From clutters to matroids

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Submitted: Nov 20, 2013; Accepted: Dec 23, 2013; Published: Jan 12, 2014
Mathematics Subject Classifications: 05B35, 05C65

Abstract

This paper deals with the question of completing a monotone increasing family of subsets Γ of a finite set Ω to obtain the dependent sets of a matroid. Specifically, we provide several natural processes for transforming the clutter Λ of the inclusion-minimal subsets of the family Γ into the set of circuits $\mathcal{C}(\mathcal{M})$ of a matroid $\mathcal{M}$ with ground set Ω. In addition, by combining these processes, we prove that all the minimal matroidal completions of the family can be obtained.

Keywords: clutter, hypergraph, matroid, circuits.

1 Introduction

A *monotone increasing* family of subsets Γ of a finite set Ω is a collection of subsets of Ω such that any superset of a set in the family Γ must be in Γ. All the inclusion-minimal elements of Γ determine a *clutter* Λ, that is, a collection of subsets of Ω none of which is a proper subset of another. Clutters are also known as *antichains*, *Sperner systems* or *simple hypergraphs*.

A wide variety of examples of monotone increasing families exist, among them the collection of the linearly dependent subsets of vectors in a vector space. A *matroid* $\mathcal{M}$ is a combinatorial object that provides an axiomatic abstraction of linear dependence on a finite set Ω. The minimal dependent sets of a matroid $\mathcal{M}$ are called *circuits*. Therefore, the family of circuits of a matroid $\mathcal{M}$ is a clutter.

*Supported by the Ministerio de Educación y Ciencia (Spain) and the European Regional Development Fund under project MTM2011-28800-C02-01, and by the Catalan Research Council under project 2009SGR1387.

We say that a clutter is *matroidal* if it corresponds to the family of circuits of a matroid. Matroidal clutters, as well as "almost matroidal" clutters, play a key role in several situations. For instance, in the context of secret-sharing schemes they become a crucial issue for providing general bounds on the optimal information rate of the scheme (see [4, 8]). In the framework of algebraic combinatorics and commutative algebra, other interesting examples can be found that deal with monomial ideals and arithmetic properties of the face ring of simplicial complexes (see [1, 9]).

Since in general a clutter is far from being matroidal, it is of interest to know how it can be transformed into a matroidal one. This paper deals with the question of finding the *matroidal completions of a clutter*.

The outline of the paper is as follows. In Section 2 we recall some definitions and basic facts about clutters and matroids. Several ways to obtain matroidal completions of clutters can be found in Section 3; namely, we present the uniform completions (Proposition 3), the I-completions (Proposition 5), and the $\mathcal{T}$-completions (Proposition 8). In addition, by means of the clutter transformations involved in these processes, a necessary condition for a clutter to be a matroid port is obtained (Proposition 9). Finally, Section 4 is devoted to analyzing the minimal matroidal completions. We characterize the clutters with only one minimal element (Theorem 12), and we show how to obtain all the minimal matroidal completions of any clutter (Theorem 13).

2 Clutters and matroids

In this section we present the definitions and basic facts concerning families of subsets, clutters and matroids that are used in the paper. (The reader is referred to [6, 10] for general references on matroid theory).

Let Ω be a finite set. Observe that if Γ is a monotone increasing family of subsets of Ω, then the collection $\min(\Gamma)$ of its inclusion-minimal elements is a clutter; while if Λ is a clutter on Ω, then the family $\Lambda^+ = \{A \subseteq \Omega : A_0 \subseteq A \text{ for some } A_0 \in \Lambda\}$ is a monotone increasing family of subsets. Clearly $\Gamma = (\min(\Gamma))^+$ and $\Lambda = \min\left(\Lambda^+\right)$. So a monotone increasing family of subsets Γ is determined uniquely by the clutter $\min(\Gamma)$, while a clutter Λ is determined uniquely by the monotone increasing family Λ^+.

Despite the foregoing, we must take into account the following lemma concerning the relationship between the inclusion and the equality of two clutters Λ_1 and Λ_2, and the inclusion and the equality of their associated monotone increasing families of subsets Λ_1^+ and Λ_2^+.

Lemma 1. *Let Λ_1, Λ_2 be two clutters on a finite set Ω. Then:*

1. $\Lambda_1 = \Lambda_2$ *if and only if* $\Lambda_1^+ = \Lambda_2^+$.

2. *If* $\Lambda_1 \subseteq \Lambda_2$ *then* $\Lambda_1^+ \subseteq \Lambda_2^+$. *The converse is not true.*

3. $\Lambda_1^+ \subseteq \Lambda_2^+$ *if and only if* $\Lambda_1 \subseteq \Lambda_2^+$.

