Sparse Distance Sets in the Triangular Lattice

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Abstract

A planar point-set X in Euclidean plane is called a k-*distance set* if there are exactly k different distances among the points in X. The function $g(k)$ denotes the maximum number of points in the Euclidean plane that is a k-distance set. In 1996, Erdős and Fishburn conjectured that for $k \geqslant 7$, every $g(k)$-point subset of the plane that determines k different distances is similar to a subset of the triangular lattice. We believe that if $g(k)$ is an increasing function of k, then the conjecture is false. We present data supporting our claim and a method of construction that unifies known optimal point configurations for $k \geqslant 3$.

1 Introduction

For a planar point-set X, let $D(X)$ denotes the set of distances between all pairs of points in X. A planar point-set X in Euclidean plane is called a k-*distance set* if $|D(X)| = k$. Let $g(k)$ denote the maximum number of points in the Euclidean plane that is a k-distance set. Only six exact values are known where $g(1) = 3$, $g(2) = 5$, $g(3) = 7$, $g(4) = 9$, and $g(5) = 12$ are determined by Erdős and Fishburn [1], and $g(6) = 13$ is determined by Wei [2].

Let R_n denote the vertices of a regular n-gon, and R_n^+ denote R_n with its center included. Clearly, $g(k) \geqslant 2k + 1$ since R_{2k+1} is a k-distance set.

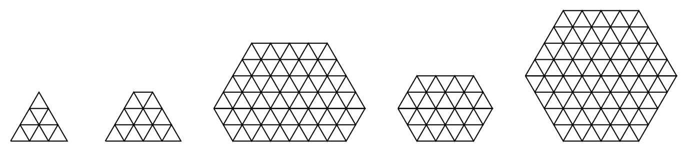
The triangular lattice is defined as

$$L_\Delta = \left\{ a(1,0) + b(1/2, \sqrt{3}/2) : a, b \in \mathbb{Z} \right\}.$$

Given positive integer a and non-negative integers r_1 and r_2 with $0 \leqslant r_2 \leqslant a+r_1-1$, let P_{a,r_1,r_2} denote a $\left(\binom{r_1}{2} - \binom{r_2}{2} + a(r_1+r_2) + r_1 - r_2 + r_1 r_2 + a\right)$-point vertically symmetric contiguous and convex subset of L_Δ where rows from top to bottom contain

$$a, a+1, a+2, \ldots, a+(r_1-1), a+r_1, a+r_1-1, a+r_1-2, \ldots, a+r_1-r_2$$

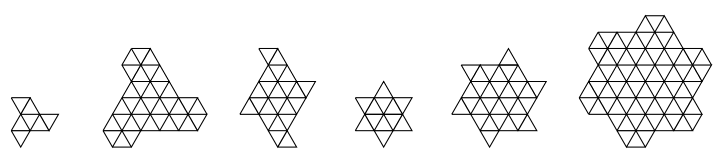
points, respectively.



$P_{1,3,0}$ $P_{2,3,0}$ $P_{5,4,2}$ $P_{4,2,2}$ $P_{5,4,4}$

Table 1: Examples of P_{a,r_1,r_2}

Let a saw-blade $B_{s,r,t}$ be an r-regular ($r \geqslant 2$) s-sided ($s \in \{3,4,6\}$) polygonal array in L_Δ with $t \leqslant r-1$ triangular blades attached on the first $t+1$ points on each side of the polygonal array. There are $\binom{r}{2} + r + 3t$, $r^2 + 4t$, and $6\binom{r}{2} + 6t + 1$ points in $B_{3,r,t}$, $B_{4,r,t}$, and $B_{6,r,t}$, respectively.



$B_{3,3,1}$ $B_{3,6,2}$ $B_{4,4,1}$ $B_{6,2,1}$ $B_{6,3,1}$ $B_{6,4,2}$

Table 2: Examples of $B_{s,r,t}$

We may generalize the notion of saw-blades and arbitrarily add triangular blades on the sides of P_{a,r_1,r_2}. Typically the tips of consecutive triangular blades are connected by straight lines of unit length. There are 2^x possible orientations with $x = 2r_1 + 2r_2 + (a-1) + (a+r_1-r_2-1) = 2a + 3r_1 + r_2 - 2$ for P_{a,r_1,r_2}.

Let us call a distance-set *sparse* if $n/k \geqslant 2.0$, and *strictly sparse* if $n/k > 2.0$. In this paper, we provide examples of sparse distance sets in L_Δ and also present lower bounds of $g(k)$ for $k \leqslant 50$. We also investigate the following conjecture of Erdős and Fishburn.

Conjecture 1 (Erdős and Fishburn [1])**.** For $k \geqslant 7$, every $g(k)$-point subset of the plane that determines k different distances is similar to a subset of L_Δ.

2 Computational results

We can observe the following about computing distance between two points in L_Δ.

Observation 1. Let two points in L_Δ be

$$x \equiv \left((2a_1+b_1)\frac{1}{2}, b_1\frac{\sqrt{3}}{2}\right) \text{ and } y \equiv \left((2a_2+b_2)\frac{1}{2}, b_2\frac{\sqrt{3}}{2}\right),$$

where $a_1, b_1, a_2, b_2 \in \mathbb{Z}$. Then the square of distance between x and y,

$$d^2(x,y) = (a_1-a_2)^2 + (a_1-a_2)(b_1-b_2) + (b_1-b_2)^2$$

involves only integer operations.

Let for $X \subset L_\Delta$ on the plane, $D^2(X)$ denote $\{d^2 : d \in D(X)\}$ and let $D^2_i(X)$ denote the i-th value in the increasing sequence with the values from $D^2(X)$.

