Unique Sequences Containing No k-Term Arithmetic Progressions

Tanbir Ahmed

Department of Computer Science and Software Engineering
Concordia University, Montréal, Canada

ta_ahmed@cs.concordia.ca

Janusz Dybizbański

Institute of Informatics
University of Gdańsk, Poland

jdybiz@inf.ug.edu.pl

Hunter Snevily*

Department of Mathematics
University of Idaho - Moscow, Idaho, USA

snevily@uidaho.edu

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Abstract

In this paper, we are concerned with calculating $r(k,n)$, the length of the longest k-Ap free subsequences in $1,2,\ldots,n$. We prove the basic inequality $r(k,n) \leqslant n - \lfloor m/2 \rfloor$, where $n = m(k-1)+r$ and $r < k-1$. We also discuss a generalization of a famous conjecture of Szekeres (as appears in Erdős and Turán [4]) and describe a simple greedy algorithm that appears to give an optimal k-AP free sequence infinitely often. We provide many exact values of $r(k,n)$ in the Appendix.

1 Introduction

Let $\langle n \rangle$ denote the sequence $1,2,\ldots,n$. A subsequence of $\langle n \rangle$ is called *k-AP free* if it does not contain any k-term arithmetic progression. Define the following:

$$\begin{aligned} r(k,n) &= \text{length of the longest } k\text{-AP free subsequences in } \langle n \rangle, \\ S(k,n) &= \{S \subseteq \{1,2,\ldots,n\} : S \text{ is } k\text{-AP free and } |S| = r(k,n)\}, \\ b(k,n) &= |S(k,n)|. \end{aligned}$$

*Hunter Snevily has passed away on November 11, 2013 after his long struggle with Parkinson's disease. We have lost a good friend and colleague. He will be greatly missed and remembered.

The study of the function $r(3, n)$ was initiated by Erdős and Turán [4]. They determined the values of $r(3, n)$ for $n \leqslant 23$ and $n = 41$, proved for $n \geqslant 8$ that

$$r(3, 2n) \leqslant n$$

and conjectured that

$$\lim_{n\to\infty} r(3, n)/n = 0,$$

which was proved in 1975 by Szemeredi [8]. Erdős and Turán also conjectured that $r(3, n) < n^{1-c}$, which was shown to be false by Salem and Spencer [6], who proved

$$r(3, n) > n^{1-c/\log\log n},$$

which was further improved by Behrend [1] to

$$r(3, n) > n^{1-c/\sqrt{\log n}}.$$

Recently, Elkin [3] has further improved this lower bound by a factor of $\Theta(\sqrt{\log n})$. The first non-trivial upper bound was due to Roth [5] who proved

$$r(3, n) < cn/\log\log n.$$

Sharma [7] showed that Erdős and Turán gave the wrong value of $r(3, 20)$ and determined the values of $r(3, n)$ for $n \leqslant 27$ and $41 \leqslant n \leqslant 43$. Recently, Dybizbański [2] has computed the exact values of $r(3, n)$ for all $n \leqslant 123$ and proved for $n \geqslant 16$ that

$$r(3, 3n) \leqslant n.$$

1.1 Szekeres' conjecture

Erdős and Turán [4] noted that there is no 3-term arithmetic progression in the sequence of all numbers $n, 0 \leqslant n \leqslant \frac{1}{2}(3^t - 1)$, which do not contain the digit 2 in the ternary scale. Hence for every $t \geqslant 1$,

$$r\left(3, (3^t + 1)/2\right) \geqslant 2^t$$

as we obtain the 3-AP-free sequence of length 2^t in $\left\langle (3^t + 1)/2 \right\rangle$ by adding 1 to each of those numbers that does not contain digit 2 in the ternary scale. Szekeres conjectured that for every $t \geqslant 1$,

$$r\left(3, (3^t + 1)/2\right) = 2^t,$$

and more generally, for any t and any prime p,

$$r\left(p, \frac{(p-2)p^t + 1}{p-1}\right) = (p-1)^t.$$

A generalization of Szekeres' conjecture will be given in Section 4.

2 A basic inequality for $r(k,n)$

Let $ap(k,n)$ denote the set of all k-APs from the numbers $1,2,\ldots,n$. Let $ap(k,n;v)$ denote the set of all k-APs, each containing v, from the numbers $1,2,\ldots,n$. Let $c(k,n)$ and $c(k,n;v)$ denote $|ap(k,n)|$ and $|ap(k,n;v)|$, respectively. Let $c_i(k,n;v)$ be the number of k-APs, each containing v as the i-th element in the AP, from the numbers $1,2,\ldots,n$. Also define $c_{max}(k,n)$ as the maximum of $c(k,n;x)$ over $x=1,2,\ldots,n$.

In this section, we determine an exact expression for $c(k,n)$ and an upper-bound for $c_{max}(k,n)$ to obtain a basic inequality for $r(k,n)$. The following observation is crucial for the proofs in this paper:

Observation 1.

$$c_j(k,n;x)=\begin{cases}\lfloor (n-x)/(k-1)\rfloor, & \text{if } j=1;\\ \lfloor (x-1)/(k-1)\rfloor, & \text{if } j=k;\\ \min\left\{\left\lfloor\frac{(n-x)}{(k-j)}\right\rfloor,\left\lfloor\frac{(x-1)}{(j-1)}\right\rfloor\right\}, & \text{otherwise.}\end{cases}$$

and $c(k,n;x)=\sum_{j=1}^{k} c_j(k,n;x)$.

Example 1. Consider $k=4$ and $n=17$. Here, $r(4,17)=11$ with an example $S=\{1,2,4,5,6,10,12,13,15,16,17\}\in S(4,17)$.

j/x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	5	5	4	4	4	3	3	3	2	2	2	1	1	1	0	0	0
2	0	1	2	3	4	5	5	4	4	3	3	2	2	1	1	0	0
3	0	0	1	1	2	2	3	3	4	4	5	5	4	3	2	1	0
4	0	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5	5
$c(4,17;x)$	5	6	7	9	11	11	13	12	12	12	13	11	11	9	7	6	5

Here $c(4,17;x)=c_1(4,17;x)+c_2(4,17;x)+c_3(4,17;x)+c_4(4,17;x)$. For example, $c(4,17;13)=1+2+4+4$ due to the following 4-APs in $ap(4,17)$ that contain 13, namely $ap(4,17;13)$:

$$\begin{array}{llll}\{13,14,15,16\}, & \{12,13,14,15\}, & \{11,13,15,17\}, & \{5,9,13,17\},\\ \{7,10,13,16\}, & \{9,11,13,15\}, & \{11,12,13,14\}, & \{1,5,9,13\},\\ \{4,7,10,13\}, & \{7,9,11,13\}, & \{10,11,12,13\}. & \end{array}$$

Observation 2. For $x=1,2,\ldots,\lfloor n/2\rfloor$, $c(k,n;x)=c(k,n;n-x+1)$, that is, the sequence $c(k,n;x)$ with $x=1,2,\ldots,n$ is symmetric.