Proof. The proofs of the statements are a straightforward consequence of the definition of Λ^+ and of the fact that $\Lambda = \min\left(\Lambda^+\right)$. So it is only necessary to present an example of clutters with $\Lambda_1 \nsubseteq \Lambda_2$ and $\Lambda_1^+ \subseteq \Lambda_2^+$. For instance, on the finite set $\Omega = \{1,2,3\}$, let us consider the clutters $\Lambda_1 = \{\{1,2\}, \{2,3\}\}$ and $\Lambda_2 = \{\{1\},\{2,3\}\}$. Then $\Lambda_1 \nsubseteq \Lambda_2$, while $\Lambda_1^+ = \{\{1,2\},\{2,3\},\{1,2,3\}\} \subseteq \{\{1\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\} = \Lambda_2^+$. □

The previous lemma leads us to consider a binary relation $\leqslant$ defined on the set of clutters on Ω. Namely, if Λ_1 and Λ_2 are two clutters on Ω, then we say that $\Lambda_1 \leqslant \Lambda_2$ if and only if $\Lambda_1^+ \subseteq \Lambda_2^+$. Therefore, $\Lambda_1 \leqslant \Lambda_2$ if and only if for all $A_1 \in \Lambda_1$ there exists $A_2 \in \Lambda_2$ such that $A_2 \subseteq A_1$. It is clear that the binary relation $\leqslant$ is reflexive and transitive. Besides, from statement (1) of the previous lemma, the relation $\leqslant$ is antisymmetric. Therefore, the binary relation $\leqslant$ is a partial order on the set of clutters of Ω. We will use this partial order throughout the paper.

There are many interesting families of clutters that can be considered. However, because of their applications, we are interested in clutters that provide matroids.

Matroids are combinatorial objects that can be axiomatized in terms of their independent sets, bases, circuits, rank function, flats, or hyperplanes. Here we present the definition in terms of circuits. A *matroid* $\mathcal{M}$ is an ordered pair $\mathcal{M} = (\Omega, \mathcal{C})$ consisting of a finite set Ω, called the *ground set* of the matroid, and a family $\mathcal{C}$ of nonempty subsets of Ω which satisfy the following three conditions:

1. $\emptyset \notin \mathcal{C}$;
2. if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$; and
3. if C_1 and C_2 are distinct members of $\mathcal{C}$ and $x \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$.

The members of the clutter $\mathcal{C}$ are the *circuits* of the matroid $\mathcal{M}$. We shall often write $\mathcal{C}(\mathcal{M})$ instead of $\mathcal{C}$. The *dependent sets* of the matroid are the supersets of the circuits, that is, the dependent sets of $\mathcal{M}$ are the members of $\mathcal{C}(\mathcal{M})^+$. Therefore, the set of dependent sets of the matroid is a monotone increasing family of subsets whose inclusion-minimal elements are its circuits. A clutter Λ is said to be a *matroidal clutter* if it is the set of circuits of a matroid, that is, if there exists a matroid $\mathcal{M}_0$ such that $\mathcal{C}(\mathcal{M}_0) = \Lambda$.

Since the set of circuits of a matroid is a clutter on the ground set of the matroid, we can consider the partial order induced by $\leqslant$ on the set of matroids with ground set Ω. Thereby, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two matroids with ground set Ω, then we say that $\mathcal{M}_1 \leqslant \mathcal{M}_2$ if and only if $\mathcal{C}(\mathcal{M}_1) \leqslant \mathcal{C}(\mathcal{M}_2)$ where $\mathcal{C}(\mathcal{M}_i)$ is the clutter of the circuits of $\mathcal{M}_i$. So, $\mathcal{M}_1 \leqslant \mathcal{M}_2$ if and only if every circuit of $\mathcal{M}_1$ contains a circuit of $\mathcal{M}_2$. In matroid theory this is equivalent to saying that the identity map on Ω is a *weak map* from the matroid $\mathcal{M}_1$ to the matroid $\mathcal{M}_2$ (see [6, Proposition 7.3.11]).

3 Matroidal completions of a clutter

The set of circuits of a matroid is a clutter, but there are clutters on a finite set Ω that are not the set of circuits of a matroid with ground set Ω. So, a natural question that arises at this point is to determine how to complete a clutter Λ to obtain a matroid; that is to say, to transform the clutter Λ into a matroidal clutter.

In order to look for matroidal completions, it is important to take into account the dependent sets of the matroid instead of the circuits. This is due to the fact that, as the following example shows, there exist clutters Λ such that $\Lambda \nsubseteq \mathcal{C}(\mathcal{M})$ for any matroid $\mathcal{M}$.

Example 2. Let $n \geqslant 4$. On the finite set $\Omega = \{1, 2, 3, \ldots, n\}$, we consider the clutter $\Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3, \ldots, n\}\}$. Observe that $\big(\{1, 2\} \cup \{1, 3\}\big) \setminus \{1\} = \{2, 3\} \subsetneq \{2, 3, \ldots, n\}$. Hence it follows that $\Lambda \nsubseteq \mathcal{C}(\mathcal{M})$ for any matroid $\mathcal{M}$.

