A criterion for the log-convexity of combinatorial sequences

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Abstract

Recently, Došlić, and Liu and Wang developed techniques for dealing with the log-convexity of sequences. In this paper, we present a criterion for the log-convexity of some combinatorial sequences. In order to prove the log-convexity of a sequence satisfying a three-term recurrence, by our method, it suffices to compute a constant number of terms at the beginning of the sequence. For example, in order to prove the log-convexity of the Apéry numbers A_n, by our method, we just need to evaluate the values of A_n for $0 \leqslant n \leqslant 6$. As applications, we prove the log-convexity of some famous sequences including the Catalan-Larcombe-French numbers. This confirms a conjecture given by Sun.

Keywords: log-convexity; three-term recurrence; combinatorial sequences

1 Introduction

A positive sequence $\{S_n\}_{n=0}^{\infty}$ is said to be log-convex (respectively log-concave) if for $n \geqslant 1$,

$$\frac{S_n}{S_{n-1}} \leqslant \frac{S_{n+1}}{S_n} \qquad \text{(respectively } \frac{S_n}{S_{n-1}} \geqslant \frac{S_{n+1}}{S_n}). \tag{1}$$

Meanwhile, the sequence $\{S_n\}_{n=0}^{\infty}$ is called strictly log-convex (log-concave) if the inequality in (1.1) is strict for all $n \geqslant 1$. In 1994, Engel [8] proved the log-convexity of the Bell numbers. Recently, Došlić [4, 5, 6], Došlić and Veljan [7], and Liu and Wang [14] developed techniques for proving the log-convexity of sequences. Došlić [4, 5, 6] presented

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several methods for dealing with the log-convexity of combinatorial sequences. He proved that the Motzkin numbers, the Fine numbers, the Franel numbers of order 3 and 4, the Apéry numbers, the large Schröder numbers, the derangements numbers and the central Delannoy numbers are log-convex. In their wonderful paper [14], Liu and Wang proved that the log-convexity is preserved under componentwise sum, under binomial convolution, and by the linear transformations given by the matrices of binomial coefficients and Stirling numbers of two kinds. Many combinatorial sequences satisfy a three-term recurrence. Liu and Wang [14] presented some criterions for the log-convexity of the sequences $\{z_n\}_{n=0}^{\infty}$ satisfying the following recurrence

$$a(n)z_{n+1} = b(n)z_n + c(n)z_{n-1}, \tag{2}$$

where $a(n)$, $b(n)$ and $c(n)$ are positive for $n \geqslant 1$, Liu and Wang [14] proved the following theorem.

Theorem 1. *Let* $\{z_n\}_{n=0}^{\infty}$ *be defined by* (2) *and*

$$\lambda_n = \frac{b(n) + \sqrt{b^2(n) + 4a(n)c(n)}}{2a(n)}. \tag{3}$$

Suppose that $z_0,\ z_1,\ z_2,\ z_3$ *is log-convex and that the inequality*

$$a(n)\lambda_{n-1}\lambda_{n+1} - b(n)\lambda(n-1) - c(n) \geqslant 0 \tag{4}$$

is true for $n \geqslant 2$. *Then the sequence* $\{z_n\}_{n=0}^{\infty}$ *is log-convex.*

Liu and Wang [14] also considered the log-convexity of the sequence $\{z_n\}_{n=0}^{\infty}$ defined by

$$(\alpha_n + \alpha_0)z_{n+1} = (\beta_1 n + \beta_0)z_n - (\gamma_1 n + \gamma_0)z_{n-1}. \tag{5}$$

for $n \geqslant 1$. They gave criterions for the log-convexity of the sequences $\{z_n\}_{n=0}^{\infty}$. Employing their criterions, they proved the log-convexity of some combinatorial sequences. Liu [13] gave sufficient conditions for the positivity of the sequences defined by (5).

