n+2}{m},\ \frac{2n}{m},\ \frac{2n-2}{m},\dots\right\},$$

a contradiction! Therefore, $m\leqslant 3$. Since $m\geqslant 2$ is odd, we must have $m=3$. Hence, $n\equiv 2 \pmod 3$ and $k\equiv 0 \pmod 3$. Similarly, if $m\mid 2n-2k$, then we deduce that $m=3$, $n\equiv 2 \pmod 3$ and $k\equiv 2 \pmod 3$. This proves the lemma. □

Proof of Lemma 2.2. For any prime $p\neq 3$, by (2.2) and (2.3), we have

$$\begin{aligned}&\operatorname{ord}_p(6k)!+\operatorname{ord}_p(6n-6k)!+\operatorname{ord}_p(2n)!+\operatorname{ord}_p(2n-2)!\\&\geqslant \operatorname{ord}_p(3k)!+\operatorname{ord}_p(3n-3k)!+\operatorname{ord}_p(3n)!+\operatorname{ord}_p(2k)!+\operatorname{ord}_p(2n-2k)!+\operatorname{ord}_p(2n-1)!.\end{aligned} \tag{2.9}$$

For $p=3$, since $\operatorname{ord}_3(3j)!=j+\operatorname{ord}_3 j!$, the inequality (2.9) reduces to

$$\operatorname{ord}_3(2n)!+\operatorname{ord}_3(2n-2)!\geqslant \operatorname{ord}_3 k!+\operatorname{ord}_3(n-k)!+\operatorname{ord}_3 n!+\operatorname{ord}_3(2n-1)!. \tag{2.10}$$

Noticing that

$$\frac{(2n)!(2n-2)!}{k!(n-k)!n!(2n-1)!}=\frac{1}{2n-1}\binom{2n}{n}\binom{n}{k}=\left(4\binom{2n-2}{n-1}-\binom{2n}{n}\right)\binom{n}{k}\in\mathbb{Z},$$

the inequality (2.10) holds. Namely, the inequality (2.9) is true for $p = 3$. This completes the proof. □

Proof of Theorem 1.2. By (2.1), one sees that

$$\begin{aligned}&\frac{1}{(2n-1)\binom{3n}{n}}\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}\\&=\frac{1}{2n-1}\frac{(6k)!(6n-6k)!}{(3k)!(3n-3k)!(3n)!}\binom{2n}{2k}\binom{n}{k},\end{aligned}$$

which is an integer divisible by $\binom{n}{k}$ in view of Lemma 2.2. □

3 Proof of Theorem 1.3

Let

$$A_{n,k}=\binom{6k}{3k}\binom{3k}{k}\binom{6(n-k)}{3(n-k)}\binom{3(n-k)}{n-k}.$$

Then, for $n \geqslant 1$ and $0 \leqslant k < n/2$, we have

$$\frac{A_{n,k}}{A_{n,k+1}}-1=\frac{(36nk+31n-36k^2-36k-5)(n-2k-1)}{(6k+5)(6k+1)(n-k)^2}\geqslant 0,$$

i.e., $A_{n,k} \geqslant A_{n,k+1}$. Since $A_{n,k} = A_{n,n-k}$, for $n \geqslant 1$, we have

$$2\binom{6n}{3n}\binom{3n}{n}=2A_{n,0}\leqslant\sum_{k=0}^{n}A_{n,k}\leqslant(n+1)A_{n,0}=(n+1)\binom{6n}{3n}\binom{3n}{n}.$$

In other words, the inequality (1.2) holds.

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