The above example leads us to the following definition. Let Λ be a clutter on a finite set Ω, and let $\mathcal{M}$ be a matroid with ground set Ω. We say that the matroid $\mathcal{M}$ is a *matroidal completion* of the clutter Λ if $\Lambda \subseteq \mathcal{C}(\mathcal{M})^+$. In other words, $\mathcal{M}$ is a matroidal completion of Λ if and only if every subset $A \in \Lambda$ is a dependent set in $\mathcal{M}$. From Lemma 1 we get that $\mathcal{M}$ is a matroidal completion of Λ if and only if $\Lambda \leqslant \mathcal{C}(\mathcal{M})$. We will write $\Lambda \leqslant \mathcal{M}$ instead of $\Lambda \leqslant \mathcal{C}(\mathcal{M})$. The set of all the matroidal completions of a clutter Λ is denoted by $\mathrm{Mat}(\Lambda)$, that is $\mathrm{Mat}(\Lambda) = \{\mathcal{M} : \Lambda \leqslant \mathcal{M}\}$. Observe that if $\emptyset \in \Lambda$ then $\mathrm{Mat}(\Lambda) = \emptyset$. So, from now on, throughout the paper we assume that $\emptyset \notin \Lambda$ if Λ is a clutter. As is shown in the next subsection, this assumption guarantees that $\mathrm{Mat}(\Lambda) \neq \emptyset$ for all clutters.

The aim of this section is to provide three methods in order to obtain matroidal completions of Λ; that is, to obtain matroids $\mathcal{M}$ in $\mathrm{Mat}(\Lambda)$. By combining these methods, the minimal matroidal completions will be studied in Section 4.

3.1 Uniform completion

The following proposition states that the family of uniform matroids provides matroidal completions of clutters. Recall that if Ω is a finite set of size $|\Omega| = \omega$ and if $m \leqslant \omega$ is a non-negative integer, then the *uniform matroid* of rank m on Ω is the matroid $\mathcal{U}_{m,\omega}$ with ground set Ω and set of circuits $\mathcal{C}(\mathcal{U}_{m,\omega}) = \{C \subseteq \Omega : |C| = m + 1\}$ if $m < \omega$ and $\mathcal{C}(\mathcal{U}_{m,\omega}) = \emptyset$ if $m = \omega$.

Proposition 3. *Let Λ be a clutter on a finite set Ω of size $|\Omega| = \omega$. Then, $\mathcal{U}_{m,\omega} \in \mathrm{Mat}(\Lambda)$ if and only if $m \leqslant s - 1$ where $s = \min\{|A| : A \in \Lambda\}$.*

Proof. The dependent sets of the uniform matroid $\mathcal{U}_{m,\omega}$ are those subsets $X \subseteq \Omega$ with $|X| \geqslant m + 1$. Therefore, $\Lambda \subseteq \mathcal{C}(\mathcal{U}_{m,\omega})^+$ if and only if $|A| \geqslant m + 1$ for all $A \in \Lambda$; that is, if and only if $s \geqslant m + 1$. $\square$

It is clear that $\mathcal{U}_{m_1,\omega} \leqslant \mathcal{U}_{m_2,\omega}$ if and only if $m_1 \geqslant m_2$. Hence, the uniform matroids in $\mathrm{Mat}(\Lambda)$ form a chain $\mathcal{U}_{s-1,\omega} \leqslant \cdots \leqslant \mathcal{U}_{1,\omega} \leqslant \mathcal{U}_{0,\omega}$ whose minimal element is $\mathcal{U}_{s-1,\omega}$. We will say that $\mathcal{U}_{s-1,\omega}$ is the *uniform completion* of Λ and it is denoted by $\mathcal{U}(\Lambda)$.

The following example shows that, in general, there are matroids in $\mathrm{Mat}(\Lambda)$ that are not uniform matroids. Moreover, from the example, it follows that in general the uniform completion $\mathcal{U}(\Lambda)$ is not a minimal matroidal completion of Λ.

Example 4. On the finite set $\Omega = \{1,2,3,4,5\}$, we consider the clutter $\Lambda = \{\{1,2\}, \{1,3\}, \{2,3,4\}, \{2,3,5\}\}$. We have that $\Lambda \subseteq \mathcal{C}(\mathcal{M})^+$ where $\mathcal{M}$ is the matroid with set of circuits $\mathcal{C}(\mathcal{M}) = \{\{1,2\}, \{1,3\}, \{2,3\}, \{1,4,5\}, \{2,4,5\}, \{3,4,5\}\}$. So $\mathcal{M}$ is a matroidal completion of Λ which is not a uniform matroid. Observe that the uniform matroids in $\mathrm{Mat}(\Lambda)$ are $\mathcal{U}_{1,4}$ and $\mathcal{U}_{0,4}$, and here $\mathcal{M} \lneqq \mathcal{U}(\Lambda) = \mathcal{U}_{1,4} \leqslant \mathcal{U}_{0,4}$.

