The Terwilliger algebra of the incidence graphs of Johnson geometry

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Abstract

In 2007, Levstein and Maldonado computed the Terwilliger algebra of the Johnson graph $J(n,m)$ when $3m \leqslant n$. It is well known that the halved graphs of the incidence graph $J(n,m,m+1)$ of Johnson geometry are Johnson graphs. In this paper, we determine the Terwilliger algebra of $J(n,m,m+1)$ when $3m \leqslant n$, give two bases of this algebra, and calculate its dimension.

Keywords: Terwilliger algebra; Johnson graph; incidence graph; Johnson geometry

1 Introduction

Let $\Gamma = (X, R)$ denote a simple connected graph with the vertex set X and the edge set R. Suppose $\mathrm{Mat}_X(\mathbb{C})$ denotes the algebra over the complex number field $\mathbb{C}$ consisting of

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all matrices whose rows and columns are indexed by elements of X. For vertices x and y, $\partial(x,y)$ denotes the *distance* between x and y, i.e., the length of a shortest path connecting x and y. Fix a vertex $x \in X$. Let $D(x) = \max\{\partial(x,y) \mid y \in X\}$ denote the *diameter with respect to* x. For each $i \in \{0,1,\ldots,D(x)\}$, let $\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}$ and define $E_i^* = E_i^*(x)$ to be the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy-entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } y \in \Gamma_i(x), \\ 0, & \text{otherwise.} \end{cases}$$

The subalgebra $\mathcal{T} = \mathcal{T}(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by the adjacency matrix A of Γ and $E_0^*, E_1^*, \ldots, E_{D(x)}^*$ is called the *Terwilliger algebra* of Γ with respect to x. Let $\mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of all column vectors whose coordinates are indexed by X. A $\mathcal{T}$*-module* is any subspace W of $\mathbb{C}^X$ such that $\mathcal{T}W \subseteq W$. We call a nonzero $\mathcal{T}$-module *irreducible* if it does not properly contain a nonzero $\mathcal{T}$-module. An irreducible $\mathcal{T}$-module W is *thin* if $\dim E_i^* W \leqslant 1$ for every i, and the graph Γ is said to be *thin with respect to* x if every irreducible $\mathcal{T}(x)$-module is thin.

Terwilliger [13, 14, 15] initiated the study of the Terwilliger algebra of an association scheme, which has been an important tool in studying structures of an association scheme. For more information, see [4, 5, 6]. The Terwilliger algebras of group schemes were discussed in [1, 2]. The Terwilliger algebras of some distance-regular graphs have been determined; see [17] for strongly regular graphs, [8] for Hypercubes, [11] for Hamming graphs, [12] for Johnson graphs, [10] for odd graphs.

Let Ω be a set of cardinality n and let $\binom{\Omega}{i}$ denote the collection of all i-subsets of Ω. Suppose m is a nonnegative integer with $m+1 \leqslant n$. The *incidence graph* $J(n,m,m+1)$ *of Johnson geometry* is a bipartite graph with a bipartition $\binom{\Omega}{m} \cup \binom{\Omega}{m+1}$, where $y \in \binom{\Omega}{m}$ and $z \in \binom{\Omega}{m+1}$ are adjacent if $y \subseteq z$. The graph $J(n,m,m+1)$ is *distance-biregular* (see [3]). It is well known that the halved graphs of $J(n,m,m+1)$ are Johnson graphs.

Levstein and Maldonado [12] determined the Terwilliger algebra of the Johnson graph $J(n,m)$ when $3m \leqslant n$. In this paper we shall determine the Terwilliger algebra of $J(n,m,m+1)$ with respect to $x \in \binom{\Omega}{m}$ when $n \geqslant 3m$. In Section 2, we introduce some useful identities for intersection matrices. In Section 3, the Terwilliger algebra of $J(n,m,m+1)$ is described. In Section 4, we give two bases of this algebra and compute its dimension.

2 Intersection matrices

In this section we shall introduce intersection matrices and some related identities.

Let V be a set of cardinality v. The *inclusion matrix* $W_{i,j}(v)$ is a binary matrix whose rows and columns are indexed by elements of $\binom{V}{i}$ and $\binom{V}{j}$, respectively, with the yz-entry defined by

$$(W_{i,j}(v))_{yz} = \begin{cases} 1, & \text{if } y \subseteq z, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$W_{i,j}(v)W_{j,k}(v) = \binom{k-i}{j-i} W_{i,k}(v). \tag{1}$$

Let $H^l_{i,j}(v)$ be a binary matrix whose rows and columns are indexed by elements of $\binom{V}{i}$ and $\binom{V}{j}$, respectively, and the yz-entry is defined by

$$(H^l_{i,j}(v))_{yz} = \begin{cases} 1, & \text{if } |y \cap z| = l, \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$C^l_{i,j}(v) = \sum_{g=l}^{\min(i,j)} \binom{g}{l} H^g_{i,j}(v). \tag{2}$$

In order to simply the notation, we write $W_{i,j}$ for $W_{i,j}(v)$ when v is clear from context, and do the same for $H^l_{i,j}(v)$ and $C^l_{i,j}(v)$. The matrices $W_{i,j}, H^l_{i,j}$ and $C^l_{i,j}$ are *intersection matrices* introduced in [7].