Motivated by these results established by Liu and Wang [14], in this paper, we investigate the log-convexity problem of the sequence $\{S_n\}_{n=0}^{\infty}$ having the following three-term recurrence

$$S_n = \frac{\sum_{i=0}^{k} a_i n^i}{\sum_{i=0}^{k} b_i n^i} S_{n-1} - \frac{\sum_{i=0}^{l} c_i n^i}{\sum_{i=0}^{l} d_i n^i} S_{n-2} \qquad (n \geqslant 2), \tag{6}$$

where $\gcd(\sum_{i=0}^{k} a_i n^i, \sum_{i=0}^{k} b_i n^i) = \gcd(\sum_{i=0}^{l} c_i n^i, \sum_{i=0}^{l} d_i n^i) = 1$ and k, l, a_k, b_k, c_l and d_l are positive numbers. The authors [19] gave a criterion for the positivity of the sequence $\{S_n\}_{n=0}^{\infty}$ defined by (6). The aim of this paper is to present a criterion for the log-convexity of some famous combinatorial sequences. By our method, in order to determine the log-convexity of the sequence $\{S_n\}_{n=0}^{\infty}$ defined by (6), it suffices to compute a constant number of terms at the beginning of the sequence $\{S_n\}_{n=0}^{\infty}$. As applications, we prove some famous combinatorial sequences are strictly log-convex. Specially, we show that the Catalan-Larcombe-French numbers $\{P_n\}_{n=0}^{\infty}$ is strictly log-convex which confirms a conjecture given by Sun [18].

In order to state our main result, we first introduce some notations. Given a polynomial $f(n)$ defined by

$$f(n)=\sum_{i=0}^{k} f_i n^i, \tag{7}$$

where f_i $(0\leqslant i\leqslant k)$ are real numbers and $f_k>0$. Define an operator L on $f(n)$ by

$$L(f(n))=\frac{1}{f_k}\sum_{0\leqslant i\leqslant k-1,\ f_i<0}|f_i|. \tag{8}$$

For example,

$$L(5n^4-2n^3+4n^2-6n-3)=\frac{11}{5}. \tag{9}$$

It is easy to see that $f(n)>0$ for $n\geqslant [L(f(n))]+1$.

Throughout this paper, we always let

$$\frac{\sum_{i=0}^{k} a_i(n+2)^i}{\sum_{i=0}^{k} b_i(n+2)^i}-\frac{\sum_{i=0}^{k} a_i(n+1)^i}{\sum_{i=0}^{k} b_i(n+1)^i}=\frac{\sum_{j=0}^{r} e_j n^j}{\sum_{t=0}^{s} h_t n^t}, \tag{10}$$

$$\frac{\sum_{i=0}^{l} c_i(n+2)^i}{\sum_{i=0}^{l} d_i(n+2)^i}-\frac{\sum_{i=0}^{l} c_i(n+1)^i}{\sum_{i=0}^{l} d_i(n+1)^i}=\frac{\sum_{j=0}^{u} p_j n^j}{\sum_{t=0}^{v} q_t n^t}, \tag{11}$$

and

$$\frac{\sum_{i=0}^{k} a_i(n+2)^i}{\sum_{i=0}^{k} b_i(n+2)^i}-\frac{\sum_{i=0}^{l} c_i(n+2)^i}{\sum_{i=0}^{l} d_i(n+2)^i}\frac{\left(\sum_{t=0}^{v} q_t n^t\right)\left(\sum_{j=0}^{r} e_j n^j\right)}{\left(\sum_{j=0}^{u} p_j n^j\right)\left(\sum_{t=0}^{s} h_t n^t\right)}$$
$$-\frac{\left(\sum_{j=0}^{u} p_j (n+1)^j\right)\left(\sum_{t=0}^{s} h_t (n+1)^t\right)}{\left(\sum_{t=0}^{v} q_t (n+1)^t\right)\left(\sum_{j=0}^{r} e_j (n+1)^j\right)}=\frac{\sum_{i=0}^{\alpha} x_i n^i}{\sum_{i=0}^{\beta} y_i n^i}, \tag{12}$$

where $h_s>0$, $q_v>0$, $y_\beta>0$ and

$$\gcd\left(\sum_{j=0}^{r} e_j n^j, \sum_{t=0}^{s} h_t n^t\right)=\gcd\left(\sum_{j=0}^{u} p_j n^j, \sum_{t=0}^{v} q_t n^t\right)=\gcd\left(\sum_{i=0}^{\alpha} x_i n^i, \sum_{i=0}^{\beta} y_i n^i\right)=1.$$

Our main result can be stated as follows.