3.2 Completion with intersections: I-transformations

In this subsection we prove that it is possible to transform a clutter Λ into a matroidal clutter by adding intersections of suitable subsets of Λ. To present our result we need to introduce some previous definitions.

Let Λ be a clutter on a finite set Ω. For a subset $X \subseteq \Omega$, we denote by $I_\Lambda(X)$ the intersection of the subsets A in Λ contained in X, (this intersection is the one involved in the characterization of the set of circuits in connected matroids, see [6, Theorem 4.3.2]). We say that a clutter Λ' is an *I-transformation* of the clutter Λ if $\Lambda' = \min\big(\Lambda \cup \{A_1 \cap A_2\}\big)$ where $A_1, A_2 \in \Lambda$ are two different subsets with $I_\Lambda(A_1 \cup A_2) \neq \emptyset$.

Proposition 5. *Let Λ be a clutter on a finite set Ω. If Λ is not matroidal then there exists a chain of clutters $\Lambda = \Lambda_0 \lneqq \Lambda_1 \lneqq \cdots \lneqq \Lambda_r$ such that the clutter Λ_i is an I-intersection of Λ_{i-1} for $i \geqslant 1$ and the clutter Λ_r is a matroidal clutter.*

Proof. First of all notice that if $A_1, A_2 \in \Lambda$ are different then, $I_\Lambda(A_1 \cup A_2) = \emptyset$ if and only if for all $x \in A_1 \cap A_2$ there exists $A_3 \in \Lambda$ with $A_3 \subseteq (A_1 \cup A_2) \setminus \{x\}$. Therefore we get that a clutter Λ is a matroidal clutter if and only if $I_\Lambda(A_1 \cup A_2) = \emptyset$ for any two different $A_1, A_2 \in \Lambda$.

The proof of the proposition follows from this equivalence. Indeed, if Λ is not a matroidal clutter, then we get that there exist two different subsets $A_1, A_2 \in \Lambda$ with $I_\Lambda(A_1 \cup A_2) \neq \emptyset$. So we can consider the clutter $\Lambda_1 = \min\big(\Lambda \cup \{A_1 \cap A_2\}\big)$, which is an I-intersection of Λ. Clearly $\Lambda \lneqq \Lambda_1$ because A_1 and A_2 are different. We now proceed in the same way with the clutter Λ_1, and so on. □

Therefore, by means of I-transformations we can transform a clutter Λ into a matroidal clutter. The matroids obtained in this way will be called *I-matroidal completions of* Λ. Next we present some examples to show that it no general result exists concerning the comparison between two different I-matroidal completions of a clutter.

Example 6. On the finite set $\Omega = \{1,2,3,4,5\}$, we consider the clutters Λ_1, Λ_2 and Λ_3 where $\Lambda_1 = \{\{1,2,3\},\{1,2,4\},\{3,4,5\}\}$, $\Lambda_2 = \{\{1,2,3\},\{1,2,4\},\{1,5\},\{2,5\}\}$ and $\Lambda_3 = \{\{1,2,3\},\{1,2,4\},\{4,5\}\}$. It is clear that Λ_1 has only one I-matroidal completion, the matroid $\mathcal{M}_1$ with set of circuits $\mathcal{C}(\mathcal{M}_1) = \{\{1,2\},\{3,4,5\}\}$. The two I-matroidal completions of the clutter Λ_2 are the matroids $\mathcal{M}_{2,1}$ and $\mathcal{M}_{2,2}$ with set of circuits $\mathcal{C}(\mathcal{M}_{2,1}) = \{\{1,2\},\{1,5\},\{2,5\}\}$ and $\mathcal{C}(\mathcal{M}_{2,2}) = \{\{1,2\},\{5\}\}$. The two I-matroidal completions of Λ_3 are the matroids $\mathcal{M}_{3,1}$ and $\mathcal{M}_{3,2}$ with set of circuits $\mathcal{C}(\mathcal{M}_{3,1}) = \{\{1,2\},\{4,5\}\}$ and $\mathcal{C}(\mathcal{M}_{3,2}) = \{\{1,2,3\},\{4\}\}$. Therefore the I-matroidal completions of Λ_2 form a chain $\mathcal{M}_{2,1} \leqslant \mathcal{M}_{2,2}$, while the I-matroidal completions of Λ_3 are not comparable, that is, $\mathcal{M}_{3,1} \nleqslant \mathcal{M}_{3,2}$ and $\mathcal{M}_{3,2} \nleqslant \mathcal{M}_{3,1}$.