Let $a_1 = b_1 = 0$. Then following is a list of the first few values of $D^2(L_\Delta)$ (in each case, d^2 is accompanied by a corresponding pair (a_2, b_2)):

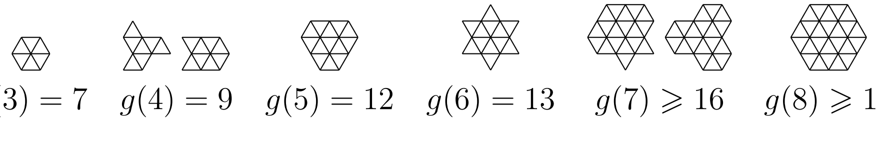
1:	1 (0, 1)	11:	25 (0, 5)	21:	52 (2, 6)	31:	81 (0, 9)	41:	112 (4, 8)
2:	3 (1, 1)	12:	27 (3, 3)	22:	57 (1, 7)	32:	84 (2, 8)	42:	117 (3, 9)
3:	4 (0, 2)	13:	28 (2, 4)	23:	61 (4, 5)	33:	91 (1, 9)	43:	121 (0,11)
4:	7 (1, 2)	14:	31 (1, 5)	24:	63 (3, 6)	34:	93 (4, 7)	44:	124 (2,10)
5:	9 (0, 3)	15:	36 (0, 6)	25:	64 (0, 8)	35:	97 (3, 8)	45:	127 (6, 7)
6:	12 (2, 2)	16:	37 (3, 4)	26:	67 (2, 7)	36:	100 (0,10)	46:	129 (5, 8)
7:	13 (1, 3)	17:	39 (2, 5)	27:	73 (1, 8)	37:	103 (2, 9)	47:	133 (1,11)
8:	16 (0, 4)	18:	43 (1, 6)	28:	75 (5, 5)	38:	108 (6, 6)	48:	139 (3,10)
9:	19 (2, 3)	19:	48 (4, 4)	29:	76 (4, 6)	39:	109 (5, 7)	49:	144 (0,12)
10:	21 (1, 4)	20:	49 (0, 7)	30:	79 (3, 7)	40:	111 (1,10)	50:	147 (2,11)

Let for $X \subset L_\Delta$, $c^2_k(X)$ denote the number of distinct pairs (x, y) in X such that $d^2(x,y) = k$. Let $c^2(X)$ denote the sequence $c^2_i(X)$ for $i = 1, 2, 3, \ldots$

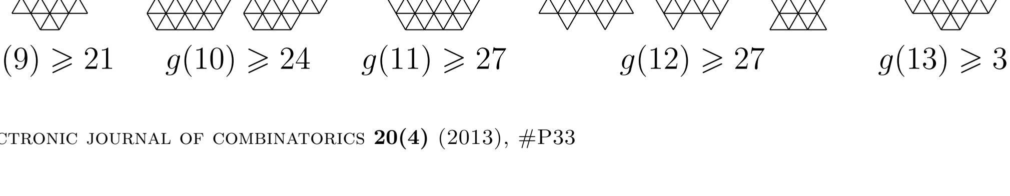
For example, $c^2(P_{1,2,0}) = \langle 9,0,3,3 \rangle$; $c^2(P_{2,1,1}) = \langle 12,0,6,3 \rangle$.

2.1 Verifying lower bounds of $g(k)$ for $3 \leqslant k \leqslant 13$

Let $L^{(n)}_\Delta$ denote a $(n^2 - n/2)$-point rectangular subset of n consecutive rows in L_Δ, with rows having $n, n-1, n, n-1, \ldots$ points, respectively. In this section, we verify the lower bounds of $g(k)$ for $3 \leqslant k \leqslant 13$ by exhaustive search in $L^{(8)}_\Delta$.



$g(3) = 7$ $g(4) = 9$ $g(5) = 12$ $g(6) = 13$ $g(7) \geqslant 16$ $g(8) \geqslant 19$

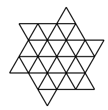


$g(9) \geqslant 21$ $g(10) \geqslant 24$ $g(11) \geqslant 27$ $g(12) \geqslant 27$ $g(13) \geqslant 31$

It can be noted that the example of $g(10) \geqslant 25$ in Figure 4 in Erdős and Fishburn [1] is actually an example of $g(11) \geqslant 25$, which can be explained using the example $P_{3,2,2}$ that we have used for $g(8) \geqslant 19$. For $P_{3,2,2}$, we have

$$D^2(P_{3,2,2}) = \{1, 3, 4, 7, 9, 12, 13, 16\}\,.$$

Let Y be the set of points when six points corresponding to the blades are added to $P_{3,2,2}$, as in Figure 4 in Erdős and Fishburn [1].



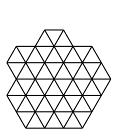
Then we have

$$D^2(Y) = \{1, 3, 4, 7, 9, 12, 13, 16, 19, 21, 28\}\,.$$

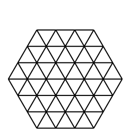
2.2 Lower bounds of $g(k)$ for $14 \leqslant k \leqslant 50$

In this section, we provide some new lower bounds of $g(k)$ for $14 \leqslant k \leqslant 50$. The lower bounds for $k = 15, 23, 34$, and 46 were previously known (Erdős and Fishburn [1]). We have considered all possible blade-orientations on the sides of P_{a,r_1,r_2} for $1 \leqslant a \leqslant 10$, $1 \leqslant r_1 \leqslant 10$, and $0 \leqslant r_2 \leqslant a + r_1 - r_2$.

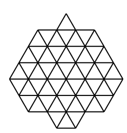
We have also conducted exhaustive searches on the $(n^2 - n/2)$-point rectangular subset of L_Δ for improving lower bounds of smaller values of k, say $k = 16$. Apparently in L_Δ, the bladed P_{a,r_1,r_2} offer the best lower bounds. In the figures below, every two points at a distance one are joined by a line.