Observe $C^0_{i,j}$ is the all-one matrix and

$$C^{\min(i,j)}_{i,j} = \begin{cases} W^{\mathrm{T}}_{j,i}, & \text{if } i > j, \\ W_{i,j}, & \text{otherwise.} \end{cases}$$

Lemma 2.1 ([7]) *Let V be a set of cardinality v. Write $W_{i,j} = W_{i,j}(v)$ and $C^l_{i,j} = C^l_{i,j}(v)$. Then*

$$C^l_{i,j} C^s_{j,k} = \sum_{h=\max(0,l+s-j)}^{\min(l,s)} \binom{v-l-s}{j-l-s+h} \binom{i-h}{l-h} \binom{k-h}{s-h} C^h_{i,k}.$$

In particular, the following hold:

(i) $W^{\mathrm{T}}_{i,j} W_{i,k} = C^i_{j,k}$;

(ii) $C^l_{i,j} W_{j,k} = \binom{k-l}{j-l} C^l_{i,k}$;

(iii) $W_{i,k} W^{\mathrm{T}}_{j,k} = \sum_{l=\max(0,i+j-k)}^{\min(i,j)} \binom{v-i-j}{k-i-j+l} C^l_{i,j}$;

(iv) $W_{i,j} C^l_{j,k} = \sum_{h=\max(0,l+j-i)}^{\min(l,i)} \binom{v-l-i}{j-l-i+h} \binom{k-h}{l-h} C^h_{i,k}$.

Fix $x \in \binom{\Omega}{m}$. We then consider the adjacency matrix A of $J(n, m, m+1)$ as a block matrix with respect to the partition $\{x\} \cup \Gamma_1(x) \cup \cdots \cup \Gamma_{D(x)}(x)$. Let $A_{i,j}$ be the submatrix of A with rows indexed by vertices of $\Gamma_i(x)$ and columns indexed by vertices of $\Gamma_j(x)$.

Lemma 2.2 *Given two vertices x, y of $J(n, m, m+1)$. If $x \in \binom{\Omega}{m}$, then*

$$\partial(x,y) = \begin{cases} 2i, & \text{if } |y| = m \text{ and } |x \cap y| = m-i, \\ 2i+1, & \text{if } |y| = m+1 \text{ and } |x \cap y| = m-i. \end{cases}$$

In particular, $D(x) = \min(2m+1, 2n-2m)$.

Proof. Immediate from [9, Lemma 2.2]. $\square$

Lemma 2.3 *Let $I_{\binom{v}{k}}$ be the identity matrix of size $\binom{v}{k}$. Then*

$$A_{i,j} = 0, \quad \text{if } \ 0 \leqslant i \leqslant j \leqslant D(x) \ \text{ and } \ i \neq j-1; \tag{3}$$

$$A_{2i,2i+1} = I_{\binom{m}{m-i}} \otimes W_{i,i+1}(n-m), \quad \text{if } \ 0 \leqslant i \leqslant \lfloor \frac{D(x)-1}{2} \rfloor; \tag{4}$$

$$A_{2i+1,2i+2} = W^{\mathrm{T}}_{m-i-1,m-i}(m) \otimes I_{\binom{n-m}{i+1}}, \quad \text{if } \ 0 \leqslant i \leqslant \lfloor \frac{D(x)}{2} \rfloor - 1, \tag{5}$$

where "$\otimes$" denotes the Kronecker product of matrices.

Proof. (3) is directed.

Pick $y \in \Gamma_{2i}(x)$, $z \in \Gamma_{2i+1}(x)$. By Lemma 2.2 we have $|y| = m$, $|z| = m+1$, $|x \cap y| = |x \cap z| = m - i$. Suppose $y = \alpha_{m-i} \cup \beta_i$, $z = \alpha'_{m-i} \cup \beta'_{i+1}$, where α_{m-i} and $\alpha'_{m-i} \in \binom{x}{m-i}$, while $\beta_i \in \binom{\Omega \setminus x}{i}$ and $\beta'_{i+1} \in \binom{\Omega \setminus x}{i+1}$. Then

$$(A_{2i,2i+1})_{yz} = 1 \Leftrightarrow \alpha_{m-i} = \alpha'_{m-i} \text{ and } \beta_i \subseteq \beta'_{i+1} \Leftrightarrow (I_{\binom{m}{m-i}} \otimes W_{i,i+1}(n-m))_{yz} = 1,$$

which leads to (4).