3.3 Completion with unions: $\mathcal{T}$-transformations

The aim of this subsection is to present some natural ways to obtain matroidal completions of a clutter Λ, that is, to obtain matroids in $\mathrm{Mat}(\Lambda)$. Unlike in the previous subsection, here we proceed in a recursive way by adding, in each step of the process, some slight modifications of the union of two distinct elements of the clutter. Our result is stated in Proposition 8, and by using these matroidal completions a necessary condition for matroid ports is presented in Proposition 9. Let us start by defining the two elementary transformations involved in the recursive process.

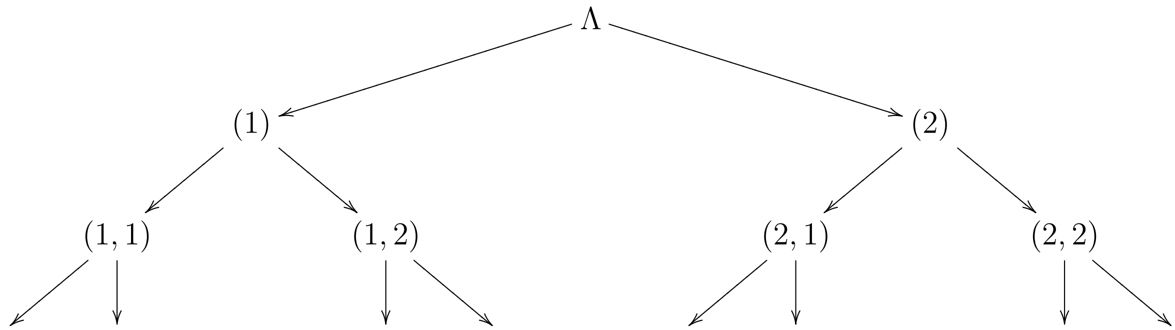
Let Λ be a clutter on a finite set Ω. We define the elementary transformations $\mathcal{T}^{(1)}(\Lambda)$ and $\mathcal{T}^{(2)}(\Lambda)$ of Λ as the clutters:

- $\mathcal{T}^{(1)}(\Lambda) = \min\big(\Lambda \cup \big\{(A_1 \cup A_2) \setminus \{x\}$ where $A_1, A_2 \in \Lambda$ are different and $x \in A_1 \cap A_2\big\}\big)$; that is, in the first elementary transformation $\mathcal{T}^{(1)}(\Lambda)$, we consider the minimal elements of the family obtained by adding to Λ those subsets that arise from the circuit condition.

- $\mathcal{T}^{(2)}(\Lambda) = \min\big(\Lambda \cup \big\{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2)$, where $A_1, A_2 \in \Lambda$ are different$\big\}\big)$; that is, in the second elementary transformation $\mathcal{T}^{(2)}(\Lambda)$, we add to Λ the subsets obtained from the union after removing the intersections $I_\Lambda(X)$ defined in the previous subsection.

Since $\mathcal{T}^{(1)}(\Lambda)$ and $\mathcal{T}^{(2)}(\Lambda)$ are clutters, we can apply the elementary transformations again. Hence, for $(i_1, i_2) \in \{1,2\} \times \{1,2\}$ we can consider the clutter $\mathcal{T}^{(i_2)}(\mathcal{T}^{(i_1)}(\Lambda))$. At this point we proceed in a recursive way. Let $r \geqslant 2$ be a non-negative integer and let $(i_1, \ldots, i_r) \in \{1,2\}^r$ be an r-tuple. Then we define the clutter $\mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda)$ by the recursion formula $\mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda) = \mathcal{T}^{(i_r)}(\mathcal{T}^{(i_1,\ldots,i_{r-1})}(\Lambda))$; that is, $\mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda)$ is the i_r elementary transformation of $\mathcal{T}^{(i_1,\ldots,i_{r-1})}(\Lambda)$.

In this way we obtain the following tree diagram of $\mathcal{T}$-transformations of the clutter Λ, where we write $(i_1, \ldots, i_r)$ instead of $\mathcal{T}^{(i_1,\ldots,i_r)}(\Lambda)$:



We will say that a clutter Λ' is a $\mathcal{T}$*-transformation* of Λ if it is obtained from Λ in this way, that is, if $\Lambda' = \mathcal{T}^{(i_1,\dots,i_r)}(\Lambda)$ for some r-tuple $(i_1,\dots,i_r)$. The next lemma points out the relationship between two $\mathcal{T}$-transformations, that is, between two clutters of the above diagram. The first statement of the lemma deals with the relationship between clutters in each branch of the diagram, whereas the last two statements deal with the comparison of clutters in a same row of the diagram, that is, the 2^r possible clutters $\mathcal{T}^{(i_1,\dots,i_r)}(\Lambda)$.

