On the codegree density of complete 3-graphs and related problems

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Abstract

Given a 3-graph F, its codegree threshold co-ex(n, F) is the largest number $d = d(n)$ such that there exists an n-vertex 3-graph in which every pair of vertices is contained in at least d triples but which contains no member of F as a subgraph. The limit

$$\gamma(F) = \lim_{n \to \infty} \frac{\text{co-ex}(n, F)}{n - 2}$$

is known to exist and is called the codegree density of F.

In this paper we generalise a construction of Czygrinow and Nagle to bound below the codegree density of complete 3-graphs: for all integers $s \geqslant 4$, the codegree density of the complete 3-graph on s vertices K_s satisfies

$$\gamma(K_s) \geqslant 1 - \frac{1}{s - 2}.$$

We then provide constructions based on Steiner triple systems which show that if this lower bound is sharp, then we do not have stability in general.

In addition we prove bounds on the codegree density for two other infinite families of 3-graphs.

Keywords: Extremal hypergraph theory; codegree density

1   Introduction

In this paper, we study codegree density for various families of 3-graphs.

1.1   Notation and definitions

Given a set A and an integer r, write $A^{(r)}$ for the collection of r-subsets of A. Also, for $n \in \mathbb{N}$ write $[n]$ for $\{1, 2, \ldots n\}$.

A *3-graph* is a pair $G = (V, E)$, where $V = V(G)$ is a set of vertices and $E = E(G) \subseteq V^{(3)}$ is a set of triples, or *3-edges*. A *subgraph* of G is a 3-graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *codegree* $d(x, y)$ of vertices $x, y \in V(G)$ is the number of 3-edges of G containing the pair $\{x, y\}$. The *minimum codegree* of G is $\delta_2(G) = \min_{xy \in V^{(2)}} d(x, y)$.

We shall also consider some *2-graphs*, or ordinary graphs, which are pairs $G = (V, E)$, with E now a set of (2-)edges, $E \subseteq V^{(2)}$.

We recall the classical definitions of the Turán number and Turán density of a 3-graph.

Definition 1. Let $n \in \mathbb{N}$, and let F be a non-empty 3-graph. The *Turán number* $\mathrm{ex}(n, F)$ of F is the largest number $e = e(n)$ such that there exists an n-vertex 3-graph with at least e triples and no copy of F as a subgraph.

An easy averaging argument shows that the sequence $\mathrm{ex}(n, F)/\binom{n}{3}$ is monotone decreasing in $[0, 1]$, and hence converges to a limit, known as the Turán density.

Definition 2. The *Turán density* of a non-empty 3-graph F is the limit

$$\pi(F) = \lim_{n \to \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{3}}.$$

The Turán density may be thought of as the asymptotically maximal proportion of 3-edges which may be present in an F-free 3-graph, and is one of the central objects of study in extremal hypergraph theory. In this paper, we are interested in another limit density, namely codegree density.

Definition 3. Let $n \in \mathbb{N}$ and let F be a non-empty 3-graph. The *codegree threshold* of F co-$\mathrm{ex}(n, F)$ is the largest number $d = d(n)$ such that there exists an n-vertex 3-graph in which every pair of vertices is contained in at least d triples but which contains no copy of F as a subgraph.

Mubayi and Zhao [15] showed that for any F, the sequence co-$\mathrm{ex}(n, F)/(n-2)$ tends to a limit as $n \to \infty$. (Note that this sequence is not monotone, as shown by Lo and Markström [13], so that the existence of a limit is not trivial.) This allows us to define the codegree density of a non-empty 3-graph.

Definition 4. The *codegree density* of a non-empty 3-graph F is defined to be the limit

$$\gamma(F) = \lim_{n \to \infty} \frac{\text{co-}\mathrm{ex}(n, F)}{n - 2}.$$

1.2 History

Codegree density was first studied by Mubayi [14], who determined it for the Fano plane F_7. Keevash [10] used hypergraph regularity to show co-ex$(n, F_7) = \lfloor n/2 \rfloor$ for n sufficiently large, with the unique extremal configuration being a complete balanced bipartite 3-graph. De Biasio and Jiang [4] later gave another proof of this fact avoiding the use of hypergraph regularity.

Mubayi and Zhao [15] showed that codegree densities are well-defined and studied various properties of γ. In particular they showed that for every $c \in [0, 1]$ there exists a family of 3-graphs $\mathcal{F}$ with $\gamma(\mathcal{F}) = c$ (so that codegree density does not 'jump'), and that the 'supersaturation' phenomenon familiar from extremal graph theory also occurs for codegree density. (See [15] for details and definitions.)