Proof. We have the following cases:

(a) For $j=1$, using Observation 1, we have

$$c_1(k,n;x)=\left\lfloor\frac{n-x}{k-1}\right\rfloor=\left\lfloor\frac{(n-x+1)-1}{k-1}\right\rfloor=c_k(k,n;n-x+1).$$

(*b*) For other values of j,

(*i*) If $\lfloor (n-x)/(k-j) \rfloor \leqslant \lfloor (x-1)/(j-1) \rfloor$, then taking $x' = n - x + 1$ and $j' = k - j + 1$, and using Observation 1,

$$c_j(k,n;x) = \left\lfloor \frac{(n-x)}{(k-j)} \right\rfloor = \left\lfloor \frac{(n-x+1)-1}{(k-j+1)-1} \right\rfloor = \left\lfloor \frac{(x'-1)}{(j'-1)} \right\rfloor = c_{j'}(k,n;x').$$

The last equality follows from the fact that

$$\left\lfloor \frac{(x'-1)}{(j'-1)} \right\rfloor \leqslant \left\lfloor \frac{(x-1)}{(j-1)} \right\rfloor = \left\lfloor \frac{(n-x')}{(k-j')} \right\rfloor.$$

(*ii*) If $\lfloor (n-x)/(k-j) \rfloor > \lfloor (x-1)/(j-1) \rfloor$, then (similar as (*i*))

$$c_j(k,n;x) = c_{k-j+1}(k,n;n-x+1).$$

Therefore, $c(k,n;x)$ with $x = 1, 2, \ldots, n$ is symmetric. □

Lemma 1. Given positive integers k and n, let $2 \leqslant j \leqslant k-1$, $n = m(k-1) + r$ with $0 \leqslant r \leqslant k-2$. Then $c_j(k,n;x)$ equals

(*a*) $\lfloor (x-1)/(j-1) \rfloor$ if $x = 1, 2, \ldots, m(j-1)$,

(*b*) $\lfloor (n-x)/(k-j) \rfloor$ if $x = n - m(k-j) + 1, n - m(k-j) + 2, \ldots, n$,

(*c*) m otherwise, that is, for $x = m(j-1) + 1, \ldots, n - m(k-j)$.

Proof. (*a*) Take $y \leqslant m(j-1)$ and assume

$$c_j(k,n;y) = \lfloor (n-y)/(k-j) \rfloor,$$

that is, $\lfloor (n-y)/(k-j) \rfloor < \lfloor (y-1)/(j-1) \rfloor$.

Since $y \leqslant m(j-1)$, we have $(y-1) \leqslant m(j-1) - 1$, and hence

$$\left\lfloor \frac{y-1}{j-1} \right\rfloor \leqslant m - 1.$$

Again, since $y \leqslant m(j-1)$, we have

$$n - y \geqslant n - m(j-1) = m(k-j) + r,$$

which implies

$$\left\lfloor \frac{n-y}{k-j} \right\rfloor \geqslant m.$$

Now, we have the following contradiction

$$m \leqslant \left\lfloor \frac{n-y}{k-j} \right\rfloor < \left\lfloor \frac{y-1}{j-1} \right\rfloor \leqslant m - 1.$$

(*b*) Take $y \geqslant n - m(k-j) + 1$ and assume

$$c_j(k, n; y) = \lfloor (y-1)/(j-1) \rfloor,$$

that is, $\lfloor (n-y)/(k-j) \rfloor > \lfloor (y-1)/(j-1) \rfloor$. Similar reasoning as (a) leads to a contradiction.

(*c*) Here, $n - m(k-j)$ can be written as $m(j-1)+r$ and $m(j-1)+1$ can be written as $n-m(k-j)-(r-1)$. There are exactly r elements in $m(j-1)+1, \ldots, n-m(k-j)$, and for any x in this range,

$$c_j(k, n; x) = \lfloor (x-1)/(j-1) \rfloor = \lfloor (n-x)/(k-j) \rfloor = m.$$

□

Lemma 2. Given positive integers k and n, let $n = m(k-1) + r$ with $0 \leqslant r \leqslant k-2$. Denote a sequence $a, a, \ldots, a$ with a repeated b times as a^b, and consider a^0 to be an empty sequence. Then for $1 \leqslant j \leqslant k$, the sequence $c_j(k, n; x)$ with $1 \leqslant x \leqslant n$ has the form

$$0^{j-1} 1^{j-1} \cdots (m-1)^{j-1} m^r (m-1)^{k-j} (m-2)^{k-j} \cdots 0^{k-j}.$$

Proof. Using Observation 1, we have for $j = 1$:

$$c_1(k, n; x) = \left\lfloor \frac{n-x}{k-1} \right\rfloor = \left\lfloor m + \frac{r-x}{k-1} \right\rfloor = m + \left\lfloor \frac{r-x}{k-1} \right\rfloor,$$

and more specifically,

$$c_1(k, n; x) \;=\; \begin{cases} m, & \text{for } x = 1, 2, \ldots, r; \\ (m-1), & \text{for } x = r+1, r+2, \ldots, r+(k-1); \\ (m-2), & \text{for } x = r+(k-1)+1, r+(k-1)+2, \ldots, r+2(k-1); \\ & \vdots, \\ 1, & \text{for } x = r+(m-2)(k-1)+1, \ldots, n-(k-1); \\ 0, & \text{for } x = r+(m-1)(k-1)+1, \ldots, n. \end{cases}$$

Hence, the sequence $c_1(k, n; x)$ with $x = 1, 2, \ldots, n$ is

$$m^r (m-1)^{k-1} (m-2)^{k-1} \cdots 1^{k-1} 0^{k-1}.$$

Similarly, for $j = k$, we have

$$c_k(k, n; x) = \left\lfloor \frac{x-1}{k-1} \right\rfloor,$$

and more specifically,

$$c_k(k, n; x) \;=\; \begin{cases} 0, & \text{for } x = 1, 2, \ldots, k-1; \\ 1, & \text{for } x = (k-1)+1, (k-1)+2, \ldots, 2(k-1); \\ 2, & \text{for } x = 2(k-1)+1, 2(k-1)+2, \ldots, 3(k-1); \\ & \vdots, \\ (m-2), & \text{for } x = (m-2)(k-1)+1, \ldots, (m-1)(k-1); \\ (m-1), & \text{for } x = (m-1)(k-1)+1, \ldots, m(k-1); \\ m, & \text{for } x = m(k-1)+1, \ldots, n. \end{cases}$$

and hence the sequence $c_k(k,n;x)$ with $x=1,2,\ldots,n$ is

$$0^{k-1}1^{k-1}\cdots(m-2)^{k-1}(m-1)^{k-1}m^r.$$

For $2\leqslant j\leqslant k-1$, by Lemma 1, we have

x	$c_j(k,n;x)$
$1,2,\ldots,j-1$	$\lfloor(x-1)/(j-1)\rfloor=0$
$(j-1)+1,(j-1)+2,\ldots,2(j-1)$	$\lfloor(x-1)/(j-1)\rfloor=1$
$\vdots$	$\vdots$
$(m-2)(j-1)+1,(m-2)(j-1)+2,\ldots,(m-1)(j-1)$	$\lfloor(x-1)/(j-1)\rfloor=m-2$
$(m-1)(j-1)+1,(m-1)(j-1)+2,\ldots,m(j-1)$	$\lfloor(x-1)/(j-1)\rfloor=m-1$
$m(j-1)+1,m(j-1)+2,\ldots,n-m(k-j)$	m
$n-m(k-j)+1,\ldots,n-(m-1)(k-j)$	$\lfloor(n-x)/(k-j)\rfloor=m-1$
$n-(m-1)(k-j)+1,\ldots,n-(m-2)(k-j)$	$\lfloor(n-x)/(k-j)\rfloor=m-2$
$\vdots$	$\vdots$
$n-2(k-j)+1,n-2(k-j)+2,\ldots,n-(k-j)$	$\lfloor(n-x)/(k-j)\rfloor=1$
$n-(k-j)+1,n-(k-j)+2,\ldots,n$	$\lfloor(n-x)/(k-j)\rfloor=0$