Lemma 7. *Let Λ be a clutter on a finite set Ω. Let $\mathcal{I} = (i_1, i_2, i_3, \dots, i_s, i_{s+1}, \dots)$ be a sequence with $i_\ell \in \{1,2\}$, and let $(j_1,\dots,j_r),(k_1,\dots,k_r) \in \{1,2\}^r$ be two r-tuples. Then, the following statements hold:*

1. $\Lambda \leqslant \mathcal{T}^{(i_1)}(\Lambda) \leqslant \mathcal{T}^{(i_1,i_2)}(\Lambda) \leqslant \cdots \leqslant \mathcal{T}^{(i_1,\dots,i_s)}(\Lambda) \leqslant \mathcal{T}^{(i_1,\dots,i_s,i_{s+1})}(\Lambda) \leqslant \cdots$*, and there exists $r \geqslant 1$ such that $\mathcal{T}^{(i_1,\dots,i_r)}(\Lambda) = \mathcal{T}^{(i_1,\dots,i_r,i_{r+1})}(\Lambda)$.*

2. *If $(j_1,\dots,j_{r-1}) = (k_1,\dots,k_{r-1})$ and $j_r \leqslant k_r$, then $\mathcal{T}^{(j_1,\dots,j_r)}(\Lambda) \leqslant \mathcal{T}^{(k_1,\dots,k_r)}(\Lambda)$. In general this inequality is not an equality.*

3. *If $(j_1,\dots,j_{r-1}) \neq (k_1,\dots,k_{r-1})$ then, in general, there is no relationship between the clutters $\mathcal{T}^{(j_1,\dots,j_r)}(\Lambda)$ and $\mathcal{T}^{(k_1,\dots,k_r)}(\Lambda)$.*

Proof. From the definitions of the two elementary transformations it follows that if Λ_0 is a clutter on Ω then $\Lambda_0 \leqslant \mathcal{T}^{(1)}(\Lambda_0)$ and $\Lambda_0 \leqslant \mathcal{T}^{(2)}(\Lambda_0)$. Therefore, we have that the iteration of the elementary transformations provides a monotone increasing sequence of clutters $\Lambda \leqslant \mathcal{T}^{(i_1)}(\Lambda) \leqslant \mathcal{T}^{(i_1,i_2)}(\Lambda) \leqslant \cdots \leqslant \mathcal{T}^{(i_1,\dots,i_s)}(\Lambda) \leqslant \mathcal{T}^{(i_1,\dots,i_s,i_{s+1})}(\Lambda) \leqslant \cdots$. The proof of statement (1) is concluded by noticing that there are only a finite number of clutters in a finite set.

Next let us prove statement (2). It is necessary to prove the inequality $\mathcal{T}^{(i_1,\dots,i_{r-1},1)}(\Lambda) \leqslant \mathcal{T}^{(i_1,\dots,i_{r-1},2)}(\Lambda)$ and, in addition, we must also show that in general this inequality is not an equality. Clearly, to do this it is enough to prove that if Λ_0 is a clutter on the finite set Ω then $\mathcal{T}^{(1)}(\Lambda_0) \leqslant \mathcal{T}^{(2)}(\Lambda_0)$ and, in addition, we must show that there are clutters Λ_0 of Ω with $\mathcal{T}^{(1)}(\Lambda_0) \neq \mathcal{T}^{(2)}(\Lambda_0)$.

First let us show that $\mathcal{T}^{(1)}(\Lambda_0) \leqslant \mathcal{T}^{(2)}(\Lambda_0)$; that is, we must demonstrate that if $X \in \mathcal{T}^{(1)}(\Lambda_0)$, then there exists $X' \in \mathcal{T}^{(2)}(\Lambda_0)$ such that $X' \subseteq X$. So let $X \in \mathcal{T}^{(1)}(\Lambda_0) = \min\big(\Lambda_0 \cup \big\{(A_1 \cup A_2) \setminus \{x\}$, where $A_1, A_2 \in \Lambda_0$ are different and $x \in A_1 \cap A_2\big\}\big)$. If $X = A \in \Lambda_0$, then from the definition of $\mathcal{T}^{(2)}(\Lambda_0)$, it follows that there exists $X' \in \mathcal{T}^{(2)}(\Lambda_0)$ with $X' \subseteq X$, as we wanted to prove. Therefore, we may assume now that $X = (A_1 \cup A_2) \setminus \{x\}$ where A_1 and A_2 are two distinct members of Λ_0 and where $x \in A_1 \cap A_2$. At this point we distinguish two cases. First suppose that $x \in I_{\Lambda_0}(A_1 \cup A_2)$. In such a case we have that $(A_1 \cup A_2) \setminus I_{\Lambda_0}(A_1 \cup A_2) \subseteq (A_1 \cup A_2) \setminus \{x\}$ and from the definition of $\mathcal{T}^{(2)}(\Lambda_0)$ we get that there exists $X' \in \mathcal{T}^{(2)}(\Lambda_0)$ with $X' \subseteq (A_1 \cup A_2) \setminus I_{\Lambda_0}(A_1 \cup A_2)$. Hence, if $x \in I_{\Lambda_0}(A_1 \cup A_2)$, then there exists $X' \in \mathcal{T}^{(2)}(\Lambda_0)$ with $X' \subseteq X$. Now assume that $x \notin I_{\Lambda_0}(A_1 \cup A_2)$. Then from the definition of $I_{\Lambda_0}(A_1 \cup A_2)$ we get that there exists $A_3 \in \Lambda_0$ with $A_3 \subseteq (A_1 \cup A_2) \setminus \{x\}$. Since $A_3 \in \Lambda_0$, there exists $X' \in \mathcal{T}^{(2)}(\Lambda_0)$ with $X' \subseteq A_3$. Therefore, if $x \notin I_{\Lambda_0}(A_1 \cup A_2)$, then there exists $X' \in \mathcal{T}^{(2)}(\Lambda_0)$ with $X' \subseteq X$. Thus, in both cases we conclude that there exists $X' \in \mathcal{T}^{(2)}(\Lambda_0)$ such that $X' \subseteq X$, that is, $\mathcal{T}^{(1)}(\Lambda_0) \leqslant \mathcal{T}^{(2)}(\Lambda_0)$, as we wanted to prove.