Theorem 2. *Let* $\{S_n\}_{n=0}^{\infty}$ *be a positive sequence and satisfy* (6). *If* $p_u > 0$, $e_r > 0$, $x_\alpha > 0$ *and there exists an integer* N_0 *such that*

$$N_0 \geqslant r_1 = \max\left\{ \left[L(\sum_{i=0}^{l} c_i n^i)\right], \left[L(\sum_{i=0}^{l} d_i n^i)\right], \left[L(\sum_{j=0}^{r} e_j n^j)\right], \left[L(\sum_{t=0}^{s} h_t n^t)\right], \right.$$
$$\left. \left[L(\sum_{j=0}^{u} p_j n^j)\right], \left[L(\sum_{t=0}^{v} q_t n^t)\right], \left[L(\sum_{i=0}^{\alpha} x_i n^i)\right], \left[L(\sum_{i=0}^{\beta} y_i n^i)\right] \right\} + 1 \quad (13)$$

and

$$\frac{S_{N_0}}{S_{N_0-1}} < \frac{S_{N_0+1}}{S_{N_0}}, \quad (14)$$

$$\frac{S_{N_0+1}}{S_{N_0}} > \frac{\left(\sum_{j=0}^{u} p_j N_0^j\right)\left(\sum_{t=0}^{s} h_t N_0^t\right)}{\left(\sum_{t=0}^{v} q_t N_0^t\right)\left(\sum_{j=0}^{r} e_j N_0^j\right)}, \quad (15)$$

then the sequence $\{S_n\}_{n=N_0}^{\infty}$ *is strictly log-convex, namely,*

$$\frac{S_n}{S_{n-1}} < \frac{S_{n+1}}{S_n}, \qquad (n \geqslant N_0). \quad (16)$$

This paper is organized as follows. We give the proof of Theorem 2 in Sections 2. As applications of Theorem 2, in Section 3, we prove the log-convexity of some famous sequences including the Catalan-Larcombe-French numbers. This confirms a conjecture given by Sun [18].

2 Proof of Theorem 2

In this section, we present the proof of Theorem 2.

Proof. By the definition of r_1, we see that for all $n \geqslant N_0 \geqslant r_1$,

$$\frac{\sum_{i=0}^{l} c_i n^i}{\sum_{i=0}^{l} d_i n^i} > 0, \quad (17)$$

$$\frac{\sum_{i=0}^{k} a_i (n+2)^i}{\sum_{i=0}^{k} b_i (n+2)^i} - \frac{\sum_{i=0}^{k} a_i (n+1)^i}{\sum_{i=0}^{k} b_i (n+1)^i} = \frac{\sum_{j=0}^{r} e_j n^j}{\sum_{t=0}^{s} h_t n^t} > 0, \quad (18)$$

$$\frac{\sum_{i=0}^{k} c_i (n+2)^i}{\sum_{i=0}^{k} d_i (n+2)^i} - \frac{\sum_{i=0}^{k} c_i (n+1)^i}{\sum_{i=0}^{k} d_i (n+1)^i} = \frac{\sum_{j=0}^{u} p_j n^j}{\sum_{t=0}^{v} q_t n^t} > 0, \quad (19)$$

and

$$\frac{\sum_{i=0}^{\alpha} x_i n^i}{\sum_{i=0}^{\beta} y_i n^i} > 0. \quad (20)$$

We first give a lower bound for $\frac{S_{n+1}}{S_n}$. Moreover, we prove that for $n \geqslant N_0$,

$$\frac{S_{n+1}}{S_n} > \frac{\left(\sum_{j=0}^{u} p_j n^j\right)\left(\sum_{t=0}^{s} h_t n^t\right)}{\left(\sum_{t=0}^{v} q_t n^t\right)\left(\sum_{j=0}^{r} e_j n^j\right)}. \tag{21}$$