Marchant, Pikhurko, Vaughan and the author [5] determined the codegree threshold of $F_{3,2} = ([5], \{123, 124, 125, 345\})$, while Pikhurko, Vaughan and the author determined the codegree density of $K_4^- = ([4], \{123, 124, 134\})$, resolving a conjecture of Nagle [16].

Nagle [16] and Czygrinow and Nagle [3] have in addition conjectured that $\gamma(K_4) = 1/2$, where K_4 denotes the complete 3-graph on 4 vertices. We describe their lower-bound construction below.

Construction 5 (Czygrinow and Nagle's construction)**.** *Let $n \in \mathbb{N}$. Let T be a tournament (an orientation of the edges of the complete 2-graph) on $[n]$. We define a 3-graph G_T on $[n]$ by setting ijk with $i < j < k$ to be a 3-edge of G_T if the ordered pairs (i, j) and (i, k) receive opposite orientations in T.*

It is easy to check that G_T has no K_4 subgraph and that by choosing T uniformly at random we obtain a 3-graph that with high probability has minimum codegree at least $n/2 - o(n)$. Czygrinow and Nagle conjectured that this was asymptotically best possible, in other words that $\gamma(K_4) = 1/2$.

No other codegree densities are known or conjectured, and like Turán's famous conjecture that $\pi(K_4) = 5/9$, the Czygrinow–Nagle conjecture remains wide open. We refer a reader to Keevash's recent survey [11] for a more complete discussion of Turán-type problems for 3-graphs.

1.3 Contribution of this paper

In this note we first describe a general construction showing that

$$\gamma(K_s) \geqslant 1 - \frac{1}{s-2}$$

for all $s \geqslant 4$, where K_s denotes the complete 3-graph on s vertices, $K_s = ([s], [s]^{(3)})$. This construction is an easy generalization of the Czygrinow–Nagle construction based on random edge-colourings of the complete 2-graph. Since the first version of this paper came out, we learned that Lo and Markström had made the same observation in an earlier preprint (see Proposition 1.1 in [13]).

Our main contribution is to show that for s congruent to 1 or 5 modulo 6, we can give different non-isomorphic constructions giving the same lower bound on $\gamma(K_s)$. These are based on Steiner triple systems, and imply that if our lower-bound is tight (as we believe) then the codegree density problem for complete 3-graphs is not stable in general: several very different near-extremal configurations exist. This mirrors the conjectured behaviour of Turán density for complete 3-graphs (see [11, 17]). In the particular case $s = 6$, we are also able to give an alternative random construction based on Ramsey numbers showing $\gamma(K_6) \geqslant 3/4$.

Finally we also give bounds on the codegree and Turán densities of two other families of 3-graphs and present a number of open problems.

Our paper is structured as follows. In Section 2, we give several different lower bound constructions for the codegree density of complete 3-graphs. In Section 3, we turn our attention to 3-graphs of the form $([t] \sqcup \{x_\star\}, \{ijx_\star : 1 \leqslant i < j \leqslant t\})$ (these correspond to complete 2-graphs on t vertices in the links of the vertices — see Section 3 for a formal definition) and give general bounds for both their codegree density and their Turán density. Finally in Section 4 we introduce co-spanned 3-graphs, and give bounds on their codegree density.

2 Complete 3-graphs

Let $n \in \mathbb{N}$ and $[n] = \{1, 2, \ldots n\}$.

Construction 6 (Colouring construction). *Let $c : [n]^{(2)} \to [s]$ be a colouring of the edges of the complete 2-graph on $[n]$ with s colours. We construct a 3-graph G_c based on this colouring in the following manner: for every triple $i, j, k \in [n]$ with $i < j < k$, we add the 3-edge ijk to $E(G_c)$ if and only if $c(ij) \neq c(ik)$.*

Remark 7. This may naturally be viewed as a generalisation of the Czygrinow–Nagle construction: we may obtain a 2-colouring of $[n]^{(2)}$ from a tournament T on $[n]$ by setting $c(ij) = 1$ for $i < j$ if ij is oriented from i to j in T, and $c(ij) = 2$ if instead ij is oriented from j to i.

Since the first version of this paper came out, we learned that Lo and Markström had also given this construction in a preprint (see Proposition 1.1 in [13]).