Similarly, (5) holds. $\square$

3 The Terwilliger algebra

Let $n \geqslant 3m$ and X denote the vertex set of $J(n,m,m+1)$. Fix $x \in \binom{\Omega}{m}$. In this section we shall determine the Terwilliger algebra $\mathcal{T} = \mathcal{T}(x)$ of $J(n,m,m+1)$. Hereafter the ground set of all matrices $C^l_{p,q}(m)$ is x and that of $C^l_{p,q}(n-m)$ is $\Omega \setminus x$.

For $i, j \in \{0, 1, \ldots, 2m+1\}$, let $\mathcal{M}_{i,j}$ be the vector space spanned by

$$C^l_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C^s_{\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil}(n-m),$$

where

$$0 \leqslant l \leqslant \min(m - \lfloor \frac{i}{2} \rfloor, m - \lfloor \frac{j}{2} \rfloor), \ 0 \leqslant s \leqslant \min(\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil).$$

Write

$$\mathcal{M} = \bigoplus_{i,j=0}^{2m+1} L(\mathcal{M}_{i,j}), \tag{6}$$

where $L(\mathcal{M}_{i,j}) = \{L(M) \in \mathrm{Mat}_X(\mathbb{C}) \mid M \in \mathcal{M}_{i,j}\}$, and

$$L(M)_{\Gamma_k(x) \times \Gamma_l(x)} = \begin{cases} M, & \text{if } k = i \text{ and } l = j, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mathcal{M}$ is a vector space. By Lemma 2.1, $\mathcal{M}$ is an algebra. In the remaining of this section we shall prove $\mathcal{T} = \mathcal{M}$.

Lemma 3.1 *The Terwilliger algebra $\mathcal{T}$ is a subalgebra of $\mathcal{M}$.*

Proof. By Lemma 2.3 we have $A \in \mathcal{M}$. For $0 \leqslant i \leqslant 2m+1$, since

$$E_i^* = E_i^*(x) = L(C^{m-\lfloor \frac{i}{2} \rfloor}_{m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{i}{2} \rfloor}(m) \otimes C^{\lceil \frac{i}{2} \rceil}_{\lceil \frac{i}{2} \rceil, \lceil \frac{i}{2} \rceil}(n-m)) \in \mathcal{M},$$

we get $\mathcal{T} \subseteq \mathcal{M}$. $\square$

For $i, j \in \{0, 1, \ldots, 2m+1\}$, let $\mathcal{T}_{i,j} = \{M_{i,j} \mid M \in \mathcal{T}\}$, where $M_{i,j}$ is the submatrix of M with rows indexed by vertices of $\Gamma_i(x)$ and columns indexed by vertices of $\Gamma_j(x)$. Since $\mathcal{T}$ is an algebra, each $\mathcal{T}_{i,j}$ is a vector space. Since $\mathcal{T} E_j^* \mathcal{T} \subseteq \mathcal{T}$, $(\mathcal{T} E_j^* \mathcal{T})_{i,k} \subseteq \mathcal{T}_{i,k}$, which gives

$$\mathcal{T}_{i,j} \mathcal{T}_{j,k} \subseteq \mathcal{T}_{i,k}. \tag{7}$$

Since A, $E_i^* \in \mathcal{T}$, we have $E_{i_1}^* A E_{i_2}^* A E_{i_3}^* \cdots A E_{i_{p-1}}^* A E_{i_p}^* \in E_{i_1}^* \mathcal{T} E_{i_p}^*$, from which it follows that

$$A_{i_1,i_2} A_{i_2,i_3} \cdots A_{i_{p-2},i_{p-1}} A_{i_{p-1},i_p} \in \mathcal{T}_{i_1,i_p}, \tag{8}$$

where $0 \leqslant i_s \leqslant 2m+1$ for any $s \in \{1, \ldots, p\}$.

Note that

$$W^{\mathrm{T}}_{m-\lfloor \frac{h}{2} \rfloor, m-\lfloor \frac{h}{2} \rfloor}(m) \otimes W_{\lceil \frac{h}{2} \rceil, \lceil \frac{h}{2} \rceil}(n-m) = I_{\binom{m}{m-\lfloor \frac{h}{2} \rfloor}} \otimes I_{\binom{n-m}{\lceil \frac{h}{2} \rceil}}.$$

By Lemma 2.3 and (1), for $h+1 \leqslant k$, one gets

$$A_{h,h+1} \cdots A_{k-1,k} = (\lfloor \frac{k}{2} \rfloor - \lfloor \frac{h}{2} \rfloor)!(\lceil \frac{k}{2} \rceil - \lceil \frac{h}{2} \rceil)! W^{\mathrm{T}}_{m-\lfloor \frac{k}{2} \rfloor, m-\lfloor \frac{h}{2} \rfloor}(m) \otimes W_{\lceil \frac{h}{2} \rceil, \lceil \frac{k}{2} \rceil}(n-m).$$