We prove (21) by induction on n. By (15), we see that (21) holds for $n = N_0$. Suppose that (21) holds for $n = m \geqslant N_0$, that is,

$$\frac{S_{m+1}}{S_m} > \frac{\left(\sum_{j=0}^{u} p_j m^j\right)\left(\sum_{t=0}^{s} h_t m^t\right)}{\left(\sum_{t=0}^{v} q_t m^t\right)\left(\sum_{j=0}^{r} e_j m^j\right)}. \tag{22}$$

It follows from (17), (18) and (22) that for $m \geqslant N_0$,

$$-\frac{\sum_{i=0}^{l} c_i(m+2)^i}{\sum_{i=0}^{l} d_i(m+2)^i}\frac{S_m}{S_{m+1}} > -\frac{\sum_{i=0}^{l} c_i(m+2)^i}{\sum_{i=0}^{l} d_i(m+2)^i}\frac{\left(\sum_{t=0}^{v} q_t m^t\right)\left(\sum_{j=0}^{r} e_j m^j\right)}{\left(\sum_{j=0}^{u} p_j m^j\right)\left(\sum_{t=0}^{s} h_t m^t\right)}. \tag{23}$$

Now, we are ready to show that (21) also holds for $n = m + 1$. Employing (6) and (23), we deduce that

$$\begin{aligned}\frac{S_{m+2}}{S_{m+1}} &= \frac{\sum_{i=0}^{k} a_i(m+2)^i}{\sum_{i=0}^{k} b_i(m+2)^i} - \frac{\sum_{i=0}^{l} c_i(m+2)^i}{\sum_{i=0}^{l} d_i(m+2)^i}\frac{S_m}{S_{m+1}} \\ &> \frac{\sum_{i=0}^{k} a_i(m+2)^i}{\sum_{i=0}^{k} b_i(m+2)^i} - \frac{\sum_{i=0}^{l} c_i(m+2)^i}{\sum_{i=0}^{l} d_i(m+2)^i}\frac{\left(\sum_{t=0}^{v} q_t m^t\right)\left(\sum_{j=0}^{r} e_j m^j\right)}{\left(\sum_{j=0}^{u} p_j m^j\right)\left(\sum_{t=0}^{s} h_t m^t\right)}.\end{aligned} \tag{24}$$

In view of (12), (20) and (24), we find that for $m \geqslant N_0$

$$\begin{aligned}&\frac{S_{m+2}}{S_{m+1}} - \frac{\left(\sum_{j=0}^{u} p_j (m+1)^j\right)\left(\sum_{t=0}^{s} h_t (m+1)^t\right)}{\left(\sum_{t=0}^{v} q_t (m+1)^t\right)\left(\sum_{j=0}^{r} e_j (m+1)^j\right)} \\ >&\frac{\sum_{i=0}^{k} a_i(m+2)^i}{\sum_{i=0}^{k} b_i(m+2)^i} - \frac{\sum_{i=0}^{l} c_i(m+2)^i}{\sum_{i=0}^{l} d_i(m+2)^i}\frac{\left(\sum_{t=0}^{v} q_t m^t\right)\left(\sum_{j=0}^{r} e_j m^j\right)}{\left(\sum_{j=0}^{u} p_j m^j\right)\left(\sum_{t=0}^{s} h_t m^t\right)} \\ &\quad - \frac{\left(\sum_{j=0}^{u} p_j (m+1)^j\right)\left(\sum_{t=0}^{s} h_t (m+1)^t\right)}{\left(\sum_{t=0}^{v} q_t (m+1)^t\right)\left(\sum_{j=0}^{r} e_j (m+1)^j\right)} \\ =&\frac{\sum_{i=0}^{\alpha} x_i m^i}{\sum_{i=0}^{\beta} y_i m^i} > 0,\end{aligned} \tag{25}$$

which implies that (21) is true for $n = m + 1$. By induction, we have proved (21) holds for $n \geqslant N_0$.