Proposition 8. *For any colouring c as above, the 3-graph G_c is K_{s+2}-free.*

Proof. Let $i_1, i_2, \ldots i_{s+2}$ be a set of $s + 2$ distinct vertices from $[n]$ with $i_1 < i_2 \ldots < i_{s+2}$. The $(s + 1)$ pairs $i_1 i_2, i_1 i_3, \ldots i_1 i_{s+2}$ are s-coloured, so by the pigeon-hole principle two of them receive the same colour, say ij_1 and ij_2, and the 3-edge $ij_1 j_2$ is therefore missing from G_c. Thus G_c is K_{s+2}-free as claimed. □

Remark 9. Construction 6 does allow copies of K_{s+2} with one triple removed to appear as a subgraph. Indeed s-colour the pairs from $[s+2]$ by setting $c(ij) = j - i \pmod{s}$ for $i < j$. The only triple of $[s+2]^{(3)}$ missing from G_c is $12(s+2)$. This suggests Construction 6 is in some sense saturated, and makes it a plausible candidate for an extremal configuration.

Theorem 10. *For all integers $s \geqslant 2$,*

$$\gamma(K_{s+2}) \geqslant 1 - \frac{1}{s}.$$

Proof. Let n be sufficiently large. Independently colour each pair from $[n]$ with an element of $[s]$ chosen uniformly at random, and let $\mathbf{c}$ denote the random colouring thus obtained.

Consider now the 3-graph $G_{\mathbf{c}}$. By Proposition 8, we know it is K_{s+2}-free. We show that with high probability it has minimum codegree

$$\delta_2(G_{\mathbf{c}}) = \left(1 - \frac{1}{s}\right)n + o(n).$$

For each pair $ij \in [n]^{(2)}$ with $i < j$, and every $k \in [n] \setminus \{ij\}$, let $X_{ij,k}$ be the Bernoulli random variable taking the value 1 if $ijk \in E(G_{\mathbf{c}})$ and 0 otherwise. The sum $X_{ij} = \sum_{k \in [n] \setminus \{ij\}} X_{ij,k}$ is then the codegree of i, j in $G_{\mathbf{c}}$.

Claim 11. *Fix $i < j$. Then $\{X_{ij,k} : k \in [n] \setminus \{ij\}\}$ forms a family of independent identically distributed Bernoulli random variables with parameter $1 - \frac{1}{s}$.*

This is a simple exercise – for completeness, a proof is included in the appendix to this paper. We can now apply a standard Chernoff bound. Fix $\varepsilon > 0$.

$$\mathbb{P}\left(\delta_2(G_{\mathbf{c}}) \leqslant (1 - \frac{1}{s} - \varepsilon)n\right) \leqslant \binom{n}{2}\mathbb{P}\left(d_{G_{\mathbf{c}}}(1,2) \leqslant (1 - \frac{1}{s} - \varepsilon)n\right)$$
$$\leqslant n^2 e^{-\frac{\varepsilon^2}{2}n}$$
$$= o(1).$$

Thus for a typical colouring $\mathbf{c}$, the minimum codegree of $G_{\mathbf{c}}$ is at least $(1 - \frac{1}{s} - \varepsilon)n$. Since $\varepsilon > 0$ was arbitrary, it follows that

$$\gamma(K_{s+2}) \geqslant 1 - \frac{1}{s},$$

as claimed. □

Our main contribution in this paper is that other, very different constructions are possible. Thus if the lower-bound on the codegree density of K_{s+2} given above is tight then in general we do not have stability for the codegree densities of complete 3-graphs.

Construction 12 (A Steiner triple system construction). *Let $s \geqslant 5$ be an integer congruent to 3 or 5 modulo 6. Let S be a Steiner triple system on $[s-2]$ — that is, a 3-graph on $[s-2]$ such that every pair of vertices is contained in exactly one 3-edge. Such systems are known to always exist, subject to the aforementioned modulo 6 condition [12].*

Given $n \in \mathbb{N}$, let $\sqcup_{i=1}^{s-2} V_i$ be a balanced $(s-2)$-partition of $[n]$ (i.e. a partition with part sizes as equal as possible). Write 'triples of the form $V_i V_j V_k$' as a shorthand for 'triples xyz with $(x, y, z) \in V_i \times V_j \times V_k$'. We define a 3-graph G_S on the vertex set $[n]$ by taking the following triples to form the 3-edge set:

- *all triples of the form $V_i V_i V_j$ for distinct $i, j \in [s-2]$*

- *all triples of the form $V_i V_j V_k$ for distinct $i, j, k \in [s-2]$ such that ijk does not belong to S.*

(This is equivalent to taking the complement of a blow-up of S and removing all triples which meet only one of the parts V_i.)

Proposition 13. *For any $n \in \mathbb{N}$, s congruent to 3 or 5 modulo 6 and any Steiner triple system S on $[s-2]$, the 3-graph G_S is K_s-free and has minimum codegree $(1-\frac{1}{s-2})n+O(1)$.*

Proof. First of all let us establish G_S is K_s-free. Note that as we have no 3-edges of the form $V_i V_i V_i$, any s-set of vertices meeting some part V_i in at least 3 vertices cannot span a K_s. Now any s-set meeting no part in more than 2 vertices must meet at least $\lceil s/2 \rceil$ parts.