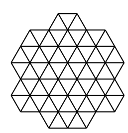
$g(14) \geqslant 34$



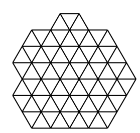
$g(15) \geqslant 37$



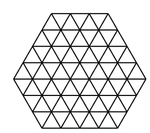
$g(16) \geqslant 36$



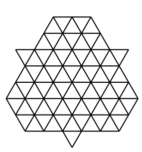
$g(17) \geqslant 42$



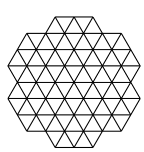
$g(18) \geqslant 45$



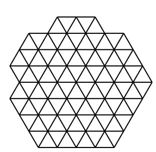
$g(19) \geqslant 48$



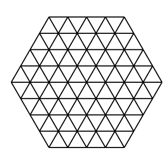
$g(20) \geqslant 49$



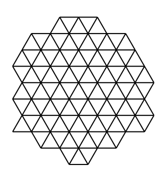
$g(21) \geqslant 55$



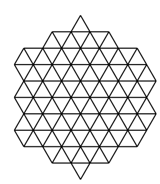
$g(22) \geqslant 58$



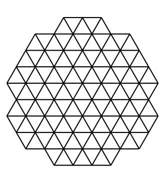
$g(23) \geqslant 61$



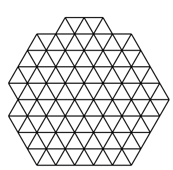
$g(24) \geqslant 63$



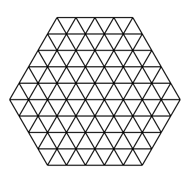
$g(25) \geqslant 64$



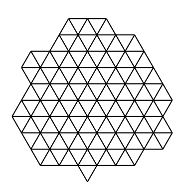
$g(26) \geqslant 69$



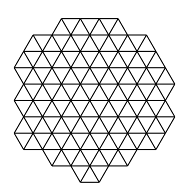
$g(27) \geqslant 72$



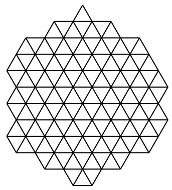
$g(28) \geqslant 75$



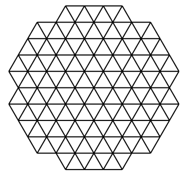
$g(29) \geqslant 73$



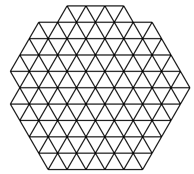
$g(30) \geqslant 79$



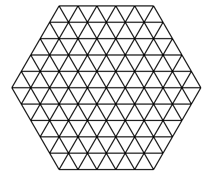
$g(31) \geqslant 80$



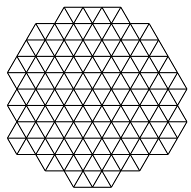
$g(32) \geqslant 85$



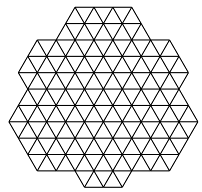
$g(33) \geqslant 88$



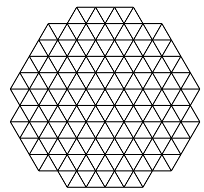
$g(34) \geqslant 91$



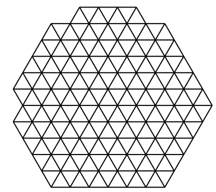
$g(35) \geqslant 96$



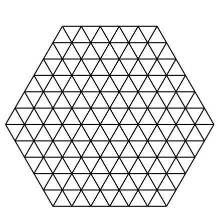
$g(36) \geqslant 96$



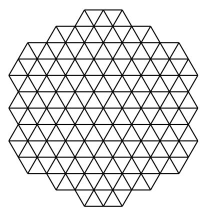
$g(37) \geqslant 102$



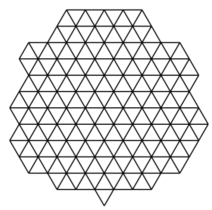
$g(38) \geqslant 105$



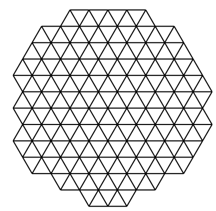
$g(39) \geqslant 108$



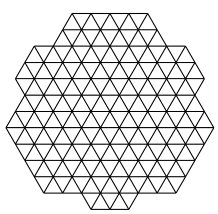
$g(40) \geqslant 109$



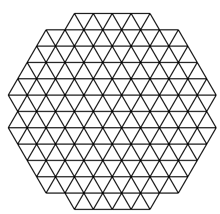
$g(41) \geqslant 109$



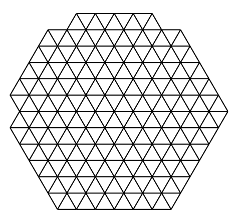
$g(42) \geqslant 115$



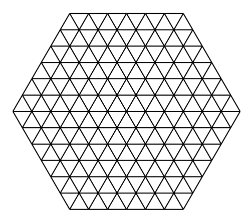
$g(43) \geqslant 115$



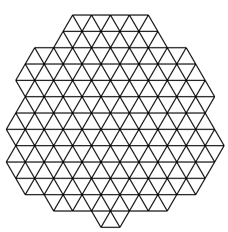
$g(44) \geqslant 121$



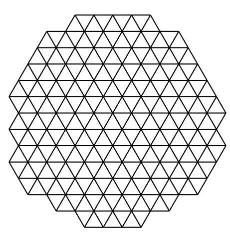
$g(45) \geqslant 124$



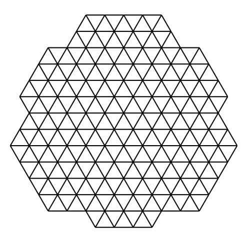
$g(46) \geqslant 127$



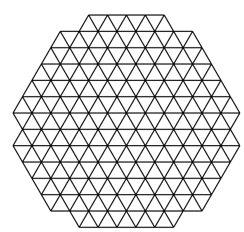
$g(47) \geqslant 129$



$g(48) \geqslant 135$



$g(49) \geqslant 135$



$g(50) \geqslant 141$

3 On properties of P_{a,r_1,r_2} and Bladed P_{a,r_1,r_2}

Let for $X \subset L_\Delta$, $b(X)$ denote the set of points in L_Δ, containing the points on the boundary of X. Let $i(X)$ denote the set of points in L_Δ, interior to X. Clearly, $|X| = b(X) + i(X)$.

Conjecture 2. The number of distinct distances between all pairs of points in P_{a,r_1,r_2} can be determined by the boundary points of P_{a,r_1,r_2} alone. (For supporting data, see Appendix E)

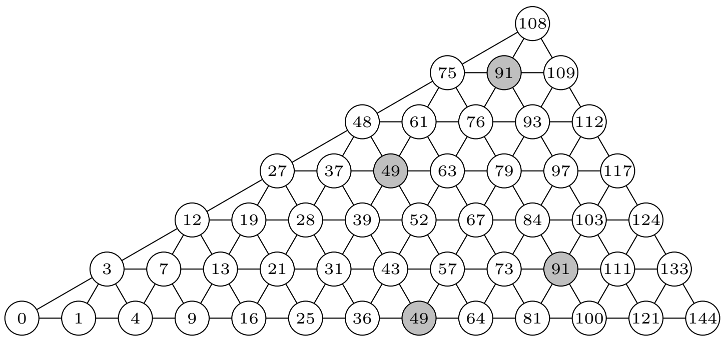
Consider the sequences X and Y, and let j be the least index such that $X_j \neq Y_j$. We say $X > Y$ if $X_j > Y_j$.

Conjecture 3. Given point sets X and Y in L_Δ with $|X| = |Y|$, if $c^2(X) > c^2(Y)$, then $i(X) \geqslant i(Y)$. (Data in Appendix D support this conjecture)

Consider the triangular array with points from $P_{r+1,r,0}$ on the boundary or interior to the triangle, say $T_{r+1,r,0}$, with vertices

$$(0,0), (2r,0), \text{ and } (3r/2, (\sqrt{3}/2)r).$$

Let $p(i,j)$ denote the j-th point on the i-th row inside $T_{r+1,r,0}$, considering the base of $T_{r+1,r,0}$ as the 0-th row. Let $f(i,j)$ be the square of distance from $(0,0)$ to $p(i,j)$. For example, in $T_{7,6,0}$ in the following figure, $f(2,5) = 52$.