Now, we turn to prove (16). We also prove (16) by induction on n. It follows from (14) that (16) holds for $n = N_0$. Assume that (16) is true for $n = m \geqslant N_0$, namely,

$$\frac{S_m}{S_{m-1}} < \frac{S_{m+1}}{S_m}. \tag{26}$$

By (17) and (26), we find that for $m \geqslant N_0$

$$\frac{\sum_{i=0}^{l} c_i(m+1)^i}{\sum_{i=0}^{l} d_i(m+1)^i} \frac{S_{m-1}}{S_m} > \frac{\sum_{i=0}^{l} c_i(m+1)^i}{\sum_{i=0}^{l} d_i(m+1)^i} \frac{S_m}{S_{m+1}}. \tag{27}$$

Employing (6), (10), (11), (19), (21) and (27), we deduce that for $m \geqslant N_0$,

$$\begin{aligned}
\frac{S_{m+2}}{S_{m+1}} - \frac{S_{m+1}}{S_m} &= \frac{\sum_{i=0}^{k} a_i(m+2)^i}{\sum_{i=0}^{k} b_i(m+2)^i} - \frac{\sum_{i=0}^{l} c_i(m+2)^i}{\sum_{i=0}^{l} d_i(m+2)^i} \frac{S_m}{S_{m+1}} \\
&\qquad - \frac{\sum_{i=0}^{k} a_i(m+1)^i}{\sum_{i=0}^{k} b_i(m+1)^i} + \frac{\sum_{i=0}^{l} c_i(m+1)^i}{\sum_{i=0}^{l} d_i(m+1)^i} \frac{S_{m-1}}{S_m} \\
&> \frac{\sum_{i=0}^{k} a_i(m+2)^i}{\sum_{i=0}^{k} b_i(m+2)^i} - \frac{\sum_{i=0}^{k} a_i(m+1)^i}{\sum_{i=0}^{k} b_i(m+1)^i} \\
&\qquad + \left(\frac{\sum_{i=0}^{l} c_i(m+1)^i}{\sum_{i=0}^{l} d_i(m+1)^i} - \frac{\sum_{i=0}^{l} c_i(m+2)^i}{\sum_{i=0}^{l} d_i(m+2)^i} \right) \frac{S_m}{S_{m+1}} \\
&= \frac{\sum_{j=0}^{r} e_j m^j}{\sum_{t=0}^{s} h_t m^t} - \frac{\sum_{j=0}^{u} p_j m^j}{\sum_{t=0}^{v} q_t m^t} \frac{S_m}{S_{m+1}} \\
&> \frac{\sum_{j=0}^{r} e_j m^j}{\sum_{t=0}^{s} h_t m^t} - \frac{\sum_{j=0}^{u} p_j m^j}{\sum_{t=0}^{v} q_t m^t} \left(\frac{\left(\sum_{t=0}^{v} q_t m^t\right) \left(\sum_{j=0}^{r} e_j m^j\right)}{\left(\sum_{j=0}^{u} p_j m^j\right) \left(\sum_{t=0}^{s} h_t m^t\right)} \right) = 0,
\end{aligned} \tag{28}$$

which implies that (16) holds for $n = m + 1$. Theorem 2 is proved by induction. This completes the proof. □

3 Applications of Theorem 2

In this section, employing the criterion given in this paper, we prove some results on the log-convexity of some combinatorial sequences.

The Catalan-Larcombe-French numbers P_n for $n \geqslant 0$ were first defined by Catalan in [2], in terms of the "Segner numbers". Catalan stated that the P_n could be defined by the following recurrence relation:

$$P_n = \frac{8(3n^2-3n+1)}{n^2}P_{n-1} - \frac{128(n-1)^2}{n^2}P_{n-2}, \tag{29}$$

for $n \geqslant 2$, with the initial values given by $P_0 = 1$ and $P_1 = 8$. Larcombe and French [12] gave a detailed account of properties of P_n, and obtained the following formulas for these numbers:

$$P_n = \sum_{k=0}^{n} \frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}} = 2^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}^2 4^{n-2k} \tag{30}$$