Hence, by (8), for $h \leqslant k$, we have

$$W^{\mathrm{T}}_{m-\lfloor \frac{k}{2} \rfloor, m-\lfloor \frac{h}{2} \rfloor}(m) \otimes W_{\lceil \frac{h}{2} \rceil, \lceil \frac{k}{2} \rceil}(n-m) \in \mathcal{T}_{h,k}. \tag{9}$$

Lemma 3.2 *For $2i+2 \leqslant j \leqslant 2m+1$ and $0 \leqslant s \leqslant i+1$, we have*

$$C^{m-\lfloor \frac{j}{2} \rfloor}_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C^{s}_{i+1, \lceil \frac{j}{2} \rceil}(n-m) \in \mathcal{T}_{2i+2,j}. \tag{10}$$

Proof. We use induction on s (s decreasing from $i+1$ to 0). Since

$$C^{m-\lfloor \frac{j}{2} \rfloor}_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C^{i+1}_{i+1, \lceil \frac{j}{2} \rceil}(n-m) = W^{\mathrm{T}}_{m-\lfloor \frac{j}{2} \rfloor, m-i-1}(m) \otimes W_{i+1, \lceil \frac{j}{2} \rceil}(n-m),$$

by (9), (10) holds for $2i+2 \leqslant j \leqslant 2m+1$ and $s = i+1$.

Assume that $C^{m-\lfloor \frac{j}{2} \rfloor}_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C^{s}_{i+1, \lceil \frac{j}{2} \rceil}(n-m) \in \mathcal{T}_{2i+2,j}$. By (7) and (8) we obtain

$$(C^{m-\lfloor \frac{j}{2} \rfloor}_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C^{s}_{i+1, \lceil \frac{j}{2} \rceil}(n-m))(A_{j,j+1} A_{j+1,j}) \in \mathcal{T}_{2i+2,j} \mathcal{T}_{j,j} \subseteq \mathcal{T}_{2i+2,j}, \tag{11}$$

$$(C^{m-\lfloor \frac{j}{2} \rfloor}_{m-i-1, m-\lfloor \frac{j}{2} \rfloor}(m) \otimes C^{s}_{i+1, \lceil \frac{j}{2} \rceil}(n-m))(A_{j,j-1} A_{j-1,j}) \in \mathcal{T}_{2i+2,j} \mathcal{T}_{j,j} \subseteq \mathcal{T}_{2i+2,j}. \tag{12}$$

When j is even, by Lemma 2.3, Lemma 2.1, (11) leads to

$$aC^{m-\frac{j}{2}}_{m-i-1,m-\frac{j}{2}}(m)\otimes C^{s}_{i+1,\frac{j}{2}}(n-m)+bC^{m-\frac{j}{2}}_{m-i-1,m-\frac{j}{2}}(m)\otimes C^{s-1}_{i+1,\frac{j}{2}}(n-m)\in\mathcal{T}_{2i+2,j},$$

where $a=(n-m-s-\frac{j}{2})(\frac{j}{2}-s+1)$ and $b=(i-s+2)(\frac{j}{2}-s+1)$. Similarly when j is odd, (12) yields that

$$a'C^{m-\lfloor\frac{j}{2}\rfloor}_{m-i-1,m-\lfloor\frac{j}{2}\rfloor}(m)\otimes C^{s}_{i+1,\lceil\frac{j}{2}\rceil}(n-m)+b'C^{m-\lfloor\frac{j}{2}\rfloor}_{m-i-1,m-\lfloor\frac{j}{2}\rfloor}(m)\otimes C^{s-1}_{i+1,\lceil\frac{j}{2}\rceil}(n-m)$$

belongs to $\mathcal{T}_{2i+2,j}$, where $a'=(n-m-s-\lceil\frac{j}{2}\rceil+1)(\lceil\frac{j}{2}\rceil-s)$ and $b'=(i-s+2)(\lceil\frac{j}{2}\rceil-s+1)$. Since $s\leqslant i+1\leqslant\lceil\frac{j}{2}\rceil$, $b\neq 0$ and $b'\neq 0$. Thus we have $C^{m-\lfloor\frac{j}{2}\rfloor}_{m-i-1,m-\lfloor\frac{j}{2}\rfloor}(m)\otimes C^{s-1}_{i+1,\lceil\frac{j}{2}\rceil}(n-m)\in\mathcal{T}_{2i+2,j}$.