It can be observed that the number of points in $T_{r+1,r,0}$ is $r^2 + 2r$. Then for $0 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant (2r+1) - 2i$,

$$f(i,j) \quad = \quad \left(\frac{3}{2}i + j - 1\right)^2 + \left(\frac{\sqrt{3}}{2}i\right)^2.$$

Let $r(i,j)$ be the number of times $f(i,j)$ is repeated in $T_{r+1,r,0}$ above $p(i,j)$. Hence for $r \geqslant 1$,

$$|D(P_{r+1,r,r})| = |D(T_{r+1,r,0})| = r^2 + 2r - \sum_{i=0}^{r} \sum_{j=1}^{2r+1-2i} [r(i,j) > 0].$$

Let $\ell_\theta(i,j)$ be the straight line passing through $p(i,j)$ making an angle $\theta \in \{0, \pi/3, 2\pi/3\}$ with the positive x-axis. Let $d_{min}(\theta, i, j, k)$ be the smallest square of distance of any point in $T_{r+1,r,0}$, on the line $\ell_\theta(i+k, j-k)$ from $(0,0)$.

Conjecture 4.

$$r(i,j) = \sum_{\substack{k \geqslant 1, j+k \leqslant 2r+1-2i, \\ d_{min}(2\pi/3, i,j,k) \leqslant f(i,j)}} \left[t(i,j,k) \in \mathbb{Z}^+\right],$$

where $t(i,j,k)$ is a solution to

$$f(i+k+(t-1), j-k-2(t-1)) - f(i,j) = 0.$$

4 Questions

Our data along with Erdős-Fishburn's results and Wei's raise the following questions:

1. Is $g(k)$ an increasing function, i.e., are our configurations for $k = 16$ and $k = 29$ optimal?

2. Is Conjecture 1 false? This can happen if the answer to Question 1 above is true, but our configuration for $k = 16$ is optimal in L_Δ. In that case, we need $g(16) \geqslant 37$ in $\mathbb{R}^2$.

3. For $k \geqslant 3$, is there always a bladed-P_{a,r_1,r_2} configuration that is optimal? So far this seems to be the case. Are they always optimal in L_Δ?

4. Is $D^2_k(L_\Delta)$ a good approximation of $g(k)$? The $D^2(L_\Delta)$-table and the lower bounds in Section 2, and the table in Appendix C say this might be the case.

5. Does the equivalent of Conjecture 2 for optimal $g(k)$ configurations hold true? See Appendix D for supporting data.

Acknowledgements

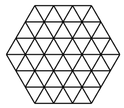
The authors would like to thank the Editor and the anonymous referee for their helpful comments and suggestions.

References

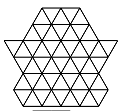
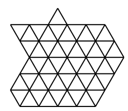
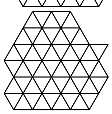
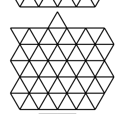
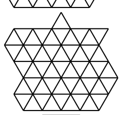
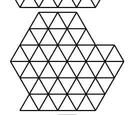
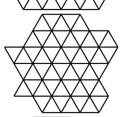
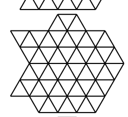
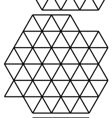
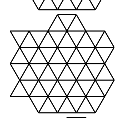
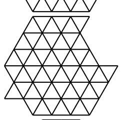
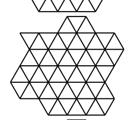
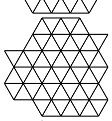
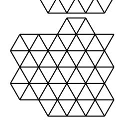
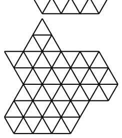
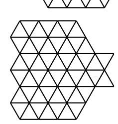
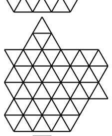
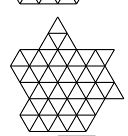
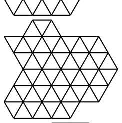
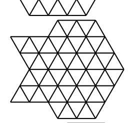
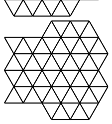
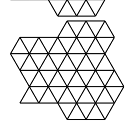
[1] P. Erdős and P. Fishburn, Maximum planar sets that determine k distances, *Discrete Math.*, **160** (1996), 115–125.

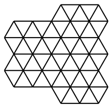
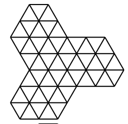
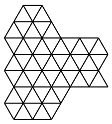
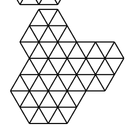
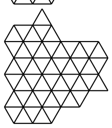
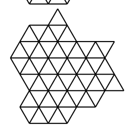
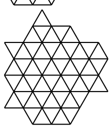
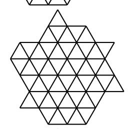
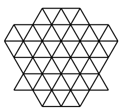
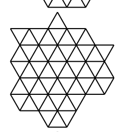
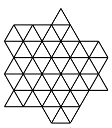
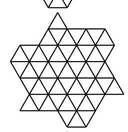
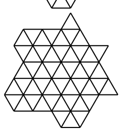
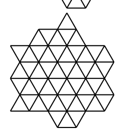
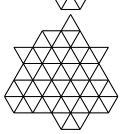
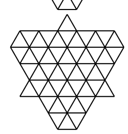
[2] X. Wei, A proof of Erdős-Fishburn's conjecture for $g(6) = 13$, *Elec. J. of Comb.*, **19(4)** (2012), #P38.