Hence, we get the sequence $c_j(k,n;x)$ for $x=1,2,\ldots,n$ as

$$0^{j-1}1^{j-1}\cdots(m-1)^{j-1}m^r(m-1)^{k-j}(m-2)^{k-j}\cdots0^{k-j}.$$

□

Corollary 1. Given positive integers k and n, let $m=\lfloor n/(k-1)\rfloor$ and $n=m(k-1)+r$. Then

$$c(k,n)=\binom{m}{2}(k-1)+mr.$$

Proof. From the definition of $c_1(k,n;x)$ in Lemma 2, for $1\leqslant x\leqslant n$, we have,

$$\begin{aligned}
c(k,n) &= \sum_{x=1}^{n-k+1} c_1(k,n;x)\\
&= \sum_{x=1}^{r} c_1(k,n;x)+\left[\sum_{x=r+1}^{r+(k-1)} c_1(k,n;x)+\right.\\
&\qquad \left.\sum_{x=r+(k-1)+1}^{r+2(k-1)} c_1(k,n;x)+\cdots+\sum_{x=r+(m-2)(k-1)+1}^{n-k+1=n-(k-1)=r+(m-1)(k-1)} c_1(k,n;x)\right]\\
&= mr+[(m-1)+(m-2)+\cdots+2+1](k-1)\\
&= \binom{m}{2}(k-1)+mr.
\end{aligned}$$

□

Corollary 2. Given positive integers k and n, let $n = m(k-1) + r$ with $r < k-1$ and $1 \leqslant j \leqslant k$. Then for $x = 1, 2, \ldots, n$,

$$c_j(k, n; x) \leqslant m.$$

Proof. Follows from Lemmas 1 and 2. □

Trivially, $c_{max}(k, n) \leqslant mk$. The following Corollary slightly improves the bound.

Corollary 3. Given positive integers k and n, let $n = m(k-1) + r$ with $r < k-1$ and $1 \leqslant j \leqslant k$. Then

$$c_{max}(k, n) \leqslant m(k-1)$$

Proof. For any $x \in \{1, 2, \ldots, n\}$, using the definitions of $c_1(k, n; x)$ and $c_k(k, n; x)$ from Lemma 2, we have,

x	$c_1(k,n;x) + c_k(k,n;x)$
$1, 2, \ldots, r$	$m + 0 = m$
$r + 0(k-1) + 1, r + 0(k-1) + 2, \ldots, k-1$	$(m-1) + 0 = m-1$
$1 + (k-1), 2 + (k-1), \ldots, r + 1(k-1)$	$(m-1) + 1 = m$
$r + 1(k-1) + 1, r + 1(k-1) + 2, \ldots, 2(k-1)$	$(m-2) + 1 = m-1$
$1 + 2(k-1), 2 + 2(k-1), \ldots, r + 2(k-1)$	$(m-2) + 2 = m$
$\vdots$	$\vdots$
$r + (m-2)(k-1) + 1, r + (m-2)(k-1) + 2, \ldots, (m-1)(k-1)$	$1 + (m-2) = m-1$
$1 + (m-1)(k-1), 2 + (m-1)(k-1), \ldots, r + (m-1)(k-1)$	$1 + (m-1) = m$
$r + (m-1)(k-1) + 1, r + (m-1)(k-1) + 2, \ldots, m(k-1)$	$0 + (m-1) = m-1$
$1 + m(k-1), 2 + m(k-1), \ldots, r + m(k-1),$	$0 + m = m$

Hence, using $c_1(k, n; x) + c_k(k, n; x) \leqslant m$ and $c_j(k, n; x) \leqslant m$ (by Corollary 2) for $1 \leqslant j \leqslant k$, we have

$$\begin{aligned} c(k, n; x) &= c_1(k, n; x) + c_k(k, n; x) + \sum_{j=2}^{k-1} c_j(k, n; x) \\ &\leqslant m + m(k-2) = m(k-1). \end{aligned}$$

Therefore, $c_{max}(k, n) \leqslant m(k-1)$. □

It can be observed that the upper bound in Corollary 3 is the best possible for $c_{max}(k, n)$. The following theorem gives an upper bound of $r(k, n)$, which is very close to actual values (see Appendix C for experimental results).

Theorem 1. Given positive integers k and n, let $m = \lfloor n/(k-1) \rfloor$ and $n = m(k-1) + r$ where $r < k-1$. Then

$$r(k, n) \leqslant n - \lfloor m/2 \rfloor.$$

Proof. Using Corollaries 2 and 3, we have

$$\begin{aligned} r(k,n) &\leqslant n - \left\lceil \frac{c(k,n)}{c_{max}(k,n)} \right\rceil \\ &\leqslant n - \left\lceil \frac{m(m-1)(k-1)/2 + mr}{m(k-1)} \right\rceil \\ &= n - \left\lceil \frac{m-1}{2} + \frac{r}{k-1} \right\rceil = n - f(m,k,r), \text{(say)} \end{aligned}$$

It can be observed that

$$f(m,k,r) = \begin{cases} y+1, & \text{if } m = 2y+1; \\ y, & \text{if } m = 2y \text{ and } 2r \leqslant k-1; \\ y+1, & \text{if } m = 2y \text{ and } 2r > k-1. \end{cases}$$

Hence,

$$r(k,n) \leqslant n - \lfloor m/2 \rfloor.$$

□

Conjecture 1. For every positive integer $k \geqslant 3$ and positive integer n, $c_{max}(k,n)$ is eventually periodic.

For example, for $k = 5$, the length of the period is 24 and the periodic increase in the value of $c_{max}(k,n)$ is 20, as indicated in the following table:

n	$c_{max}(5,n)$	n	$c_{max}(5,n)$	n	$c_{max}(5,n)$	n	$c_{max}(5,n)$	n	$c_{max}(5,n)$	
25	20	49	40	73	60	97	80	121	100	$\cdots$
26	20	50	40	74	60	98	80	122	100	$\cdots$
27	20	51	40	75	60	99	80	123	100	$\cdots$
28	21	52	41	76	61	100	81	124	101	$\cdots$
29	22	53	42	77	62	101	82	125	102	$\cdots$
30	22	54	42	78	62	102	82	126	102	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
47	37	71	57	95	77	119	97	143	117	$\cdots$
48	37	72	57	96	77	120	97	144	117	$\cdots$

Conjecture 2. Given an odd positive integer k, the size of the largest subset U of $\{1, 2, \ldots, n\}$ for any positive integer n, with each $x \in U$ having the same $c(k,n;x)$, is bounded from above by a constant $f(k) \leqslant k^2$.

The implications of Conjecture 2 being true is as follows:
Let $w = c_{max}(k,n)$ and consider the largest ℓ such that

$$g(k,\ell,w) = f(k)\,(w + (w-1) + \cdots + (w - (\ell - 1)))$$

has the property $c(k,n) - g(k,\ell,w) \geqslant f(k)\ell(\ell-1)/2$. Then

$$\begin{aligned} r(k,n) &\leqslant n - \left\lceil \frac{c(k,n)}{w} \right\rceil \leqslant n - \left\lceil \frac{g(k,\ell,w) + f(k)\ell(\ell-1)/2}{w} \right\rceil \\ &= n - f(k)\ell. \end{aligned}$$

See Appendix A for data supporting Conjectures 1 and 2.

3 Unimodality lemmas

A sequence is called *unimodal* if it is first increasing and then decreasing. In this section, we prove some lemmas on sequences regarding $c(k, n; x)$ for $k \geqslant 3$.

Lemma 3. Given positive integers k and n, for any $j \in \{2, 3, \ldots, k-1\}$, the sequence $c_j(k, n; x)$ with $x = 1, 2, \ldots, n$ is unimodal.

Proof. Follows directly from Lemma 2. □

Lemma 4. The sequence $c(3, n; i)$ with $i = 1, 2, \ldots, n$ is unimodal.