We claim that every set of at least $\lceil s/2 \rceil$ vertices from $[s-2]$ must span at least one 3-edge of S. Indeed label such a set as $X = \{x_1, x_2, \ldots, x_{\lceil s/2 \rceil}\}$. Suppose for contradiction that X is an independent set in S. Then since S is a Steiner triple system, for each of x_i, $i = 2, \ldots \lceil s/2 \rceil$, there exists a unique y_i such that $x_1 x_i y_i$ is a 3-edge of S, and moreover these y_i are distinct (else the pair $x_1 y_i$ would be contained in more than one 3-edge). Thus we would need

$$\left\lceil \frac{s}{2} \right\rceil - 1 \leqslant |[s-2] \setminus X| = s - \left\lceil \frac{s}{2} \right\rceil - 2,$$

a contradiction.

Thus any s-set of vertices meeting at least $\lceil s/2 \rceil$ different parts V_i must meet three distinct part V_i, V_j, V_k such that ijk is a 3-edge of S. By construction, no triple of the form $V_i V_j V_k$ is a 3-edge of G_S, and thus our s-set misses at least one 3-edge. It follows that G_S is K_s-free, as claimed.

Now let us compute its codegree. Consider two vertices v, v' of G_S. If they belong to the same part V_i, then for every vertex $w \notin V_i$, $vv'w$ is a 3-edge of G_S, and thus the codegree of v and v' is at least $(1 - \frac{1}{s-2})n + O(1)$ (since our partition was balanced).

On the other hand suppose that $v \in V_i$ and $v' \in V_{i'}$ for some distinct $i, i' \in [s-2]$. Then for any $w \in V_i \sqcup V_{i'}$, $vv'w$ is a 3-edge of G_S. In addition, let us denote by i'' the unique member of $[s-2]$ such that $ii'i''$ is a 3-edge of the Steiner triple system S. Then for all $j \in [s-2] \setminus \{i, i', i''\}$ and all $w \in V_j$, $vv'w$ is a 3-edge of G_S. Thus the codegree of v, v' is again at least $(1 - \frac{1}{s-2})n + O(1)$. $\qquad\square$

Remark 14. The construction above based on Steiner triple system gives a rather large number of non-isomorphic constructions: while the Fano plane is the unique (up to isomorphism) Steiner triple system on 7 vertices, there are for example 11,084,874,829 non-isomorphic Steiner triple systems on 19 vertices (see Section 4.5 in [2]).

Remark 15. The 3-graphs arising from Construction 12 have large (linear-sized) independent vertex sets, whereas it is easy to show using Chernoff bounds that typical 3-graphs arising from Construction 6 and random colourings c do not. Thus our two constructions are genuinely different.

For $s = 6$, we can give another construction based on random colourings and the Ramsey number of the triangle.

Construction 16 (A Ramsey-based construction). *Let $n \in \mathbb{N}$. Given a colouring $c : [n]^{(2)} \to \{1, 2\}$, let G_c be the 3-graph on $[n]$ with 3-edges consisting of all triples not inducing a monochromatic triangle with respect to c.*

Since the Ramsey number for monochromatic triangles in 2-coloured graphs is $R(3, 3) = 6$, every 6-set of vertices in G_c must be missing at least one 3-edge, so that G_c is K_6-free as required. The expected codegree of a pair of vertices in a typical colouring c is $\frac{3}{4}(n-1)$. Applying the same probabilistic tools as in Theorem 10, we can easily recover from this another proof of $\gamma(K_6) \geqslant 3/4$.

Remark 17. Typical instances of Construction 16 and Construction 6 with $s = 4$ are genuinely different. Indeed let c and c' be respectively a 2- and a 4-colouring of $[n]^{(2)}$, chosen uniformly at random. Consider a 5-set $U = \{u_1, u_2, u_3, u_4, u_5\}$ from $[n]$ with $1 \leqslant u_1 < u_2 < \ldots < u_5 \leqslant n$.

The probability that U induces a copy of K_5 in G_c is the probability that c decomposes the pairs from U into two monochromatic 5-cycles, one with colour 1 and the other with colour 2. This occurs with probability

$$\#\{\text{decompositions}\} \times 2^{-10} = 3 \times 2^{-7}.$$

On the other hand for $i = 1, 2, 3$, let A_i be the event that for every $j, j' : i < j < j' \leqslant 5$, c' assigns different colours to $u_i u_j$ and $u_i u_{j'}$; this is exactly the event that all 3-edges of the form $u_i u_j u_{j'}$ with $i < j < j'$ are in the 3-graph $G_{c'}$ obtained from c' by applying Construction 6.