A 15-distance sets with 37-points in $L_{\Delta}^{(8)}$

X	$c^2(X)$
	$\langle 90, 0, 72, 69, 0, 0, 108, 0, 48, 0, 0, 39, 72, 0, 0, 27, 0, 0, 48, 0, 36, 0, 0, 0, 12, 0, 12, 18, 0, 0, 12, 0, 0, 0, 0, 3 \rangle$

B 16-distance sets with 36-points in $L_{\Delta}^{(8)}$

X	$c^2(X)$	X	$c^2(X)$
	$\langle 85, 0, 69, 64, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 27, 0, 0, 42, 0, 36, 0, 0, 0, 12, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 3, 4 \rangle$		$\langle 85, 0, 68, 64, 0, 0, 102, 0, 45, 0, 0, 36, 66, 0, 0, 28, 0, 0, 42, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 0, 6, 0, 4 \rangle$
	$\langle 86, 0, 70, 65, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 27, 0, 0, 42, 0, 36, 0, 0, 0, 12, 0, 9, 17, 0, 0, 11, 0, 0, 0, 0, 2, 2 \rangle$		$\langle 85, 0, 70, 64, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 28, 0, 0, 41, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 0, 5, 0, 2 \rangle$
	$\langle 85, 0, 69, 64, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 28, 0, 0, 41, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 0, 5, 0, 3 \rangle$		$\langle 86, 0, 69, 65, 0, 0, 103, 0, 45, 0, 0, 37, 67, 0, 0, 26, 0, 0, 44, 0, 34, 0, 0, 0, 11, 0, 10, 17, 0, 0, 11, 0, 0, 0, 0, 3, 2 \rangle$
	$\langle 85, 0, 70, 64, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 27, 0, 0, 42, 0, 36, 0, 0, 0, 12, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 2, 4 \rangle$		$\langle 85, 0, 69, 64, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 0, 4, 0, 3 \rangle$
	$\langle 86, 0, 71, 65, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 27, 0, 0, 42, 0, 36, 0, 0, 0, 12, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 1, 2 \rangle$		$\langle 85, 0, 71, 64, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 0, 4, 0, 1 \rangle$
	$\langle 85, 0, 69, 65, 0, 0, 102, 0, 46, 0, 0, 36, 68, 0, 0, 25, 0, 0, 44, 0, 34, 0, 0, 0, 12, 0, 11, 16, 0, 0, 12, 0, 0, 0, 0, 3, 0, 0, 2 \rangle$		$\langle 85, 0, 70, 63, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 27, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 18, 0, 0, 10, 0, 0, 0, 0, 0, 4, 0, 2 \rangle$
	$\langle 85, 0, 71, 64, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 27, 0, 0, 42, 0, 36, 0, 0, 0, 12, 0, 9, 17, 0, 0, 11, 0, 0, 0, 0, 1, 4 \rangle$		$\langle 85, 0, 71, 63, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 27, 0, 0, 41, 0, 36, 0, 0, 0, 13, 0, 9, 18, 0, 0, 10, 0, 0, 0, 0, 0, 5, 0, 1 \rangle$
	$\langle 84, 0, 67, 62, 0, 0, 101, 0, 43, 0, 0, 36, 65, 0, 0, 26, 0, 0, 43, 0, 36, 0, 0, 0, 12, 0, 11, 20, 0, 0, 12, 0, 0, 0, 0, 0, 7, 0, 5 \rangle$		$\langle 85, 0, 70, 64, 0, 0, 102, 0, 45, 0, 0, 37, 66, 0, 0, 26, 0, 0, 44, 0, 34, 0, 0, 0, 11, 0, 10, 18, 0, 0, 12, 0, 0, 0, 0, 2, 4 \rangle$
	$\langle 84, 0, 69, 63, 0, 0, 102, 0, 45, 0, 0, 36, 66, 0, 0, 27, 0, 0, 42, 0, 36, 0, 0, 0, 12, 0, 9, 18, 0, 0, 12, 0, 0, 0, 0, 0, 6, 0, 3 \rangle$		$\langle 84, 0, 68, 63, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 27, 0, 0, 40, 0, 36, 0, 0, 0, 12, 0, 9, 18, 0, 0, 12, 0, 0, 0, 0, 0, 4, 0, 4 \rangle$
	$\langle 85, 0, 69, 63, 0, 0, 102, 0, 43, 0, 0, 36, 66, 0, 0, 27, 0, 0, 42, 0, 36, 0, 0, 0, 13, 0, 11, 18, 0, 0, 10, 0, 0, 0, 0, 0, 6, 0, 3 \rangle$		$\langle 85, 0, 68, 64, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 0, 4, 0, 4 \rangle$
	$\langle 85, 0, 70, 64, 0, 0, 104, 0, 43, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 11, 16, 0, 0, 10, 0, 0, 0, 0, 0, 4, 0, 2 \rangle$		$\langle 85, 0, 69, 63, 0, 0, 103, 0, 43, 0, 0, 36, 67, 0, 0, 27, 0, 0, 41, 0, 36, 0, 0, 0, 13, 0, 11, 18, 0, 0, 10, 0, 0, 0, 0, 0, 5, 0, 3 \rangle$

	⟨85, 0, 70, 62, 0, 0, 102, 0, 43, 0, 0, 36, 66, 0, 0, 26, 0, 0, 42, 0, 36, 0, 0, 0, 13, 0, 11, 20, 0, 0, 10, 0, 0, 0, 0, 0, 6, 0, 2⟩		⟨84, 0, 66, 60, 0, 0, 96, 0, 39, 0, 0, 36, 60, 0, 0, 24, 0, 0, 48, 0, 36, 0, 0, 0, 12, 0, 15, 24, 0, 0, 12, 0, 0, 0, 0, 0, 12, 0, 6⟩
	⟨84, 0, 68, 61, 0, 0, 99, 0, 41, 0, 0, 36, 63, 0, 0, 25, 0, 0, 45, 0, 36, 0, 0, 0, 12, 0, 13, 22, 0, 0, 12, 0, 0, 0, 0, 0, 9, 0, 4⟩		⟨85, 0, 68, 62, 0, 0, 100, 0, 41, 0, 0, 36, 64, 0, 0, 26, 0, 0, 44, 0, 36, 0, 0, 0, 13, 0, 13, 20, 0, 0, 10, 0, 0, 0, 0, 0, 8, 0, 4⟩
	⟨84, 0, 70, 62, 0, 0, 101, 0, 43, 0, 0, 36, 65, 0, 0, 26, 0, 0, 43, 0, 36, 0, 0, 0, 12, 0, 11, 20, 0, 0, 12, 0, 0, 0, 0, 0, 7, 0, 2⟩		⟨84, 0, 69, 63, 0, 0, 103, 0, 43, 0, 0, 36, 67, 0, 0, 27, 0, 0, 41, 0, 36, 0, 0, 0, 12, 0, 11, 18, 0, 0, 12, 0, 0, 0, 0, 0, 5, 0, 3⟩
	⟨84, 0, 71, 64, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 28, 0, 0, 41, 0, 36, 0, 0, 0, 12, 0, 9, 16, 0, 0, 12, 0, 0, 0, 0, 0, 5, 0, 1⟩		⟨84, 0, 70, 64, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 12, 0, 9, 16, 0, 0, 12, 0, 0, 0, 0, 0, 4, 0, 2⟩
	⟨85, 0, 71, 64, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 1, 4⟩		⟨84, 0, 71, 63, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 17, 0, 0, 11, 0, 0, 0, 0, 1, 6⟩
	⟨84, 0, 70, 63, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 2, 6⟩		⟨84, 0, 69, 63, 0, 0, 105, 0, 45, 0, 0, 36, 69, 0, 0, 27, 0, 0, 39, 0, 36, 0, 0, 0, 12, 0, 9, 18, 0, 0, 12, 0, 0, 0, 0, 0, 3, 0, 3⟩
	⟨84, 0, 69, 63, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 28, 0, 0, 40, 0, 36, 0, 0, 0, 13, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 3, 6⟩		⟨84, 0, 71, 63, 0, 0, 104, 0, 45, 0, 0, 36, 68, 0, 0, 29, 0, 0, 38, 0, 36, 0, 0, 0, 14, 0, 9, 16, 0, 0, 10, 0, 0, 0, 0, 1, 6⟩
	⟨84, 0, 70, 63, 0, 0, 103, 0, 45, 0, 0, 36, 67, 0, 0, 29, 0, 0, 38, 0, 36, 0, 0, 0, 14, 0, 9, 17, 0, 0, 11, 0, 0, 0, 0, 2, 6⟩		⟨84, 0, 69, 63, 0, 0, 102, 0, 45, 0, 0, 36, 66, 0, 0, 30, 0, 0, 36, 0, 36, 0, 0, 0, 15, 0, 9, 18, 0, 0, 12, 0, 0, 0, 0, 3, 6⟩