and

$$P_n = \frac{1}{n!}\sum_{r+s=n}\binom{2r}{r}\binom{2s}{s}\frac{(2r)!(2s)!}{r!s!} = \sum_{r+s=n}\frac{\binom{2r}{r}^2\binom{2s}{s}^2}{\binom{n}{r}} \tag{31}$$

for $n \geqslant 0$. The first few P_n are 1, 8, 80, 896, 10816, 137728. This is the sequence A053175 in Sloane's database [16]. The sequence $\{P_n\}_{n=0}^{\infty}$ is also related to the theory of modular forms; see [20].

Recently, Sun [18] conjectured that

Conjecture 3. The sequences $\{P_{n+1}/P_n\}_{n=0}^{\infty}$ and $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$ are strictly increasing.

Employing Theorem 2, we prove that

Corollary 4. *Conjecture 3 is true.*

Proof. By (13), we find $r_1 = 3$. Set $N_0 = 3$. It is easy to check that (14) and (15) hold for $N_0 = 3$. By Theorem 2, we see that the sequence $\{P_n\}_{n=3}^{\infty}$ is strictly log-convex. It is a routine to verify that $\frac{P_{i+1}}{P_i} > \frac{P_i}{P_{i-1}}$ for $1 \leqslant i \leqslant 3$. Thus, the sequence $\{P_n\}_{n=0}^{\infty}$ is strictly log-convex and the sequence $\{P_{n+1}/P_n\}_{n=0}^{\infty}$ is strictly increasing, namely,

$$\frac{P_{n+1}}{P_n} > \frac{P_n}{P_{n-1}}, \qquad n \geqslant 1. \tag{32}$$

By (32) and the fact $P_0 = 1$, we deduce that

$$P_n = P_0 \prod_{i=1}^{n} \frac{P_i}{P_{i-1}} < \left(\frac{P_{n+1}}{P_n}\right)^n, \tag{33}$$

which implies that

$$P_n^{n+1} < P_{n+1}^n. \tag{34}$$

It follows from (34) that the sequences $\{\sqrt[n]{P_n}\}_{n=1}^{\infty}$ is strictly increasing. This completes the proof. □

The Apéry number A_n is defined by

$$A_n = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} A_{n-1} - \frac{(n-1)^3}{n^3} A_{n-2}, \qquad n \geqslant 2, \tag{35}$$

with $A_0 = 1$ and $A_1 = 5$. The Apéry numbers play a key role in Apéry's proof of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$; see [1]. The log-convexity of $\{A_n\}_{n=0}^{\infty}$ was proved by Došlić [4]. Chen and Xia [3] proved that the sequence $\{A_n\}_{n=0}^{\infty}$ is 2-log-convex. Now, we present another proof of the log-convexity of $\{A_n\}_{n=0}^{\infty}$. Set $k = l = 3$, $a_3 = 34$ and $b_3 = c_3 = d_3 = 1$ in Theorem 2. By the definition of r_1, we obtain $r_1 = 5$. Set $N_0 = 5$. We can check that (14) and (15) hold for $N_0 = 5$. Thus, by Theorem 2, the sequence $\{A_n\}_{n=5}^{\infty}$ is log-convex. We can also verify that $\frac{A_{i+1}}{A_i} > \frac{A_i}{A_{i-1}}$ for $1 \leqslant i \leqslant 5$. Thus, the following corollary is true.

Corollary 5. *The sequence $\{A_n\}_{n=0}^{\infty}$ is strictly log-convex.*

The central Delannoy number D_n is defined by

$$D_n = \frac{3(2n-1)}{n} D_{n-1} - \frac{n-1}{n} D_{n-2}, \qquad n \geqslant 2, \tag{36}$$

with $D_0 = 1$ and $D_1 = 3$; see [15]. Došlić [4] , and Liu and Wang [14] proved the log-convexity of the sequence $\{D_n\}_{n=0}^{\infty}$. By (13), we find $r_1 = 2$. Let $N_0 = 2$. It is easy to check that (14) and (15) hold for $N_0 = 2$. The following corollary follows from Theorem 2 and the fact $\frac{D_2}{D_1} > \frac{D_1}{D_0}$.