Observe that the characteristic functions of the A_i events form a family of independent random variables, since they depend on the colours assigned by c' to disjoint edge-sets, and that the probability that U induces a copy of K_5 in $G_{c'}$ is exactly the probability that $\bigcap_{i=1}^{3} A_i$ occurs. This probability is thus

$$\mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) = \frac{4!}{4^4} \frac{4!/1!}{4^3} \frac{4!/2!}{4^2}$$
$$= \frac{3^3}{2^{10}}.$$

It follows in particular that K_5 subgraphs have different frequencies in G_c and $G_{c'}$ for typical c, c'. The lower-bound constructions for $\gamma(K_6)$ arising from Construction 6 and Construction 16 are thus genuinely different.

An obvious question to ask is whether the lower bound we provide is tight.

Question 18. Is $\gamma(K_s) = 1 - \frac{1}{s-2}$ for all $s \geqslant 4$?

For comparison, let us note that the Turán density of K_s is conjectured to be $1 - \frac{4}{(s-1)^2}$, with many known non-isomorphic constructions attaining that bound [11]. However the

standard examples contain pairs of vertices with codegree only $1 - \frac{2}{s-1}$ proportion of the maximum possible.

It would also be interesting to know whether the Czygrinow–Nagle construction is the only example of a K_4-free 3-graph with codegree density $1/2 + o(1)$.

Question 19. Are all K_4-free configurations with codegree density $1/2 + o(1)$ 'close' to a Czygrinow–Nagle construction?

3 Suspensions of complete 2-graphs

Let G be a 3-graph. Given a vertex $x \in V(G)$, we may form a 2-graph in a natural way by considering the pairs of vertices v, v' making a 3-edge with x in G.

Definition 20. The *link graph* of $x \in V(G)$ is the 2-graph

$$G_x = \left(V(G) \setminus \{x\}, \{vv' : \ xvv' \in E(G)\}\right).$$

For every 2-graph H we can consider the 3-graph corresponding to the presence of an H-subgraph in a link-graph.

Definition 21. Given a 2-graph H, let $S(H)$ denote the *suspension* of H, that is, the 3-graph with vertex set $V(H) \sqcup \{x_\star\}$ and 3-edges

$$\{x_\star vv' : \ vv' \in E(H)\}.$$

So for example the complete 3-graph on 4 vertices with one 3-edge removed, K_4^-, may be thought of as the suspension $S(K_3^{(2)})$ of the ordinary triangle $K_3^{(2)}$.

As mentioned in the introduction, Pikhurko, Vaughan and the author [6] recently showed $\gamma(K_4^-) = 1/4$, with the extremal configuration obtained by taking as the 3-edges the oriented triangles in a random tournament. It is rather natural to ask what $\gamma(S(K_s^{(2)}))$ may be in general, where $K_s^{(2)}$ is the complete graph on s vertices,

$$K_s^{(2)} = ([s], \{ij : 1 \leqslant i < j \leqslant s\}).$$

Problem 22. Give bounds for $\gamma(S(K_s^{(2)}))$.

Note that by Turán's theorem [18] and averaging we have $\gamma(S(K_s^{(2)})) \leqslant 1 - \frac{1}{s-1}$. We give below a construction (which we do not believe is sharp in general) which shows that this trivial upper bound is not off by more than a multiplicative factor of $1 - \frac{1}{s-2}$.

Construction 23 (Rainbow triangles)**.** *Let $c : [n]^{(2)} \to [s-1]$ be a colouring of the edges of the complete 2-graph on $[n]$ with $s - 1$ colours. We construct a 3-graph G_c based on this colouring in the following manner: for every triple $i, j, k \in [n]$, add the 3-edge ijk to $E(G_c)$ if and only if $c(ij), c(ik), c(jk)$ are distinct — that is, if each of the sides of the triangle ijk receives a different colour. We call such a triangle a 'rainbow triangle'.*

Proposition 24. *For every colouring c, the 3-graph G_c is $S(K_s^{(2)})$-free.*

Proof. Let $v_0, v_1, \ldots, v_s$ be an $s+1$-set of vertices from $[n]$. Then by the pigeon-hole principle, there exist $1 \leqslant i < j \leqslant s$ such that $c(v_0 v_i) = c(v_0 v_j)$. It follows that the triangle $v_0 v_i v_j$ is not rainbow, and hence that $v_1, v_2 \ldots v_s$ do not induce a copy of $K_s^{(2)}$ in the link graph of v_0 in G_c. Since $v_0, v_1, \ldots, v_s$ was arbitrary, it follows that G_c is $S(K_s^{(2)})$-free, as claimed. $\qquad\square$