Proof. From Observation 1,

$$c_j(3,n;x) = \begin{cases} \lfloor (n-x)/2 \rfloor, & \text{if } j = 1; \\ x - 1, & \text{if } j = 2 \text{ and } x \leqslant \lfloor n/2 \rfloor; \\ n - x, & \text{if } j = 2 \text{ and } x > \lfloor n/2 \rfloor; \\ \lfloor (x-1)/2 \rfloor, & \text{if } j = 3. \end{cases}$$

By Observation 2, $c(3, n; i)$ equals $c(3, n; n-i+1)$ for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$.

Now, we consider the following two cases:

1. $(n = 2m)$. For $i = 1, 2, \ldots, m-1$, we have

   $$c_2(3,n;i+1) = c_2(3,n;i) + 1,$$

   and also for $i = 1, 2, \ldots, m$,

   $$c_1(3,n;i) + c_3(3,n;i) = \left\lfloor \frac{2m-i}{2} \right\rfloor + \left\lfloor \frac{i-1}{2} \right\rfloor = \left\lfloor m - \frac{i}{2} \right\rfloor + \left\lfloor \frac{i}{2} - \frac{1}{2} \right\rfloor.$$

   If $i = 2j$ $(j \geqslant 1)$, then

   $$c_1(3,n;i) + c_3(3,n;i) = (m-j) + \left\lfloor j - \frac{1}{2} \right\rfloor = (m-j) + (j-1) = m-1.$$

   If $i = 2j+1$ $(j \geqslant 0)$, then

   $$\begin{aligned} c_1(3,n;i) + c_3(3,n;i) &= \left\lfloor m - j - \frac{1}{2} \right\rfloor + \left\lfloor \frac{(2j+1)-1}{2} \right\rfloor \\ &= (m-j-1) + j = m-1. \end{aligned}$$

   Therefore, for $i = 1, 2, \ldots, m-1$,

   $$c(3,n;i+1) = (m-1) + c_2(3,n;i) + 1 = c(3,n;i) + 1.$$

2. $(n = 2m + 1)$. For $i = 1, 2, \ldots, m$, we have

$$c_2(3, n; i + 1) = c_2(3, n; i) + 1,$$

and also

$$c_1(3, n; i) + c_3(3, n; i) = \left\lfloor \frac{2m + 1 - i}{2} \right\rfloor + \left\lfloor \frac{i - 1}{2} \right\rfloor = \left\lfloor m + \frac{1}{2} - \frac{i}{2} \right\rfloor + \left\lfloor \frac{i}{2} - \frac{1}{2} \right\rfloor.$$

If $i = 2j$ $(j \geqslant 1)$, then

$$c_1(3, n; i) + c_3(3, n; i) = \left\lfloor m + \frac{1}{2} - j \right\rfloor + \left\lfloor j - \frac{1}{2} \right\rfloor = (m - j) + (j - 1) = m - 1.$$

If $i = 2j + 1$ $(j \geqslant 0)$, then

$$\begin{aligned} c_1(3, n; i) + c_3(3, n; i) &= \left\lfloor m + \frac{1}{2} - j - \frac{1}{2} \right\rfloor + \left\lfloor \frac{(2j + 1) - 1}{2} \right\rfloor \\ &= (m - j) + j = m. \end{aligned}$$

Therefore, for $i = 1, 2, \ldots, m$,

- If i is odd, then

$$\begin{aligned} c(3, n; i + 1) &= c_1(3, n; i + 1) + c_2(3, n; i + 1) + c_3(3, n; i + 1) \\ &= c_2(3, n; i) + 1 + (m - 1) = c_2(3, n; i) + m \\ &= c_2(3, n; i) + c_1(3, n; i) + c_3(3, n; i) = c(3, n; i). \end{aligned}$$

- If i is even, then

$$\begin{aligned} c(3, n; i + 1) &= c_1(3, n; i + 1) + c_2(3, n; i + 1) + c_3(3, n; i + 1) \\ &= c_2(3, n; i) + 1 + m = c_2(3, n; i) + (m - 1) + 2 \\ &= c_2(3, n; i) + c_1(3, n; i) + c_3(3, n; i) + 2 = c(3, n; i) + 2. \end{aligned}$$

Hence, $c(3, n; i)$ with $i = 1, 2, \ldots, n$ is unimodal. □

Lemma 5. For $k \geqslant 4$, there are infinitely many n such that the sequence $c(k, n; i)$ with $i = 1, 2, \ldots, n$ is unimodal.

Proof. We show that $c(k, n; i)$ for $1 \leqslant i \leqslant n$ with $n = lcm\{1, 2, \ldots, k - 1\} \cdot m$ (where $m \geqslant 1$) is unimodal. Since n is even, $n/2$ is an integer. By Observation 2, the sequence $c(k, n; i)$ with $1 \leqslant i \leqslant n$ is symmetric. So assume $i \leqslant n/2$. Let $lcm\{1, 2, \ldots, k - 1\}$ be equal to $h_r \cdot r$ with $2 \leqslant r \leqslant k - 1$, and $i \equiv s \pmod{k - 1}$ with $t = \lfloor i/(k - 1) \rfloor$. Now,

$$\begin{aligned} c_1(k, n; i) &= \left\lfloor \frac{(n - i)}{(k - 1)} \right\rfloor = \left\lfloor mh_{k-1} - \frac{i}{(k - 1)} \right\rfloor = \left\lfloor mh_{k-1} - t - \frac{s}{k - 1} \right\rfloor \\ c_1(k, n; i + 1) &= \left\lfloor \frac{(n - i - 1)}{(k - 1)} \right\rfloor = \left\lfloor mh_{k-1} - \frac{i + 1}{(k - 1)} \right\rfloor = \left\lfloor mh_{k-1} - t - \frac{s + 1}{k - 1} \right\rfloor \end{aligned}$$

Therefore,

$$c_1(k,n;i+1) = \begin{cases} c_1(k,n;i) - 1, & \text{if } s = 0; \\ c_1(k,n;i), & \text{otherwise.} \end{cases}$$

Similarly,

$$c_k(k,n;i+1) = \begin{cases} c_k(k,n;i) + 1, & \text{if } s = 0; \\ c_k(k,n;i), & \text{otherwise.} \end{cases}$$

Hence, $c_1(k,n;i) + c_k(k,n;i)$ remains constant for $1 \leqslant i \leqslant n$.

Again, for $1 \leqslant i \leqslant n/2$, we have

$$i - 1 \leqslant n - i.$$

For $2 \leqslant j \leqslant k-1$, we want to show

$$c_j(k,n;i+1) + c_{k-j+1}(k,n;i+1) \geqslant c_j(k,n;i) + c_{k-j+1}(k,n;i).$$

Assume $j > \lfloor k/2 \rfloor$. This implies $k - j \leqslant j - 1$. Since $i - 1 \leqslant n - i$, we have

$$\left\lfloor \frac{(i-1)}{(j-1)} \right\rfloor \leqslant \left\lfloor \frac{(n-i)}{(k-j)} \right\rfloor.$$

So for $1 \leqslant i \leqslant n/2 - 1$, and considering $i \equiv s \pmod{j-1}$ and $t = \lfloor i/(j-1) \rfloor$, we have,

$$\begin{aligned} c_j(k,n;i+1) &= \left\lfloor \frac{i}{j-1} \right\rfloor = t \\ c_j(k,n;i) &= \left\lfloor \frac{i-1}{j-1} \right\rfloor = \left\lfloor t + \frac{s-1}{j-1} \right\rfloor = \begin{cases} t-1, & \text{if } s = 0; \\ t, & \text{if } s \geqslant 1. \end{cases} \end{aligned}$$