Corollary 6. *The sequence $\{D_n\}_{n=0}^{\infty}$ is strictly log-convex.*

The little Schröer number s_n is defined by

$$s_n = \frac{3(2n-1)}{n+1} s_{n-1} - \frac{n-2}{n+1} s_{n-2}, \qquad n \geqslant 2, \tag{37}$$

with $s_0 = 1$ and $s_1 = 1$; see [9, 17]. Došlić [4] , and Liu and Wang [14] proved the log-convexity of the sequence $\{s_n\}_{n=0}^{\infty}$. It is easy to see that $r_1 = 3$. Let $N_0 = 3$. We can check that (14) and (15) hold for $N_0 = 3$. The following corollary follows from Theorem 2 and the fact $\frac{s_3}{s_2} > \frac{s_2}{s_1} > \frac{s_1}{s_0}$.

Corollary 7. *The sequence $\{s_n\}_{n=0}^{\infty}$ is strictly log-convex.*

Let R_n be the number of the set of all tree-like polyhexes with $n+1$ hexagons. The sequence $\{R_n\}_{n=0}^{\infty}$ satisfies the recurrence

$$R_n = \frac{3(2n-1)}{n+1} R_{n-1} - \frac{5(n-2)}{n+1} R_{n-2}, \qquad n \geqslant 2, \tag{38}$$

with $R_0 = 1$ and $R_1 = 1$; see [11]. The sequence $\{R_n\}_{n=0}^{\infty}$ is the sequence A002212 in Sloane's database [16]. Liu and Wang [14] proved the log-convexity of the sequence $\{R_n\}_{n=0}^{\infty}$. Let $N_0 = 3$. Employing Theorem 2 and evaluating the values of R_2, R_3 and R_4, we can prove the following corollary.

Corollary 8. *The sequence* $\{R_n\}_{n=0}^{\infty}$ *is strictly log-convex.*

Let w_n be the number of walks on cubic lattice with n steps, starting and finishing on the $x-y$ plane and never going below it. The sequence $\{w_n\}_{n=0}^{\infty}$ has three-term recurrence relation

$$w_n = \frac{4(2n+1)}{n+2}w_{n-1} - \frac{12(n-1)}{n+2}w_{n-2}, \qquad n \geqslant 2, \tag{39}$$

with $w_0 = 1$ and $w_1 = 4$; see [10]. The sequence $\{w_n\}_{n=0}^{\infty}$ is the sequence A005572 in Sloane's database [16]. Liu and Wang [14] proved the log-convexity of the sequence $\{w_n\}_{n=0}^{\infty}$. Set $N_0 = 2$. The following corollary follows from Theorem 2 and the fact $\frac{w_{i+1}}{w_i} > \frac{w_i}{w_{i-1}}$ for $i = 1,\ 2$.

Corollary 9. *The sequence* $\{w_n\}_{n=0}^{\infty}$ *is strictly log-convex.*

Let F_n be defined by

$$F_n = \frac{4n^4 - n^3 - n^2 + 3n + 2}{n^4 + 2n^2 - 1}F_{n-1} - \frac{2n^3 - 5n^2 - n + 1}{2n^3 - 3n^2 + 2n}F_{n-2}, \qquad n \geqslant 2, \tag{40}$$

with $F_0 = 1$ and $F_1 = 1$. By (13), we find $r_1 = 42$. Set $N_0 = 42$. It is easy to check that (14) and (15) hold for $N_0 = 42$. We can also verify that $\frac{F_{i+1}}{F_i} > \frac{F_i}{F_{i-1}}$ for $3 \leqslant i \leqslant 42$. Hence, we can prove the following corollary.

Corollary 10. *The sequence* $\{F_n\}_{n=2}^{\infty}$ *is strictly log-convex.*

To conclude this paper, we remark that the method presented in this paper can be used to prove the log-convexity of some combinatorial sequences satisfied longer recurrence relations. The principle is the same.

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