To finish the proof of statement (2) we must show that there are clutters Λ_0 with $\mathcal{T}^{(1)}(\Lambda_0) \neq \mathcal{T}^{(2)}(\Lambda_0)$. Let us consider the clutter $\Lambda_0 = \{\{1,2,3\},\{1,2,4,\dots,n\}\}$ of the finite set $\Omega = \{1,2,3,4,\dots,n\}$ where $n \geqslant 4$. Then we have that $\mathcal{T}^{(1)}(\Lambda_0) = \{\{1,2,3\}, \{1,2,4,\dots,n\},\{1,3,4,\dots,n\},\{2,3,4,\dots,n\}\}$, whereas $\mathcal{T}^{(2)}(\Lambda_0) = \{\{1,2,3\}, \{1,2,4,\dots,n\},\{3,4,\dots,n\}\}$. So in this case we have that $\mathcal{T}^{(1)}(\Lambda_0) \lneqq \mathcal{T}^{(2)}(\Lambda_0)$. This completes the proof of statement (2).

To finish, let us prove statement (3); namely, we are going to show that in general neither the lexicographic order nor the reverse lexicographic order between r-tuples is preserved by applying $\mathcal{T}$-transformations and, moreover, we prove that there exist clutters Λ for which the $\mathcal{T}$-transformations $\mathcal{T}^{(i_1,\dots,i_r)}(\Lambda)$ and $\mathcal{T}^{(j_1,\dots,j_r)}(\Lambda)$ are not comparable. On the finite set $\Omega = \{1,2,3,4,5\}$, we consider the clutters $\Lambda_1 = \{\{1,2,3\},\{2,3,4\},\{4,5\}\}$, $\Lambda_2 = \{\{1,2,3\},\{1,4,5\},\{3,5\}\}$ and $\Lambda_3 = \{\{1,2,3\},\{2,3,4\},\{3,4,5\}\}$. For the clutter Λ_1 we have that $\mathcal{T}^{(1,2)}(\Lambda_1) = \{\{1,5\},\{4,5\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{2,3,5\}\}$ and that $\mathcal{T}^{(2,1)}(\Lambda_1) = \{\{1,4\},\{1,5\},\{4,5\},\{1,2,3\},\{2,3,4\},\{2,3,5\}\}$, and therefore we have the inequality $\mathcal{T}^{(1,2)}(\Lambda_1) \leqslant \mathcal{T}^{(2,1)}(\Lambda_1)$. However, since the transformations $(1,2)$ and $(2,1)$ of the clutter Λ_2 are $\mathcal{T}^{(1,2)}(\Lambda_2) = \{\{2,4\},\{3,5\},\{1,2,3\},\{1,2,5\},\{1,3,4\},\{1,4,5\}\}$ and $\mathcal{T}^{(2,1)}(\Lambda_2) = \{\{3,5\},\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,4,5\}, \{2,3,4\},\{2,4,5\}\}$, then $\mathcal{T}^{(2,1)}(\Lambda_2) \leqslant \mathcal{T}^{(1,2)}(\Lambda_2)$. On the other hand, the $\mathcal{T}$- transformations $(1,2)$ and $(2,1)$ of the clutter Λ_3 are $\mathcal{T}^{(1,2)}(\Lambda_3) = \{\{1,5\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{2,3,5\},\{2,4,5\}, \{3,4,5\}\}$ and $\mathcal{T}^{(2,1)}(\Lambda_3) = \{\{1,4\},\{2,5\},\{1,2,3\},\{1,3,5\},\{2,3,4\},\{3,4,5\}\}$, and so we have that $\mathcal{T}^{(1,2)}(\Lambda_3) \nleqslant \mathcal{T}^{(2,1)}(\Lambda_3)$ and that $\mathcal{T}^{(2,1)}(\Lambda_3) \nleqslant \mathcal{T}^{(1,2)}(\Lambda_3)$. This completes the proof of the lemma. $\square$