Take $j' = k - j + 1$, and then $j' - 1 \leqslant k - j'$. If $\lfloor (i-1)/(j'-1) \rfloor \leqslant \lfloor (n-i)/(k-j') \rfloor$, then

$$0 \leqslant c_{j'}(k,n;i+1) - c_{j'}(k,n;i) \leqslant 1,$$

else

$$\begin{aligned} c_{j'}(k,n;i) &= \left\lfloor \frac{n-i}{k-j'} \right\rfloor = \left\lfloor \frac{n-i}{j-1} \right\rfloor = \left\lfloor mh_{j-1} - t - \frac{s}{j-1} \right\rfloor \\ &= \begin{cases} (mh_{j-1} - t), & \text{if } s = 0; \\ (mh_{j-1} - t - 1), & \text{otherwise.} \end{cases} \\ c_{j'}(k,n;i+1) &= \left\lfloor \frac{n-i-1}{k-j'} \right\rfloor = \left\lfloor mh_{j-1} - t - \frac{s+1}{j-1} \right\rfloor = (mh_{j-1} - t - 1) \end{aligned}$$

Therefore,

$$c_j(k,n;i+1) + c_{j'}(k,n;i+1) \geqslant c_j(k,n;i) + c_{j'}(k,n;i).$$

So the sequence $c(k,n;x)$ with $1 \leqslant x \leqslant n/2$ is non-decreasing and hence the sequence $c(k,n;x)$ with $1 \leqslant x \leqslant n$ is unimodal for infinitely many n. □

4 Uniqueness conjectures

In this section, we generalize Szekeres' conjecture and provide a construction for the lower bound. We also provide a construction algorithm for $r(k,n)$. Define

$$J(k,L) = \{(n,m) : n \leqslant L, r(k,n) = m \text{ and } b(k,n) = 1\}.$$

We have the following experimental data, based on which we formulate Conjectures 3 and 4:

$$\begin{aligned}
J(3,123) &= \{(2,2),(5,4),(14,8),(30,12),(41,16),(74,22),(84,24),(104,28),\\
&\quad (114,30),(122,32)\},\\
J(5,105) &= \{(2,2),(3,3),(4,4),(9,8),(14,12),(19,16),(44,32),(69,48),(94,64)\},\\
J(7,139) &= \{(2,2),(3,3),(4,4),(5,5),(6,6),(13,12),(20,18),(27,24),(34,30),\\
&\quad (41,36),(90,72),(139,108)\},\\
J(11,117) &= \{(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),(21,20),\\
&\quad (32,30),(43,40),(54,50),(65,60),(76,70),(87,80),(98,90),(109,100)\},\\
J(13,161) &= \{(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9),(10,10),(11,11),\\
&\quad (12,12),(25,24),(38,36),(51,48),(64,60),(77,72),(90,84),(103,96),\\
&\quad (116,108),(129,120),(142,132),(155,144)\}.
\end{aligned}$$

Conjecture 3 (The Uniqueness Conjecture)**.** Consider a prime $p \geqslant 3$ and an integer $t \geqslant 1$. Then for $1 \leqslant i \leqslant p-1$,

$$r\left(p, \frac{(ip-i-1)p^t+1}{(p-1)}\right) = i \cdot (p-1)^t,$$

and $b(p,x) = 1$ where $1 \leqslant x \leqslant p-2$ or else

$$x = \frac{(ip-i-1)p^t+1}{(p-1)}.$$

It can be observed that Szekeres' conjecture is a special case of Conjecture 3 with $i = 1$.

Conjecture 4 (Strong Uniqueness Conjecture)**.** Consider a prime $p > 3$ and an integer $t \geqslant 1$. Then $b(p,x) = 1$ if and only if $1 \leqslant x \leqslant p-2$ or else

$$x = \frac{(ip-i-1)p^t+1}{(p-1)}$$

with $1 \leqslant i \leqslant p-1$.

See Appendix C for data supporting Conjectures 3 and 4.

4.1 Construction for the lower-bound of Conjecture 3

For a prime $p > 3$ and $1 \leqslant i \leqslant p-1$, take

$$n = \frac{(ip-i-1)p^t+1}{p-1} = ip^t - p^{t-1} - p^{t-2} - \cdots - p - 1.$$

We can construct a p-AP free subset of $\{1, 2, \ldots, n\}$ of size $i \cdot (p-1)^t$ as follows:

$$\begin{aligned}
T_0 &= \{1, 2, \ldots, n\}, \\
T_1 &= T_0 - \{j : j \equiv 0 \pmod{p}\} = T_0 - S_0, \\
T_2 &= T_1 - \left\{j_1 p^2 - p + j_2 : 1 \leqslant j_1 \leqslant \lfloor n/p^2 \rfloor, 1 \leqslant j_2 \leqslant p-1\right\} = T_1 - S_1, \\
T_3 &= T_2 - \left\{j_1 p^3 - p^2 + j_2 p - j_3 : 1 \leqslant j_1 \leqslant \lfloor n/p^3 \rfloor, 1 \leqslant j_2, j_3 \leqslant p-1\right\} = T_2 - S_2, \\
T_4 &= T_3 - \left\{j_1 p^4 - p^3 + j_2 p^2 - j_3 p + j_4 : 1 \leqslant j_1 \leqslant \lfloor n/p^4 \rfloor, 1 \leqslant j_2, j_3, j_4 \leqslant p-1\right\} \\
&= T_3 - S_3, \\
&\vdots, \\
T_t &= T_{t-1} - \left\{j_1 p^t - p^{t-1} + \sum_{\ell=2}^{t} p^{t-\ell} j_\ell (-1)^\ell : 1 \leqslant j_1 \leqslant \lfloor n/p^t \rfloor, 1 \leqslant j_2, j_3, \ldots, j_t \leqslant p-1\right\} \\
&= T_{t-1} - S_{t-1}.
\end{aligned}$$

It can be observed that

$$\begin{aligned}
|S_0| &= \lfloor n/p \rfloor = ip^{t-1} - p^{t-2} - \cdots - p - 2, \\
|S_1| &= (p-1)\lfloor n/p^2 \rfloor = (p-1)\left(ip^{t-2} - p^{t-3} - \cdots - p - 2\right), \\
|S_2| &= (p-1)^2 \lfloor n/p^3 \rfloor = (p-1)^2\left(ip^{t-3} - p^{t-4} - \cdots - p - 2\right), \\
&\vdots, \\
|S_{t-1}| &= (p-1)^{t-1} \lfloor n/p^t \rfloor = (p-1)^{t-1}(i-1).
\end{aligned}$$

Lemma 6. The S_ℓ's for $0 \leqslant \ell \leqslant t-1$ are disjoint.