C An approximation for $g(k)$

We present lower bounds for $g(k)$ with r-regular hexagonal arrays $P_{r,r-1,r-1}$ (with $n = 6\binom{r}{2} + 1$ points) for $2 \leqslant r \leqslant 57$, and also compare with the corresponding values of $D_k^2(L_\Delta)$.

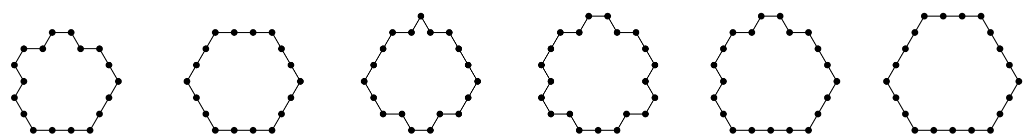
r	n	k	$D_k^2(L_\Delta)$	$\lvert n - D_k^2(L_\Delta)\rvert$	r	n	k	$D_k^2(L_\Delta)$	$\lvert n - D_k^2(L_\Delta)\rvert$
2	7	3	4	3	3	19	8	16	3
4	37	15	36	1	5	61	23	61	0
6	91	34	93	2	7	127	46	129	2
8	169	59	175	6	9	217	74	228	11
10	271	90	289	18	11	331	109	351	20
12	397	129	432	35	13	469	150	508	39
14	547	172	592	45	15	631	196	684	53
16	721	222	784	63	17	817	249	889	72
18	919	277	999	80	19	1027	308	1123	96
20	1141	339	1252	111	21	1261	372	1393	132
22	1387	405	1531	144	23	1519	440	1669	150

24	1657	477	1821	164	25	1801	516	1987	186
26	1951	557	2164	213	27	2107	598	2331	224
28	2269	639	2511	242	29	2437	685	2707	270
30	2611	731	2908	297	31	2791	776	3099	308
32	2977	822	3303	326	33	3169	871	3508	339
34	3367	925	3747	380	35	3571	976	3969	398
36	3781	1031	4225	444	37	3997	1088	4476	479
38	4219	1139	4699	480	39	4447	1197	4963	516
40	4681	1257	5223	542	41	4921	1315	5488	567
42	5167	1376	5776	609	43	5419	1440	6064	645
44	5677	1504	6352	675	45	5941	1569	6643	702
46	6211	1635	6933	722	47	6487	1706	7267	780
48	6769	1777	7588	819	49	7057	1844	7903	846
50	7351	1914	8221	870	51	7651	1992	8587	936

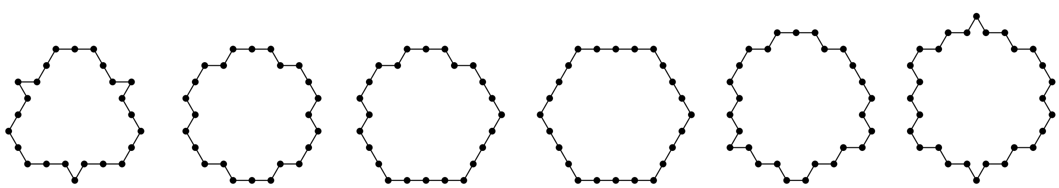
We have corrected some errors in computation in Erdős and Fishburn [1] of the number of distinct distances in $P_{r,r-1,r-1}$ for $14 \leqslant r \leqslant 21$.

D Examples of $|D(X)| = |D(B(X))|$

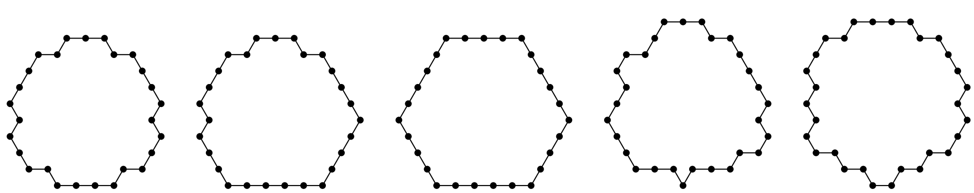
In this section, for $14 \leqslant k \leqslant 50$ (we have verified for $3 \leqslant k \leqslant 13$ as well), we use the point set X that we used in Section 2.2 to prove $g(k) \geqslant |X|$.