Hence the desired result follows. $\square$

Lemma 3.3 *The algebra $\mathcal{M}$ is a subalgebra of $\mathcal{T}$.*

Proof. During this proof we will omit the symbol (m) from matrices in front of "$\otimes$", and omit $(n-m)$ from matrices behind "$\otimes$".

In order to get the desired conclusion, we only need to show that $\mathcal{M}_{i,j}\subseteq\mathcal{T}_{i,j}$ for $i,j\in\{0,1,\ldots,2m+1\}$. Write $\mathcal{M}^{\mathrm{T}}_{i,j}=\{M^{\mathrm{T}}\mid M\in\mathcal{M}_{i,j}\}$ and $\mathcal{T}^{\mathrm{T}}_{i,j}=\{M^{\mathrm{T}}\mid M\in\mathcal{T}_{i,j}\}$. Since $\mathcal{M}_{j,i}=\mathcal{M}^{\mathrm{T}}_{i,j}$ and $\mathcal{T}_{j,i}=\mathcal{T}^{\mathrm{T}}_{i,j}$, it suffices to prove $\mathcal{M}_{i,j}\subseteq\mathcal{T}_{i,j}$ for $i\leqslant j$. We use induction on i.

Step 1. We show that $\mathcal{M}_{0,j}\subseteq\mathcal{T}_{0,j}$ for $0\leqslant j\leqslant 2m+1$.

According to (6), the subspace $\mathcal{M}_{0,j}$ is spanned by $C^{l}_{m,m-\lfloor\frac{j}{2}\rfloor}\otimes C^{0}_{0,\lceil\frac{j}{2}\rceil}$, where $0\leqslant l\leqslant m-\lfloor\frac{j}{2}\rfloor$. Since

$$C^{l}_{m,m-\lfloor\frac{j}{2}\rfloor}\otimes C^{0}_{0,\lceil\frac{j}{2}\rceil}=\binom{m-\lfloor\frac{j}{2}\rfloor}{l}W^{\mathrm{T}}_{m-\lfloor\frac{j}{2}\rfloor,m}\otimes W_{0,\lceil\frac{j}{2}\rceil}$$

for any $l\in\{0,1,\ldots,m-\lfloor\frac{j}{2}\rfloor\}$, we get $\mathcal{M}_{0,j}\subseteq\mathcal{T}_{0,j}$ from (9).

Step 2. Assume that $\mathcal{M}_{p,j}\subseteq\mathcal{T}_{p,j}$ for $p\leqslant 2i$. We will show that $\mathcal{M}_{2i+1,j}\subseteq\mathcal{T}_{2i+1,j}$ and $\mathcal{M}_{2i+2,j}\subseteq\mathcal{T}_{2i+2,j}$.

Step 2.1. We show that $\mathcal{M}_{2i+1,j}\subseteq\mathcal{T}_{2i+1,j}$ for $2i+1\leqslant j\leqslant 2m+1$.

It suffices to prove

$$C^{l}_{m-i,m-\lfloor\frac{j}{2}\rfloor}\otimes C^{s}_{i+1,\lceil\frac{j}{2}\rceil}\in\mathcal{T}_{2i+1,j},\tag{13}$$

where $0\leqslant l\leqslant m-\lfloor\frac{j}{2}\rfloor$, $0\leqslant s\leqslant i+1$.

By induction hypothesis,

$$C^{l}_{m-i,m-\lfloor\frac{j}{2}\rfloor}\otimes C^{s}_{i,\lceil\frac{j}{2}\rceil}\in\mathcal{M}_{2i,j}\subseteq\mathcal{T}_{2i,j},$$

for $0 \leqslant l \leqslant m - \lfloor \frac{j}{2} \rfloor,\ 0 \leqslant s \leqslant i$. Since

$$A^{\mathrm{T}}_{2i,2i+1} = I_{\binom{m}{m-i}} \otimes W^{\mathrm{T}}_{i,i+1} \in \mathcal{M}^{\mathrm{T}}_{2i,2i+1} \subseteq \mathcal{T}^{\mathrm{T}}_{2i,2i+1},$$

we have

$$(I_{\binom{m}{m-i}} \otimes W^{\mathrm{T}}_{i,i+1})(C^{l}_{m-i,m-\lfloor \frac{j}{2} \rfloor} \otimes C^{s}_{i,\lceil \frac{j}{2} \rceil}) \in \mathcal{T}^{\mathrm{T}}_{2i,2i+1}\mathcal{T}_{2i,j} \subseteq \mathcal{T}_{2i+1,j}.$$

By Lemma 2.1, (13) holds for $0 \leqslant l \leqslant m - \lfloor \frac{j}{2} \rfloor$ and $0 \leqslant s \leqslant i$.

Next we shall show that (13) holds for $0 \leqslant l \leqslant m - \lfloor \frac{j}{2} \rfloor$ and $s = i+1$.