Proof. Each element in S_0 is divisible by p, and no element in any other S_ℓ is divisible by p. So S_0 is disjoint from every other S_ℓ. For $2 \leqslant \ell \leqslant t-1$,

$$S_\ell = \left\{(xp + (-1)^{\ell-1} j_{\ell+1}) \in T_0 : x \in S_{\ell-1}, 1 \leqslant j_{\ell+1} \leqslant p-1\right\},$$

and hence $S_\ell \cap S_u = \emptyset$ for $1 \leqslant u \leqslant \ell - 1$. □

Lemma 7. For a prime $p > 3$ and $1 \leqslant i \leqslant p-1$,

$$|T_t| = i \cdot (p-1)^t.$$

Proof. We can write the summation $\sum_{j=0}^{t-1}|S_j|$ as follows:

$$\begin{aligned}\sum_{j=0}^{t-1}|S_j| &= ip^{t-1}\left(\sum_{a=0}^{t-1}\binom{a}{0}\right)+\cdots+(-1)^{\ell-1}ip^{t-\ell}\left(\sum_{a=\ell-1}^{t-1}\binom{a}{\ell-1}\right)+\cdots-i-\sum_{\ell=0}^{t-1}p^{\ell},\\ &= ip^{t-1}\binom{t}{1}+\cdots+(-1)^{\ell-1}ip^{t-\ell}\binom{t}{\ell}+\cdots-i-\sum_{\ell=0}^{t-1}p^{\ell},\\ &= i\sum_{\ell=1}^{t}\binom{t}{\ell}p^{t-\ell}(-1)^{\ell-1}-\sum_{\ell=0}^{t-1}p^{\ell}.\end{aligned}$$

The fact

$$\sum_{a=\ell-1}^{t-1}\binom{a}{\ell-1}=\binom{t}{\ell}$$

can be easily proven using induction on t and using the fact that

$$\binom{t}{\ell}+\binom{t}{\ell-1}=\binom{t+1}{\ell}.$$

Now, we have

$$\begin{aligned}|T_t| &= n-\sum_{j=0}^{t-1}|S_j|,\\ &= ip^t-\sum_{\ell=0}^{t-1}p^{\ell}-\left(i\sum_{\ell=1}^{t}\binom{t}{\ell}p^{t-\ell}(-1)^{\ell-1}-\sum_{\ell=0}^{t-1}p^{\ell}\right),\\ &= ip^t+i\sum_{\ell=1}^{t}\binom{t}{\ell}p^{t-\ell}(-1)^{\ell}=i\cdot(p-1)^t.\end{aligned}$$

□

Lemma 8. Given a prime $p>3$, $n=ip^t-\sum_{\ell=0}^{t-1}p^{\ell}$ with $1\leqslant i\leqslant p-1$, and the set $T=\{1,2,\ldots,n\}$; the set T_1 contains no p-AP with

$$d\in\{1\leqslant d_1\leqslant\lfloor(n-1)/(p-1)\rfloor : d_1\not\equiv 0 \pmod p\}.$$

Proof. Assume T_1 contains a p-AP $a, a+d, \ldots, a+(p-1)d$. Here $a\not\equiv 0 \pmod p$. Suppose $a\equiv j \pmod p$ for some $1\leqslant j\leqslant p-1$. Then $a+d(p-z)\equiv 0 \pmod p$ for some $1\leqslant z\leqslant p-1$ when $dz\equiv j \pmod p$.

For each $d\in\{1\leqslant d_1\leqslant\lfloor(n-1)/(p-1)\rfloor : d_1\not\equiv 0 \pmod p\}$,

$$\bigcup_{z=1}^{p-1}\{dz \pmod p\}=\{1,2,\ldots,p-1\}$$

and so there exists $1\leqslant z\leqslant p-1$ for any $1\leqslant j\leqslant p-1$ such that $dz\equiv j \pmod p$. But, this is a contradiction as there is no number in T_1 which is divisible by p. Hence T_1 contains no p-AP with $d\in\{1\leqslant d_1\leqslant\lfloor(n-1)/(p-1)\rfloor : d_1\not\equiv 0 \pmod p\}$. □

Lemma 9. The set T_t is p-AP free.

Proof. By construction, T_t contains no p-AP with $d \in \{1, p, p^2, \dots, p^t\}$. By Lemma 8, T_t does not contain a p-AP with any other d. Hence T_t is p-AP free. □

4.2 A construction algorithm for $r(k, n)$

In this section, we propose a greedy algorithm for construction of k-AP free subsequence of $1, 2, \dots, n$. We call this algorithm *Bi-symmetric Greedy Algorithm* (BGA) as it builds a fully symmetric subsequence that is k-AP free.

1. Take $T = \{1, n\}$.
2. Choose the smallest $j \in \{1, 2, \dots, n\} - T$ such that $T \cup \{j, n-j+1\}$ is k-AP free. Set $T = T \cup \{j, n-j+1\}$.
3. Repeat step 2 until no such j can be found.
4. Output T.

Clearly,

$$r(k, n) \geqslant |BGA(k, n)|.$$

From experimental data, we have the following observation:

Observation 3. Consider a prime $p > 3$. Then $|BGA(p, x)| = x$ if $1 \leqslant x \leqslant p-2$, or else for $1 \leqslant i \leqslant p-1$ and $t \geqslant 1$,

$$\left| BGA\left(p, \frac{(ip - i - 1)p^t + 1}{(p-1)}\right) \right| = i \cdot (p-1)^t.$$

See Appendix B for supporting data.

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A Computed values

$f(3) = 4$, $c_{max}(3, n-4) = c_{max}(3, n) + 4$ for $n \geqslant 8$

n	$c_{max}(3,n)$	$\|U\|$	n	$c_{max}(3,n)$	$\|U\|$	n	$c_{max}(3,n)$	$\|U\|$	
4	2	2	8	6	2	12	10	2	⋯
5	4	**4**	9	8	**4**	13	12	**4**	⋯
6	4	2	10	8	2	14	12	2	⋯
7	5	**4**	11	9	**4**	15	13	**4**	⋯

$f(5) = 8$, $c_{max}(5, n-24) = c_{max}(5, n) + 20$ for $n \geqslant 62$

n	$c_{max}(5,n)$	$\|U\|$	n	$c_{max}(5,n)$	$\|U\|$	n	$c_{max}(5,n)$	$\|U\|$	
38	29	6	62	49	6	86	69	6	⋯
39	30	6	63	50	6	87	70	6	⋯
40	31	6	64	51	6	88	71	6	⋯
41	32	6	65	52	6	89	72	6	⋯
42	33	**8**	66	53	**8**	90	73	**8**	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	47	4	84	67	4	108	87	4	⋯
61	49	**8**	85	69	**8**	109	89	**8**	⋯

$f(7) = 14$, $c_{max}(7, n-120) = c_{max}(7, n) + 94$ for $n \geqslant 467$

n	$c_{max}(7,n)$	$\|U\|$	n	$c_{max}(7,n)$	$\|U\|$	n	$c_{max}(7,n)$	$\|U\|$	
347	268	10	467	362	10	587	456	10	⋯
348	269	10	468	363	10	588	457	10	⋯
349	270	10	469	364	10	589	458	10	⋯
350	271	10	470	365	10	590	459	10	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
358	278	**14**	478	372	**14**	598	466	**14**	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
465	361	10	585	455	10	705	549	10	⋯
466	361	**14**	586	455	**14**	706	549	**14**	⋯

$f(9) = 20$, $c_{max}(9, n-6720) = c_{max}(9, n) + 5104$ for $n \geqslant 13365$

n	$c_{max}(9,n)$	$\|U\|$	n	$c_{max}(9,n)$	$\|U\|$	n	$c_{max}(9,n)$	$\|U\|$	
6645	5043	14	13365	10147	14	20085	15251	14	⋯
6646	5045	14	13366	10149	14	20086	15253	14	⋯
6647	5045	16	13367	10149	16	20087	15253	16	⋯
6648	5045	14	13368	10149	14	20088	15253	14	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
6691	5078	**20**	13411	10182	**20**	20131	15286	**20**	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
13363	10146	14	20083	15250	14	26803	20354	14	⋯
13364	10146	16	20084	15250	16	26804	20354	16	⋯