Corollary 25.
$$\gamma(S(K_s^{(2)})) \geqslant (1 - \frac{1}{s-1})(1 - \frac{2}{s-1}).$$

Proof. Picking c uniformly at random and applying Construction 23, we have that the expected codegree of any pair of vertices in G_c is $(1 - \frac{1}{s-1})(1 - \frac{2}{s-1})(n-2)$. Applying the same probabilistic tools as in Theorem 10, we obtain from this a proof that $\gamma(S(K_s^{(2)})) \geqslant (1 - \frac{1}{s-1})(1 - \frac{2}{s-1})$. $\qquad\square$

Remark 26. Since $\gamma(S(K_3^{(2)})) = \gamma(K_4^-) = \frac{1}{4}$, we know this bound on $\gamma(S(K_s^{(2)}))$ fails to be sharp for $s = 3, 4$. Given this, it seems unlikely that this construction is sharp for $s \geqslant 5$.

We note that the analogue of Problem 22 for Turán density is also open.

Problem 27. Give bounds for $\pi(S(K_s^{(2)}))$.

The Turán density of $S(K_3^{(2)}) = K_4^-$ and $S(K_4^{(2)})$ are conjectured to be 2/7 and 1/2 respectively, with the lower-bounds coming from recursive constructions due to Frankl and Füredi [8] and Bollobás, Leader and Malvenuto [1] respectively. Close to matching upper bounds were obtained using flag algebras by Vaughan and the author [7], suggesting the lower bounds are best possible.

Below we give a generalisation of Bollobás, Leader and Malvenuto's construction for all integers $s \geqslant 4$ which are not divisible by 3, which we conjecture is best possible.

Construction 28 (Iterated complements of Steiner triple systems). *Let $s \geqslant 2$ be an integer congruent to 1 or 2 modulo 3. Then $2s - 1$ is congruent to 1 or 3 modulo 6. Let S be a Steiner triple system on $[2s-1]$ — such a system is known to always exist, subject to the aforementioned modulo 6 condition [12].*

Given $n \in \mathbb{N}$, we construct a 3-graph G_S in an iterated fashion as follows. First of all, take a balance partition of $[n]$ into $2s-1$ parts $V_1, V_2 \ldots V_{2s-1}$. Now take as 3-edges all triples of the form $V_i V_j V_k$ with $1 \leqslant i < j < k \leqslant 2s-1$ and $ijk \notin E(S)$. (This is equivalent to taking a blow-up of the complement of S.) Finally, repeat this construction inside each of the $2s-1$ parts $V_1, V_2 \ldots V_{2s-1}$.

Proposition 29. *The 3-graph G_s is $S(K_s^{(2)})$-free and contains $\left(1 - \frac{2}{s}\right)\binom{n}{3} + O(n^2)$ 3-edges.*

Proof. Suppose for contradiction that we have an $(s+1)$-set of vertices $v_0, v_1 \ldots v_s$ in G_S such that $v_1, \ldots v_s$ span a copy of $K_s^{(2)}$ in the link-graph of v_0.

Without loss of generality we may assume that $v_0 \in V_{2s-1}$. We may also assume that at least one vertex v_i, $1 \leqslant i \leqslant s$, lies in a different part from v_0 — without loss of generality let us say we have $v_1 \in V_1$.

Since S is a Steiner triple system, for every $i \in [2s-2]$, we have a unique $j \in [2s-2] \backslash \{j\}$ such that $ij(2s-1)$ is a 3-edge of S. This defines a matching on $[2s-2]$; by relabelling the parts if necessary, we may assume that this matching is $i(s-1+i) : \ i \in [s-1]$.

Thus if $u \in V_i$ and $u' \in V_{s-1+i}$ then $uu'v_0$ it not a 3-edge of G_S. By construction, if $u \in V_{2s-1}$ then uv_0v_1 is not a 3-edge of G_S; also, if u, u' lie in the same part V_i, $i \in [2s-2]$, then $uu'v_0$ is not a 3-edge of G_S.

We must thus have $v_1, v_2 \ldots v_s$ lie in s distinct parts from $V_1, V_2, \ldots V_{2s-2}$ (or else we do not have a copy of $S(K_s^{(2)})$). But placing a vertex inside part V_i for some $1 \leqslant i \leqslant (s-1)$ forbids us from placing any vertex in part V_{s-1+i}, and vice-versa. Since $2 \times s - 1 > |[2s-2]| = 2s-2$, this contradicts the pigeon-hole principle. The 3-graph G_S is thus $S(K_s^{(2)})$-free, as claimed.