By (9), for $j \leqslant k \leqslant 2m+1$,

$$(W^{\mathrm{T}}_{m-\lfloor \frac{j}{2} \rfloor,m-i} \otimes W_{i+1,\lceil \frac{j}{2} \rceil})(W^{\mathrm{T}}_{m-\lfloor \frac{k}{2} \rfloor,m-\lfloor \frac{j}{2} \rfloor} \otimes W_{\lceil \frac{j}{2} \rceil,\lceil \frac{k}{2} \rceil})(W_{m-\lfloor \frac{k}{2} \rfloor,m-\lfloor \frac{j}{2} \rfloor} \otimes W^{\mathrm{T}}_{\lceil \frac{j}{2} \rceil,\lceil \frac{k}{2} \rceil})$$

belongs to $\mathcal{T}_{2i+1,j}$. By Lemma 2.1,

$$aC^{m-\lfloor \frac{k}{2} \rfloor}_{m-i,m-\lfloor \frac{j}{2} \rfloor} \otimes \left(\sum_{h=\max(0,i+1+\lceil \frac{j}{2} \rceil-\lceil \frac{k}{2} \rceil)}^{i+1} \binom{n-m-i-1-\lceil \frac{j}{2} \rceil}{\lceil \frac{k}{2} \rceil - i - 1 - \lceil \frac{j}{2} \rceil + h} C^{h}_{i+1,\lceil \frac{j}{2} \rceil} \right) \tag{14}$$

belongs to $\mathcal{T}_{2i+1,j}$, where $a = \binom{\lfloor \frac{k}{2} \rfloor - i}{\lfloor \frac{k}{2} \rfloor - \lfloor \frac{j}{2} \rfloor}\binom{\lceil \frac{k}{2} \rceil - i - 1}{\lceil \frac{j}{2} \rceil - i - 1} \neq 0$. Since (13) holds for $0 \leqslant l \leqslant m - \lfloor \frac{j}{2} \rfloor$ and $0 \leqslant s \leqslant i$, one has

$$\binom{n-m-i-1-\lceil \frac{j}{2} \rceil}{\lceil \frac{k}{2} \rceil - \lceil \frac{j}{2} \rceil} C^{m-\lfloor \frac{k}{2} \rfloor}_{m-i,m-\lfloor \frac{j}{2} \rfloor} \otimes C^{i+1}_{i+1,\lceil \frac{j}{2} \rceil} \in \mathcal{T}_{2i+1,j}.$$

Since $0 \leqslant 2i+1 \leqslant j \leqslant k-1 \leqslant 2m-1$ and $n \geqslant 3m$, we get

$$n-m-i-1-\lceil \frac{j}{2} \rceil \geqslant n-m-m-\lceil \frac{j}{2} \rceil \geqslant m - \lceil \frac{j}{2} \rceil \geqslant \lceil \frac{k}{2} \rceil - \lceil \frac{j}{2} \rceil \geqslant 0,$$

and so $\binom{n-m-i-1-\lceil \frac{j}{2} \rceil}{\lceil \frac{k}{2} \rceil - \lceil \frac{j}{2} \rceil} \neq 0$. Hence (13) holds for $0 \leqslant l \leqslant m - \lfloor \frac{j}{2} \rfloor$ and $s = i+1$.

Step 2.2. We show that $\mathcal{M}_{2i+2,j} \subseteq \mathcal{T}_{2i+2,j}$ for $2i+2 \leqslant j \leqslant 2m+1$.

It suffices to prove

$$C^{l}_{m-i-1,m-\lfloor \frac{j}{2} \rfloor} \otimes C^{s}_{i+1,\lceil \frac{j}{2} \rceil} \in \mathcal{T}_{2i+2,j}, \quad 0 \leqslant l \leqslant m - \lfloor \frac{j}{2} \rfloor,\ 0 \leqslant s \leqslant i+1. \tag{15}$$

By the inductive assumption, for $0 \leqslant l \leqslant m - \lfloor \frac{j}{2} \rfloor$ and $0 \leqslant s \leqslant i+1$,

$$C^{l}_{m-i,m-\lfloor \frac{j}{2} \rfloor} \otimes C^{s}_{i+1,\lceil \frac{j}{2} \rceil} \in \mathcal{M}_{2i+1,j} \subseteq \mathcal{T}_{2i+1,j}.$$

Since

$$A^{\mathrm{T}}_{2i+1,2i+2} = W_{m-i-1,m-i} \otimes I_{\binom{n-m}{i+1}} \in \mathcal{T}^{\mathrm{T}}_{2i+1,2i+2},$$

by (7) we have

$$(W_{m-i-1,m-i} \otimes I_{\binom{n-m}{i+1}})(C^l_{m-i,m-\lfloor\frac{j}{2}\rfloor} \otimes C^s_{i+1,\lceil\frac{j}{2}\rceil}) \in \mathcal{T}^{\mathrm{T}}_{2i+1,2i+2}\mathcal{T}_{2i+1,j} \subseteq \mathcal{T}_{2i+2,j}. \tag{16}$$