$f(11) = 26$, $f(13) = 36$

B BGA results

$p = 3$

$t = 1,\ i = 1,\ |BGA(3, 2)| = 2$
$BGA(3, 2)| = \{1, 2\}$
$t = 2,\ i = 1,\ |BGA(3, 5)| = 4$
$BGA(3, 5)| = \{1, 2, 4, 5\}$
$t = 3,\ i = 1,\ |BGA(3, 14)| = 8$
$BGA(3, 14)| = \{1, 2, 4, 5, 10, 11, 13, 14\}$
$t = 4,\ i = 1,\ |BGA(3, 41)| = 16$
$BGA(3, 41)| = \{1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41\}$
$t = 5,\ i = 1,\ |BGA(3, 122)| = 32$
$BGA(3, 122)| = \{1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 82, 83, 85, 86, 91, 92, 94, 95, 109, 110, 112, 113, 118, 119, 121, 122\}$
$t = 6,\ i = 1,\ |BGA(3, 365)| = 64$
$BGA(3, 365)| = \{1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 82, 83, 85, 86, 91, 92, 94, 95, 109, 110, 112, 113, 118, 119, 121, 122, 244, 245, 247, 248, 253, 254, 256, 257, 271, 272, 274, 275, 280, 281, 283, 284, 325, 326, 328, 329, 334, 335, 337, 338, 352, 353, 355, 356, 361, 362, 364, 365\}$
$t = 7,\ i = 1,\ |BGA(3, 1094)| = 128$
$BGA(3, 1094)| = \{1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41, 82, 83, 85, 86, 91, 92, 94, 95, 109, 110, 112, 113, 118, 119, 121, 122, 244, 245, 247, 248, 253, 254, 256, 257, 271, 272, 274, 275, 280, 281, 283, 284, 325, 326, 328, 329, 334, 335, 337, 338, 352, 353, 355, 356, 361, 362, 364, 365, 730, 731, 733, 734, 739, 740, 742, 743, 757, 758, 760, 761, 766, 767, 769, 770, 811, 812, 814, 815, 820, 821, 823, 824, 838, 839, 841, 842, 847, 848, 850, 851, 973, 974, 976, 977, 982, 983, 985, 986, 1000, 1001, 1003, 1004, 1009, 1010, 1012, 1013, 1054, 1055, 1057, 1058, 1063, 1064, 1066, 1067, 1081, 1082, 1084, 1085, 1090, 1091, 1093, 1094\}$

$p = 5$

$t = 1,\ i = 1,\ |BGA(5, 4)| = 4$
$BGA(5, 4)| = \{1, 2, 3, 4\}$
$t = 1,\ i = 2,\ |BGA(5, 9)| = 8$
$BGA(5, 9)| = \{1, 2, 3, 4, 6, 7, 8, 9\}$
$t = 1,\ i = 3,\ |BGA(5, 14)| = 12$
$BGA(5, 14)| = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14\}$
$t = 2,\ i = 1,\ |BGA(5, 19)| = 16$
$BGA(5, 19)| = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19\}$
$t = 2,\ i = 2,\ |BGA(5, 44)| = 32$
$BGA(5, 44)| = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 26, 27, 28, 29, 31, 32, 33, 34, 36, 37, 38, 39, 41, 42, 43, 44\}$
$t = 2,\ i = 3,\ |BGA(5, 69)| = 48$
$BGA(5, 69)| = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 26, 27, 28, 29, 31, 32, 33, 34, 36, 37, 38, 39, 41, 42, 43, 44, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 66, 67, 68, 69\}$
$t = 3,\ i = 1,\ |BGA(5, 94)| = 64$

$BGA(5, 94)| = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 26, 27, 28, 29, 31, 32, 33, 34, 36, 37, 38, 39, 41, 42, 43, 44, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 66, 67, 68, 69, 76, 77, 78, 79, 81, 82, 83, 84, 86, 87, 88, 89, 91, 92, 93, 94\}$

$t = 3$, $i = 2$, $|BGA(5, 219)| = 128$

$BGA(5, 219)| = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 26, 27, 28, 29, 31, 32, 33, 34, 36, 37, 38, 39, 41, 42, 43, 44, 51, 52, 53, 54, 56, 57, 58, 59, 61, 62, 63, 64, 66, 67, 68, 69, 76, 77, 78, 79, 81, 82, 83, 84, 86, 87, 88, 89, 91, 92, 93, 94, 126, 127, 128, 129, 131, 132, 133, 134, 136, 137, 138, 139, 141, 142, 143, 144, 151, 152, 153, 154, 156, 157, 158, 159, 161, 162, 163, 164, 166, 167, 168, 169, 176, 177, 178, 179, 181, 182, 183, 184, 186, 187, 188, 189, 191, 192, 193, 194, 201, 202, 203, 204, 206, 207, 208, 209, 211, 212, 213, 214, 216, 217, 218, 219\}$

$p = 7$

$t = 1$, $i = 1$, $|BGA(7, 6)| = 6$

$BGA(7, 6)| = \{1, 2, 3, 4, 5, 6\}$

$t = 1$, $i = 2$, $|BGA(7, 13)| = 12$

$BGA(7, 13)| = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13\}$

$t = 1$, $i = 3$, $|BGA(7, 20)| = 18$

$BGA(7, 20)| = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20\}$

$t = 1$, $i = 4$, $|BGA(7, 27)| = 24$

$BGA(7, 27)| = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27\}$

$t = 1$, $i = 5$, $|BGA(7, 34)| = 30$

$BGA(7, 34)| = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34\}$

$t = 2$, $i = 1$, $|BGA(7, 41)| = 36$

$BGA(7, 41)| = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41\}$

$t = 2$, $i = 2$, $|BGA(7, 90)| = 72$

$BGA(7, 90)| = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 85, 86, 87, 88, 89, 90\}$

$t = 2$, $i = 3$, $|BGA(7, 139)| = 108$

$BGA(7, 139)| = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40, 41, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 61, 62, 64, 65, 66, 67, 68, 69, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 85, 86, 87, 88, 89, 90, 99, 100, 101, 102, 103, 104, 106, 107, 108, 109, 110, 111, 113, 114, 115, 116, 117, 118, 120, 121, 122, 123, 124, 125, 127, 128, 129, 130, 131, 132, 134, 135, 136, 137, 138, 139\}$

$p = 11$

$t = 1$, $i = 1$, $|BGA(11, 10)| = 10$

$BGA(11, 10)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$t = 1$, $i = 2$, $|BGA(11, 21)| = 20$

$BGA(11, 21)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$

$t = 1,\ i = 3,\ |BGA(11, 32)| = 30$
$BGA(11, 32)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32\}$
$t = 1,\ i = 4,\ |BGA(11, 43)| = 40$
$BGA(11, 43)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43\}$
$t = 1,\ i = 5,\ |BGA(11, 54)| = 50$
$BGA(11, 54)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54\}$
$t = 1,\ i = 6,\ |BGA(11, 65)| = 60$
$BGA(11, 65)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65\}$
$t = 1,\ i = 7,\ |BGA(11, 76)| = 70$
$BGA(11, 76)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76\}$
$t = 1,\ i = 8,\ |BGA(11, 87)| = 80$
$BGA(11, 87)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87\}$
$t = 1,\ i = 9,\ |BGA(11, 98)| = 90$
$BGA(11, 98)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98\}$
$t = 2,\ i = 1,\ |BGA(11, 109)| = 100$
$BGA(11, 109)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109\}$
$t = 2,\ i = 2,\ |BGA(11, 230)| = 200$
$BGA(11, 230)| = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 177, 178, 179, 180, 181, 182, 183, 184, 185, 186, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230\}$