Now the number of 3-edges contained in G_S is

$$|E(G_S)| = \left(\binom{2s-1}{3} - |E(S)| \right) \left(\frac{n}{2s-1} \right)^3 \times \left(1 + (2s-1) \times \frac{1}{(2s-1)^3} \right.$$
$$\left. + (2s-1)^2 \times \frac{1}{(2s-1)^6} + \cdots \right) + O(n^2)$$
$$= \left(\frac{2s-4}{2s} + o(1) \right) \binom{n}{3},$$

as required. $\qquad\square$

Corollary 30. *Let $s \in \mathbb{N}$ be congruent to 1 or 2 modulo 3. Then*

$$\pi(S(K_s^{(2)})) \geqslant 1 - \frac{2}{s}.$$

Conjecture 31. The lower bound given above in Corollary 30 is sharp.

By Turán's theorem and averaging, we have that

$$\pi(S(K_s^{(2)})) \leqslant 1 - \frac{1}{s-1}$$

for all $s \geqslant 2$. On the other hand Construction 28 shows that $\pi(S(K_s^{(2)})) \geqslant 1 - \frac{1}{2s} + O\left(\frac{1}{s^2}\right)$. It seems likely that $\pi(S(K_s^{(2)})) = 1 - \frac{C}{s} + O\left(\frac{1}{s^2}\right)$ for some constant $C \in [1, 2]$. As a first step towards a general result, it would be interesting to identify the correct value of C (or indeed prove that such a constant exists!).

Conjecture 32 (Weakening of Conjecture 31). For every $\varepsilon > 0$ there exists $s_0 = s_0(\varepsilon)$ such that for all $s \geqslant s_0$ we have

$$\pi(S(K_s^{(2)})) \leqslant 1 - \frac{2-\varepsilon}{s}.$$

Similarly, we would like to know if

$$\gamma(S(K_s^{(2)})) = 1 - \frac{C'}{s} + O\left(\frac{1}{s^2}\right)$$

for some constant $C' \in [1,3]$, and to know the value of C' (if it does exist).

4   Co-spanned complete 3-graphs

In this section we give bounds on the codegree densities of another family of 3-graphs.

Definition 33. Given a 3-graph H, let $F(H)$ denote the *co-spanned H*, that is, the 3-graph with vertex set $V(H) \sqcup \{x_\star, y_\star\}$ and 3-edges

$$\{x_\star y_\star v : \ v \in V(H)\} \sqcup E(H).$$

So for example the 3-graph $F_{3,2} = ([5], \{123, 124, 125, 345\})$ may be thought of as the co-spanned K_3.

As mentioned in the Introduction Marchant, Pikhurko, Vaughan and the author [5] showed $\gamma(F_{3,2}) = 1/3$ and determined the extremal configurations, which are obtained by taking an almost balanced tripartition of the vertex set $\sqcup_{i=1}^3 V_i$, putting in all triples of the form $V_i V_i V_{i+1}$ (winding round modulo 3) and adding a small number ($O(n^2)$) of 3-edges of the form $V_1 V_2 V_3$.

Problem 34. Find a good lower bound construction for $\gamma(F(K_t))$.

Clearly for any 3-graph H we must have $\gamma(F(H)) \geqslant \gamma(H)$. We also have the following upper-bound.

Proposition 35.
$$\gamma(F(H)) \leqslant \frac{1}{2 - \gamma(H)}.$$

Proof. Assume that $\gamma(H) < 1$, for otherwise we have nothing to prove. Let γ be a real number with $\frac{1}{2-\gamma(H)} < \gamma < 1$. Suppose G is a 3-graph on $[n]$ with $\delta_2(G) \geqslant \gamma n$. Consider any pair $xy \in [n]^{(2)}$, and let $\Gamma(x,y) = \{z \in [n] : \ xyz \in E(G)\}$ be their joint neighbourhood.

Set $\alpha = |\Gamma(x,y)|/n$ By our codegree assumption, we have $\alpha \geqslant \gamma$. Also by the codegree assumption, every pair of vertices $z, z' \in \Gamma(x,y)$ must have at least $\gamma n - (1-\alpha)n$ neighbours inside $\Gamma(x,y)$.