By Lemma 2.1,

$$W_{m-i-1,m-i}C^l_{m-i,m-\lfloor\frac{j}{2}\rfloor} = (i+1-l)C^l_{m-i-1,m-\lfloor\frac{j}{2}\rfloor} + (m-\lfloor\frac{j}{2}\rfloor - l + 1)C^{l-1}_{m-i-1,m-\lfloor\frac{j}{2}\rfloor}.$$

Thus (16) implies that

$$((i+1-l)C^l_{m-i-1,m-\lfloor\frac{j}{2}\rfloor} + (m-\lfloor\frac{j}{2}\rfloor - l + 1)C^{l-1}_{m-i-1,m-\lfloor\frac{j}{2}\rfloor}) \otimes C^s_{i+1,\lceil\frac{j}{2}\rceil} \tag{17}$$

belongs to $\mathcal{T}_{2i+2,j}$, where $0 \leqslant l \leqslant m - \lfloor\frac{j}{2}\rfloor$, $0 \leqslant s \leqslant i+1$. Since the coefficient of $C^{l-1}_{m-i-1,m-\lfloor\frac{j}{2}\rfloor} \otimes C^s_{i+1,\lceil\frac{j}{2}\rceil}$ in (17) is $m - \lfloor\frac{j}{2}\rfloor - l + 1 \neq 0$, by Lemma 3.2 we get (15).

Hence the desired result follows. □

Theorem 3.4 *Fix $x \in \binom{\Omega}{m}$. Let $\mathcal{T}$ be the Terwilliger algebra of $J(n,m,m+1)$ with respect to x and $\mathcal{M}$ be the algebra defined in (6). If $n \geqslant 3m$, then $\mathcal{T} = \mathcal{M}$.*

Proof. Combining Lemmas 3.1 and 3.3, the desired result follows. □

The condition $n \geqslant 3m$ guarantees the coefficient of $C^{m-\lfloor\frac{k}{2}\rfloor}_{m-i,m-\lfloor\frac{j}{2}\rfloor} \otimes C^{i+1}_{i+1,\lceil\frac{j}{2}\rceil}$ in (14) is non-zero. It seems to be interesting to determine the Terwilliger algebra of $J(n,m,m+1)$ without this assumption.

Theorem 3.5 ([16, Theorem 13]) *Let $\Gamma = (X,R)$ be a graph and $\mathcal{T}$ be the Terwilliger algebra of Γ with respect to a vertex x. If $E^*_i\mathcal{T}E^*_i$ is symmetric for any $i \in \{0,1,\ldots,D(x)\}$, then Γ is thin with respect to x.*

Corollary 3.6 *With reference to Theorem 3.4, $J(n,m,m+1)$ is thin with respect to x.*

Proof. By Theorem 3.4, for any $i \in \{0,1,\ldots,D(x)\}$, the subspace $E^*_i\mathcal{T}E^*_i$ is spanned by

$$L(C^l_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{i}{2}\rfloor}(m) \otimes C^s_{\lceil\frac{i}{2}\rceil,\lceil\frac{i}{2}\rceil}(n-m)),$$

where $0 \leqslant l \leqslant m - \lfloor\frac{i}{2}\rfloor, 0 \leqslant s \leqslant \lceil\frac{i}{2}\rceil$. Since each element of $E^*_i\mathcal{T}E^*_i$ is symmetric, we get the conclusion from Theorem 3.5. □

4 Two bases of the Terwilliger algebra

In this section we shall determine two bases of the Terwilliger algebra $\mathcal{T}$ in Theorem 3.4. Set

$$G_{i,j} = \{g \mid H^g_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m) \neq 0\},\ R_{i,j} = \{r \mid H^r_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m) \neq 0\}.$$

Theorem 4.1 *Let $\mathcal{T}$ be as in Theorem 3.4. Then*

$$\{L(H^g_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m) \otimes H^r_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m)),\ g \in G_{i,j},\ r \in R_{i,j}\}_{i,j=0}^{2m+1} \tag{18}$$

as well as

$$\{L(C^l_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m) \otimes C^s_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m)),\ l \in G_{i,j},\ s \in R_{i,j}\}_{i,j=0}^{2m+1} \tag{19}$$

are two bases of $\mathcal{T}$.