C Computed values of $r(k,n)$

n	$b(n)$	lower bound $\lvert BGA(3,n)\rvert$	$r(3,n)$	upper bound (Theorem 1)	n	$b(n)$	lower bound $\lvert BGA(3,n)\rvert$	$r(3,n)$	upper bound (Theorem 1)
4	2	2	3	3	54	2	16	18	41
5	1	4	4	4	58	2	16	19	44
9	4	4	5	7	63	2	16	20	48
11	7	6	6	9	71	4	18	21	54
13	6	6	7	10	74	1	18	22	56
14	1	8	8	11	82	10	18	23	62
20	2	8	9	15	84	1	18	24	63
24	2	8	10	18	92	14	22	25	69
26	2	10	11	20	95	8	24	26	72
30	1	10	12	23	100	2	24	27	75
32	2	12	13	24	104	1	24	28	78
36	2	12	14	27	111	6	26	29	84
40	20	14	15	30	114	1	28	30	86
41	1	16	16	31	121	70	30	31	91
51	14	16	17	39	122	1	32	32	92

n	$b(n)$	lower bound $\lvert BGA(4,n)\rvert$	$r(4,n)$	upper bound (Theorem 1)	n	$b(n)$	lower bound $\lvert BGA(4,n)\rvert$	$r(4,n)$	upper bound (Theorem 1)
5	3	4	4	5	30	2	16	18	25
6	2	4	5	5	33	6	16	19	28
8	4	6	6	7	34	2	18	20	29
9	2	6	7	8	37	6	18	21	31
10	1	8	8	9	40	14	18	22	34
13	3	9	9	11	43	38	18	23	36
15	2	8	10	13	45	2	20	24	38
17	10	10	11	15	48	2	20	25	40
19	24	12	12	16	50	3	22	26	42
21	10	12	13	18	53	12	22	27	45
23	12	13	14	20	54	1	24	28	45
25	2	14	15	21	58	6	26	29	49
27	8	14	16	23	60	1	26	30	50
28	4	14	17	24					

n	$b(n)$	lower bound $\|BGA(5,n)\|$	$r(5,n)$	upper bound (Theorem 1)	n	$b(n)$	lower bound $\|BGA(5,n)\|$	$r(5,n)$	upper bound (Theorem 1)
6	4	4	5	6	52	2	32	35	46
7	3	6	6	7	54	1300	32	36	48
8	2	6	7	7	56	3508	32	37	49
9	1	8	8	8	57	1736	36	38	50
11	6	8	9	10	58	768	32	39	51
12	4	10	10	11	59	256	40	40	52
13	2	8	11	12	61	512	36	41	54
14	1	12	12	13	62	192	38	42	55
16	4	12	13	14	63	64	38	43	56
17	3	12	14	15	64	16	36	44	56
18	2	14	15	16	66	32	40	45	58
19	1	16	16	17	67	12	40	46	59
24	40	16	17	21	68	4	42	47	60
25	2	16	18	22	69	1	48	48	61
27	70	18	19	24	76	2012	42	49	67
28	12	18	20	25	77	1202	44	50	68
29	2	20	21	26	78	640	44	51	69
31	5	20	22	28	79	256	48	52	70
33	266	22	23	29	81	768	44	53	71
34	81	24	24	30	82	432	48	54	72
36	236	24	25	32	83	216	48	55	73
37	115	26	26	33	84	81	52	56	74
38	48	24	27	34	86	216	50	57	76
39	16	28	28	35	87	108	52	58	77
41	32	26	29	36	88	48	50	59	77
42	12	28	30	37	89	16	56	60	78
43	4	30	31	38	91	32	54	61	80
44	1	32	32	39	92	12	56	62	81
49	2	32	33	43	93	4	56	63	82
51	18	32	34	45	94	1	64	64	83

n	$b(n)$	lower bound $\lvert BGA(7,n)\rvert$	$r(7,n)$	upper bound (Theorem 1)	n	$b(n)$	lower bound $\lvert BGA(7,n)\rvert$	$r(7,n)$	upper bound (Theorem 1)
8	6	6	7	8	51	76	38	40	47
9	5	8	8	9	52	22	38	41	48
10	4	8	9	10	53	2	40	42	49
11	3	10	10	11	55	242	42	43	51
12	2	10	11	11	56	8	42	44	52
13	1	12	12	12	57	4	42	45	53
15	12	12	13	14	59	218	44	46	55
16	9	14	14	15	60	54	46	47	55
17	6	13	15	16	61	10	46	48	56
18	4	16	16	17	63	32	48	49	58
19	2	16	17	18	64	14	48	50	59
20	1	18	18	19	65	2	50	51	60
22	8	18	19	21	67	19807	52	52	62
23	6	20	20	22	68	10294	52	53	63
24	4	20	21	22	69	4103	54	54	64
25	3	20	22	23	71	18522	54	55	66
26	2	20	23	24	72	11541	56	56	66
27	1	24	24	25	73	6914	54	57	67
29	6	22	25	27	74	3888	56	58	68
30	5	24	26	28	75	1944	56	59	69
31	4	24	27	29	76	729	60	60	70
32	3	24	28	30	78	2918	58	61	72
33	2	26	29	31	79	1621	62	62	73
34	1	30	30	32	80	864	60	63	74
36	6	30	31	33	81	432	60	64	75
37	5	28	32	34	82	192	62	65	76
38	4	30	33	35	83	64	66	66	77
39	3	32	34	36	85	192	64	67	78
40	2	30	35	37	86	80	64	68	79
41	1	36	36	38	87	32	66	69	80
46	18	36	37	43	88	12	68	70	81
48	9	36	38	44	89	4	66	71	82
50	392	36	39	46	90	1	72	72	83

n	$b(n)$	lower bound $\lvert BGA(11,n)\rvert$	$r(11,n)$	upper bound (Theorem 1)	n	$b(n)$	lower bound $\lvert BGA(11,n)\rvert$	$r(11,n)$	upper bound (Theorem 1)
12	10	10	11	12	61	5	54	56	58
13	9	12	12	13	62	4	52	57	59
14	8	12	13	14	63	3	54	58	60
15	7	14	14	15	64	2	56	59	61
16	6	14	15	16	65	1	60	60	62
17	5	16	16	17	67	10	56	61	64
18	4	16	17	18	68	9	58	62	65
19	3	18	18	19	69	8	58	63	66
20	2	18	19	19	70	7	62	64	67
21	1	20	20	20	71	6	62	65	68
23	30	20	21	22	72	5	60	66	69
24	25	22	22	23	73	4	62	67	70
25	20	21	23	24	74	3	62	68	71
26	16	24	24	25	75	2	66	69	72
27	12	24	25	26	76	1	70	70	73
28	9	26	26	27	78	10	66	71	75
29	6	25	27	28	79	9	66	72	76
30	4	28	28	29	80	8	68	73	76
31	2	28	29	30	81	7	68	74	77
32	1	30	30	31	82	6	68	75	78
34	20	30	31	33	83	5	72	76	79
35	16	32	32	34	84	4	72	77	80
36	12	32	33	35	85	3	74	78	81
37	9	32	34	36	86	2	74	79	82
38	6	34	35	37	87	1	80	80	83
39	5	34	36	38	89	10	76	81	85
40	4	36	37	38	90	9	78	82	86
41	3	36	38	39	91	8	76	83	87
42	2	38	39	40	92	7	80	84	88
43	1	40	40	41	93	6	80	85	89
45	10	39	41	43	94	5	80	86	90
46	9	40	42	44	95	4	82	87	91
47	8	40	43	45	96	3	80	88	92
48	7	42	44	46	97	2	80	89	93
49	6	42	45	47	98	1	90	90	94
50	5	44	46	48	100	10	84	91	95
51	4	43	47	49	101	9	84	92	96
52	3	44	48	50	102	8	88	93	97
53	2	46	49	51	103	7	86	94	98
54	1	50	50	52	104	6	88	95	99
56	10	48	51	54	105	5	90	96	100
57	9	50	52	55	106	4	90	97	101
58	8	50	53	56	107	3	92	98	102
59	7	50	54	57	108	2	92	99	103
60	6	52	55	57	109	1	100	100	104