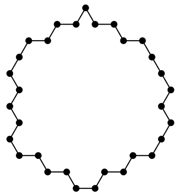
$g(14) \geqslant 34 = |X|$, $|B(X)| = 18$, $|D(B(X))| = 14$ — $g(15) \geqslant 37 = |X|$, $|B(X)| = 18$, $|D(B(X))| = 15$ — $g(16) \geqslant 36 = |X|$, $|B(X)| = 19$, $|D(B(X))| = 16$ — $g(17) \geqslant 42 = |X|$, $|B(X)| = 21$, $|D(B(X))| = 17$ — $g(18) \geqslant 45 = |X|$, $|B(X)| = 21$, $|D(B(X))| = 18$ — $g(19) \geqslant 48 = |X|$, $|B(X)| = 21$, $|D(B(X))| = 19$



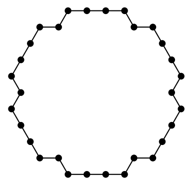
$g(20) \geqslant 49 = |X|$, $|B(X)| = 24$, $|D(B(X))| = 20$ — $g(21) \geqslant 55 = |X|$, $|B(X)| = 24$, $|D(B(X))| = 21$ — $g(22) \geqslant 58 = |X|$, $|B(X)| = 24$, $|D(B(X))| = 22$ — $g(23) \geqslant 61 = |X|$, $|B(X)| = 24$, $|D(B(X))| = 23$ — $g(24) \geqslant 63 = |X|$, $|B(X)| = 27$, $|D(B(X))| = 24$ — $g(25) \geqslant 64 = |X|$, $|B(X)| = 28$, $|D(B(X))| = 25$



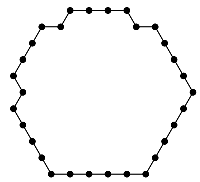
$g(26) \geqslant 69 = |X|$, $|B(X)| = 27$, $|D(B(X))| = 26$ — $g(27) \geqslant 72 = |X|$, $|B(X)| = 27$, $|D(B(X))| = 27$ — $g(28) \geqslant 75 = |X|$, $|B(X)| = 27$, $|D(B(X))| = 28$ — $g(29) \geqslant 73 = |X|$, $|B(X)| = 29$, $|D(B(X))| = 29$ — $g(30) \geqslant 79 = |X|$, $|B(X)| = 30$, $|D(B(X))| = 30$



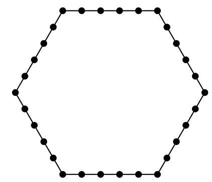
$g(31) \geqslant 80 = |X|$
$|B(X)| = 31$
$|D(B(X))| = 31$



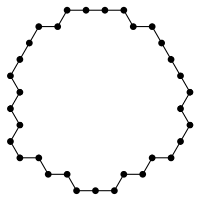
$g(32) \geqslant 85 = |X|$
$|B(X)| = 30$
$|D(B(X))| = 32$



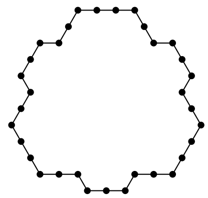
$g(33) \geqslant 88 = |X|$
$|B(X)| = 30$
$|D(B(X))| = 33$



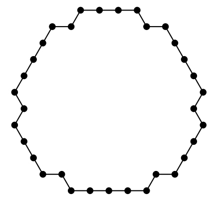
$g(34) \geqslant 91 = |X|$
$|B(X)| = 30$
$|D(B(X))| = 34$



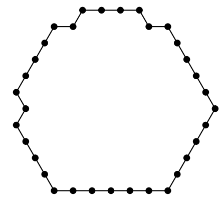
$g(35) \geqslant 96 = |X|$
$|B(X)| = 33$
$|D(B(X))| = 35$



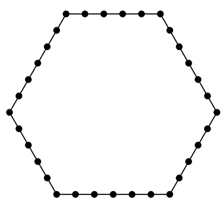
$g(36) \geqslant 96 = |X|$
$|B(X)| = 33$
$|D(B(X))| = 36$



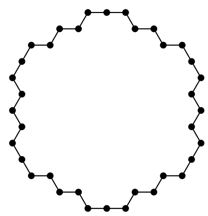
$g(37) \geqslant 102 = |X|$
$|B(X)| = 33$
$|D(B(X))| = 37$



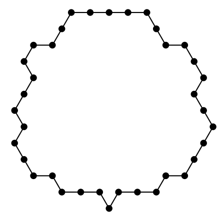
$g(38) \geqslant 105 = |X|$
$|B(X)| = 33$
$|D(B(X))| = 38$



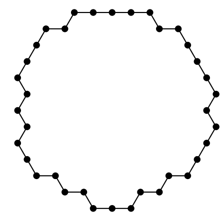
$g(39) \geqslant 108 = |X|$
$|B(X)| = 33$
$|D(B(X))| = 39$



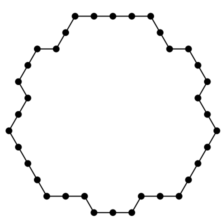
$g(40) \geqslant 109 = |X|$
$|B(X)| = 36$
$|D(B(X))| = 40$



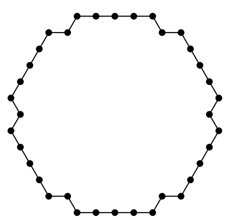
$g(41) \geqslant 109 = |X|$
$|B(X)| = 36$
$|D(B(X))| = 41$



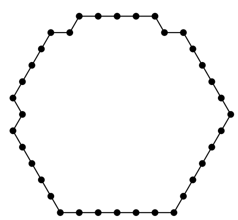
$g(42) \geqslant 115 = |X|$
$|B(X)| = 36$
$|D(B(X))| = 42$



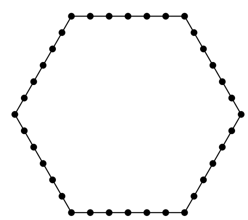
$g(43) \geqslant 115 = |X|$
$|B(X)| = 36$
$|D(B(X))| = 43$



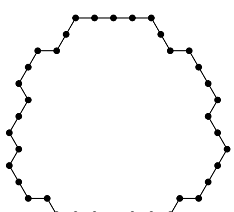
$g(44) \geqslant 121 = |X|$
$|B(X)| = 36$
$|D(B(X))| = 44$



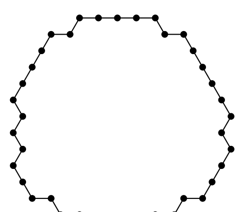
$g(45) \geqslant 124 = |X|$
$|B(X)| = 36$
$|D(B(X))| = 45$



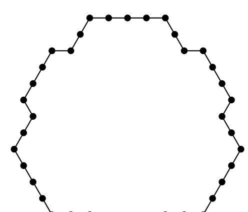
$g(46) \geqslant 127 = |X|$
$|B(X)| = 36$
$|D(B(X))| = 46$



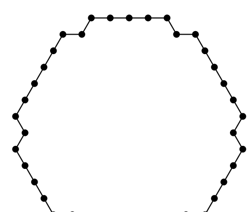
$g(47) \geqslant 129 = |X|$
$|B(X)| = 39$
$|D(B(X))| = 47$