Proof. Without loss of generality, suppose $i \leqslant j$. We have $H^l_{i,j}(v) \neq 0$ if and only if $\max(0, i+j-v) \leqslant l \leqslant \min(i,j)$, so $\lceil\frac{i}{2}\rceil - |R_{i,j}| + 1 \leqslant r \leqslant \lceil\frac{i}{2}\rceil$ when $r \in R_{i,j}$. By (2) we obtain

$$C^r_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m) = \sum_{h=r}^{\lceil\frac{i}{2}\rceil} \binom{h}{r} H^h_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m), \tag{20}$$

which implies that $H^r_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m)$ is a linear combination of $\{C^s_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m)\}_{s\in R_{i,j}}$ for any $r \in R_{i,j}$. Similarly, $H^g_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m)$ can be expressed as a linear combination of $\{C^l_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m)\}_{l\in G_{i,j}}$ for any $g \in G_{i,j}$. Hence every element of

$$\{H^g_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m) \otimes H^r_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m)\}_{g\in G_{i,j},r\in R_{i,j}} \tag{21}$$

belongs to $\mathcal{M}_{i,j}$. Again by (2), for $0 \leqslant l \leqslant m - \lfloor\frac{j}{2}\rfloor$ and $0 \leqslant s \leqslant \lceil\frac{i}{2}\rceil$,

$$\begin{aligned} & C^l_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m) \otimes C^s_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m) \\ = & \Big(\sum_{g=l}^{m-\lfloor\frac{j}{2}\rfloor} \binom{g}{l} H^g_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m)\Big) \otimes \Big(\sum_{r=s}^{\lceil\frac{i}{2}\rceil} \binom{r}{s} H^r_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m)\Big). \end{aligned}$$

Observe that (21) are linearly independent, so (21) is a basis of $\mathcal{M}_{i,j}$. Therefore (18) is a basis of $\mathcal{T}$.

Furthermore, by (20) we get $\{C^l_{m-\lfloor\frac{i}{2}\rfloor,m-\lfloor\frac{j}{2}\rfloor}(m) \otimes C^s_{\lceil\frac{i}{2}\rceil,\lceil\frac{j}{2}\rceil}(n-m)\}_{l\in G_{i,j},s\in R_{i,j}}$ is also a basis of $\mathcal{M}_{i,j}$, from which it follows that (19) is a basis of $\mathcal{T}$. $\square$

Corollary 4.2 *With reference to Theorem 3.4 we get the dimension of $\mathcal{T}$ is*

$$\dim \mathcal{T} = \begin{cases} \frac{1}{12}(m+1)(m+2)(m+3)(3m+10)-4, & \text{if } n=3m, \\ \frac{1}{12}(m+1)(m+2)(m+3)(3m+10)-1, & \text{if } n=3m+1, \\ \frac{1}{12}(m+1)(m+2)(m+3)(3m+10), & \text{if } n \geqslant 3m+2. \end{cases}$$

Proof. By Theorem 4.1,

$$\begin{aligned} \dim \mathcal{T} &= \sum_{i,j=0}^{2m+1} |G_{i,j}||R_{i,j}| \\ &= \sum_{i,j=0}^{2m+1} (\min(m-\lfloor \frac{i}{2} \rfloor, m-\lfloor \frac{j}{2} \rfloor) - \max(0, m-\lfloor \frac{i}{2} \rfloor - \lfloor \frac{j}{2} \rfloor)+1) \\ &\quad \times (\min(\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil) - \max(0, \lceil \frac{i}{2} \rceil + \lceil \frac{j}{2} \rceil - n+m)+1). \end{aligned}$$

By zigzag calculation, we get the desired result. □

5 Concluding Remark

We conclude this paper with the following remarks:

(i) Let Ω be a set of cardinality n and let $J(n,m)$ be the Johnson graph based on Ω with $n \geqslant 3m$. Fix an m-subset x of Ω. Let $\mathcal{T}' = \mathcal{T}'(x)$ and $\mathcal{T} = \mathcal{T}(x)$ be the Terwilliger algebra of $J(n,m)$ and $J(n,m,m+1)$ with respect to x, respectively. Since $\bigoplus_{i,j=0}^{m} E^*_{2i}(x)\mathcal{T}E^*_{2j}(x)$ is an algebra, $\{L(H^g_{m-i,m-j}(m) \otimes H^r_{i,j}(n-m)),\ g \in G_{2i,2j},\ r \in R_{2i,2j}\}_{i,j=0}^{m}$ is a basis of $\bigoplus_{i,j=0}^{m} E^*_{2i}(x)\mathcal{T}E^*_{2j}(x)$ by Theorem 4.1. By [12, Definition 4.2, Lemma 4.4, Theorem 5.9] this basis coincides with that of $\mathcal{T}'$, which implies that $\mathcal{T}' \simeq \bigoplus_{i,j=0}^{m} E^*_{2i}(x)\mathcal{T}E^*_{2j}(x)$.

(ii) Using the same method, the Terwilliger algebra of $J(n,m,m+1)$ with respect to an $(m+1)$-subset may be determined.

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