Thus the restriction $G' = G[\Gamma(x, y)]$ of the 3-graph G to the vertices in $\Gamma(x, y)$ has codegree

$$\frac{\delta_2(G')}{|V(G')|} \geqslant 1 - \frac{1-\gamma}{\alpha}$$

$$\geqslant 1 - \frac{1-\gamma}{\gamma} \qquad \text{since } \alpha \geqslant \gamma.$$

Since $\gamma > \frac{1}{2-\gamma(H)}$, we have $1 - \frac{1-\gamma}{\gamma} > \gamma(H)$. In particular if n is large enough, G' contains a copy of H, which, when taken with x, y in G gives rise to a copy of $F(H)$.

It follows that $\gamma(F(H)) \leqslant \frac{1}{2-\gamma(H)}$ as claimed. $\qquad\square$

Thus if the answer to Question 18 is positive (as we believe) and $\gamma(K_t) = 1 - 1/(s-2)$ for all $s \geqslant 4$, then

$$1 - \frac{1}{s-2} \leqslant \gamma(F(K_s)) \leqslant 1 - \frac{1}{s-1}$$

for all $s \geqslant 3$.

Question 36. Is $\gamma(F(K_s)) = 1 - \frac{1}{s} + O\left(\frac{1}{s^2}\right)$?

Let us finish by noting that the Turán density analogue of Problem 34 is also open. The Turán number and extremal configurations ('one-way' bipartite 3-graphs) for $F_{3,2}$ are known [9], but we do not know of any constructions or conjectures for $\pi(F(K_s))$ besides the ones showing $\pi(K_s) \geqslant 1 - \frac{4}{(s-1)^2}$.

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Appendix

Here we give for completeness a proof of the claim we used in the proof of Theorem 10.

Claim 37. *Fix $i < j$. Then $\{X_{ij,k} : k \in [n] \setminus \{ij\}\}$ forms a family of independent identically distributed Bernoulli random variables with parameter $1 - \frac{1}{s}$.*

Proof. Let $K \sqcup K'$ be a partitition of $[n] \setminus \{ij\}$. Then,

$$\mathbb{P}\left(X_{ij,k} = 1 \ \forall k \in K, \ X_{ij,k'} = 0 \ \forall k' \in K'\right)$$

$$= \sum_{c=1}^{s} \mathbb{P}(\mathbf{c}(ij) = c)\mathbb{P}\left(X_{ij,k} = 1 \ \forall k \in K, \ X_{ij,k'} = 0 \ \forall k' \in K' | \mathbf{c}(ij) = c\right)$$

$$= \sum_{c=1}^{s} \frac{1}{s} \prod_{k \in K} \mathbb{P}(X_{ij,k} = 1|\mathbf{c}(ij) = c) \prod_{k' \in K'} \mathbb{P}(X_{ij,k'} = 0|\mathbf{c}(ij) = c)$$

$$= \prod_{k \in K} \mathbb{P}(X_{ij,k} = 1|\mathbf{c}(ij) = 1) \prod_{k' \in K'} \mathbb{P}(X_{ij,k'} = 0|\mathbf{c}(ij) = 1).$$

Here in the second equality above we used the fact that conditional on the value of $\mathbf{c}(ij)$, the random variable $X_{ij,k}$ depends only on the random variables $\mathbf{c}(ik), \mathbf{c}(jk)$. Since each edge is coloured independently, we have that the conditional random variables $X_{ij,k}|\mathbf{c}(ij) = c$ are independent. In the third equality, we use the fact that the problem is symmetric with respect to our s colours.

Now for any $k \in K$,

$$\mathbb{P}(X_{ij,k} = 1) = \sum_{c=1}^{s} \mathbb{P}(\mathbf{c}(ij) = c)\mathbb{P}(X_{ij,k} = 1|\mathbf{c}(ij) = c)$$

$$= s \times \frac{1}{s}\mathbb{P}(X_{ij,k} = 1|\mathbf{c}(ij) = 1),$$

again using the symmetry in the colours, and similarly for any $k' \in K'$

$$\mathbb{P}(X_{ij,k'} = 0) = \mathbb{P}(X_{ij,k'} = 0|\mathbf{c}(ij) = 1).$$

Thus we have

$$\mathbb{P}\left(X_{ij,k} = 1 \ \forall k \in K, \ X_{ij,k'} = 0 \ \forall k' \in K'\right) = \prod_{k \in K} \mathbb{P}(X_{ij,k} = 1) \prod_{k' \in K'} \mathbb{P}(X_{ij,k'} = 0),$$

for any $K \sqcup K' = [n] \setminus \{ij\}$, proving that $\{X_{ij,k} : k \in [n] \setminus \{ij\}\}$ forms a family of independent random variables as claimed. By construction, they are identically distributed Bernoulli random variables with parameter $(1 - \frac{1}{s})$, concluding the proof of our claim. □