$g(48) \geqslant 135 = |X|$
$|B(X)| = 39$
$|D(B(X))| = 48$



$g(49) \geqslant 135 = |X|$
$|B(X)| = 39$
$|D(B(X))| = 49$



$g(50) \geqslant 141 = |X|$
$|B(X)| = 39$
$|D(B(X))| = 50$

E Data supporting conjecture 2

X	$\|X\|$	$\|D(X)\|$	$\|B(X)\|$	$\|D(B(X))\|$	X	$\|X\|$	$\|D(X)\|$	$\|B(X)\|$	$\|D(B(X))\|$
$P_{1,1,0}$	3	1	3	1	$P_{1,1,1}$	4	2	4	2
$P_{1,2,0}$	6	3	6	3	$P_{1,2,1}$	8	4	7	4
$P_{1,2,2}$	9	5	8	5	$P_{1,3,0}$	10	5	9	5
$P_{1,3,1}$	13	7	10	7	$P_{1,3,2}$	15	8	11	8
$P_{1,3,3}$	16	9	12	9	$P_{1,4,0}$	15	8	12	8
$P_{1,4,1}$	19	10	13	10	$P_{1,4,2}$	22	12	14	12
$P_{1,4,3}$	24	13	15	13	$P_{1,4,4}$	25	14	16	14
$P_{2,0,1}$	3	1	3	1	$P_{2,1,0}$	5	3	5	3
$P_{2,1,1}$	7	3	6	3	$P_{2,1,2}$	8	4	7	4
$P_{2,2,0}$	9	5	8	5	$P_{2,2,1}$	12	5	9	5
$P_{2,2,2}$	14	7	10	7	$P_{2,2,3}$	15	8	11	8
$P_{2,3,0}$	14	8	11	8	$P_{2,3,1}$	18	8	12	8
$P_{2,3,2}$	21	10	13	10	$P_{2,3,3}$	23	12	14	12
$P_{2,3,4}$	24	13	15	13	$P_{2,4,0}$	20	11	14	11
$P_{2,4,1}$	25	11	15	11	$P_{2,4,2}$	29	14	16	14
$P_{2,4,3}$	32	16	17	16	$P_{2,4,4}$	34	18	18	18
$P_{2,4,5}$	35	19	19	19	$P_{3,0,1}$	5	3	5	3
$P_{3,0,2}$	6	3	6	3	$P_{3,1,0}$	7	5	7	5
$P_{3,1,1}$	10	5	8	5	$P_{3,1,2}$	12	5	9	5
$P_{3,1,3}$	13	7	10	7	$P_{3,2,0}$	12	8	10	8
$P_{3,2,1}$	16	8	11	8	$P_{3,2,2}$	19	8	12	8
$P_{3,2,3}$	21	10	13	10	$P_{3,2,4}$	22	12	14	12
$P_{3,3,0}$	18	11	13	11	$P_{3,3,1}$	23	11	14	11
$P_{3,3,2}$	27	11	15	11	$P_{3,3,3}$	30	14	16	14
$P_{3,3,4}$	32	16	17	16	$P_{3,3,5}$	33	18	18	18
$P_{3,4,0}$	25	15	16	15	$P_{3,4,1}$	31	15	17	15
$P_{3,4,2}$	36	15	18	15	$P_{3,4,3}$	40	18	19	18
$P_{3,4,4}$	43	21	20	21	$P_{3,4,5}$	45	23	21	23
$P_{3,4,6}$	46	25	22	25	$P_{4,0,1}$	7	5	7	5
$P_{4,0,2}$	9	5	8	5	$P_{4,0,3}$	10	5	9	5
$P_{4,1,0}$	9	7	9	7	$P_{4,1,1}$	13	8	10	8
$P_{4,1,2}$	16	8	11	8	$P_{4,1,3}$	18	8	12	8
$P_{4,1,4}$	19	10	13	10	$P_{4,2,0}$	15	11	12	11
$P_{4,2,1}$	20	11	13	11	$P_{4,2,2}$	24	11	14	11
$P_{4,2,3}$	27	11	15	11	$P_{4,2,4}$	29	14	16	14
$P_{4,2,5}$	30	16	17	16	$P_{4,3,0}$	22	15	15	15
$P_{4,3,1}$	28	15	16	15	$P_{4,3,2}$	33	15	17	15
$P_{4,3,3}$	37	15	18	15	$P_{4,3,4}$	40	18	19	18
$P_{4,3,5}$	42	21	20	21	$P_{4,3,6}$	43	23	21	23
$P_{4,4,0}$	30	19	18	19	$P_{4,4,1}$	37	19	19	19
$P_{4,4,2}$	43	19	20	19	$P_{4,4,3}$	48	19	21	19
$P_{4,4,4}$	52	22	22	22	$P_{4,4,5}$	55	25	23	25
$P_{4,4,6}$	57	28	24	28	$P_{4,4,7}$	58	30	25	30
$P_{5,0,1}$	9	7	9	7	$P_{5,0,2}$	12	8	10	8
$P_{5,0,3}$	14	8	11	8	$P_{5,0,4}$	15	8	12	8
$P_{5,1,0}$	11	9	11	9	$P_{5,1,1}$	16	11	12	11
$P_{5,1,2}$	20	11	13	11	$P_{5,1,3}$	23	11	14	11
$P_{5,1,4}$	25	11	15	11	$P_{5,1,5}$	26	14	16	14
$P_{5,2,0}$	18	14	14	14	$P_{5,2,1}$	24	15	15	15
$P_{5,2,2}$	29	15	16	15	$P_{5,2,3}$	33	15	17	15
$P_{5,2,4}$	36	15	18	15	$P_{5,2,5}$	38	18	19	18

$P_{5,2,6}$	39	21	20	21	$P_{5,3,0}$	26	19	17	19
$P_{5,3,1}$	33	19	18	19	$P_{5,3,2}$	39	19	19	19
$P_{5,3,3}$	44	19	20	19	$P_{5,3,4}$	48	19	21	19
$P_{5,3,5}$	51	22	22	22	$P_{5,3,6}$	53	25	23	25
$P_{5,3,7}$	54	28	24	28	$P_{5,4,0}$	35	23	20	23
$P_{5,4,1}$	43	23	21	23	$P_{5,4,2}$	50	23	22	23
$P_{5,4,3}$	56	23	23	23	$P_{5,4,4}$	61	23	24	23
$P_{5,4,5}$	65	27	25	27	$P_{5,4,6}$	68	31	26	31
$P_{5,4,7}$	70	34	27	34	$P_{5,4,8}$	71	37	28	37