Now let us consider the "stable" value $\mathcal{T}_*^{(\mathcal{I})}(\Lambda)$ of the monotone increasing sequence of clutters $\Lambda \leqslant \mathcal{T}^{(i_1)}(\Lambda) \leqslant \cdots \leqslant \mathcal{T}^{(i_1,\dots,i_r)}(\Lambda) \leqslant \mathcal{T}^{(i_1,\dots,i_r,i_{r+1})}(\Lambda) \leqslant \cdots$ obtained from Λ by using the sequence $\mathcal{I} = (i_1, i_2, \dots, i_r, i_{r+1}, \dots\ )$ where $i_\ell \in \{1,2\}$; that is, if

$r_0 = \min\{r \,:\, \mathcal{T}^{(i_1,\dots,i_r)}(\Lambda) = \mathcal{T}^{(i_1,\dots,i_r i_{r+1})}(\Lambda)\}$, then:

$$\begin{array}{ll} \mathcal{T}_*^{(\mathcal{I})}(\Lambda) = \Lambda & \text{if } \mathcal{T}^{(i_1)}(\Lambda) = \Lambda, \\ \mathcal{T}_*^{(\mathcal{I})}(\Lambda) = \mathcal{T}^{(i_1,\dots,i_{r_0})}(\Lambda) & \text{if } \mathcal{T}^{(i_1)}(\Lambda) \neq \Lambda. \end{array}$$

The main result of this subsection is the following proposition, which states that for any sequence $\mathcal{I}$ the clutter $\mathcal{T}_*^{(\mathcal{I})}(\Lambda)$ defines a matroid.

Proposition 8. *Let Λ be a clutter on a finite set Ω. Let $\mathcal{I} = (i_1, i_2, i_3, \dots)$ be a sequence where $i_\ell \in \{1,2\}$. Then, the clutter $\mathcal{T}_*^{(\mathcal{I})}(\Lambda)$ is a matroidal clutter with $\Lambda \leqslant \mathcal{T}_*^{(\mathcal{I})}(\Lambda)$. Furthermore, if $r_0 = \min\{r \,:\, \mathcal{T}^{(i_1,\dots,i_r)}(\Lambda) = \mathcal{T}^{(i_1,\dots,i_r,i_{r+1})}(\Lambda)\}$, then for $s \leqslant r_0 - 1$ the clutter $\mathcal{T}^{(i_1,\dots,i_s)}(\Lambda)$ is not matroidal.*

Proof. From Lemma 7 we get that $\Lambda \leqslant \mathcal{T}^{(i_1,\dots,i_\ell)}(\Lambda)$ for any ℓ-tuple $(i_1, \dots, i_\ell)$. Therefore, $\Lambda \leqslant \mathcal{T}_*^{(\mathcal{I})}(\Lambda)$. Recall that by definition $\mathcal{T}^{(i_1,\dots,i_\ell)}(\Lambda) = \mathcal{T}^{(i_\ell)}(\mathcal{T}^{(i_1,\dots,i_{\ell-1})}(\Lambda))$. Therefore, to prove the proposition it is enough to demonstrate that if Λ_0 is a clutter on Ω, then $\Lambda_0 = \mathcal{T}^{(1)}(\Lambda_0)$ if and only if Λ_0 is a matroidal clutter, and that $\Lambda_0 = \mathcal{T}^{(2)}(\Lambda_0)$ if and only if Λ_0 is a matroidal clutter. In other words, we must prove that $\Lambda_0 = \mathcal{T}^{(2)}(\Lambda_0)$ if and only if $\Lambda_0 = \mathcal{T}^{(1)}(\Lambda_0)$, if and only if Λ_0 is a matroidal clutter.

First let us show that if $\Lambda_0 = \mathcal{T}^{(2)}(\Lambda_0)$ then $\Lambda_0 = \mathcal{T}^{(1)}(\Lambda_0)$. By applying statements (1) and (2) of Lemma 7 we get that $\Lambda_0 \leqslant \mathcal{T}^{(1)}(\Lambda_0)$ and that $\mathcal{T}^{(1)}(\Lambda_0) \leqslant \mathcal{T}^{(2)}(\Lambda_0)$. Hence, if $\Lambda_0 = \mathcal{T}^{(2)}(\Lambda_0)$, then $\Lambda_0 \leqslant \mathcal{T}^{(1)}(\Lambda_0) \leqslant \mathcal{T}^{(2)}(\Lambda_0) = \Lambda_0$ and so $\Lambda_0 = \mathcal{T}^{(1)}(\Lambda_0)$.

Now let us show that if $\Lambda_0 = \mathcal{T}^{(1)}(\Lambda_0)$ then Λ_0 is a matroidal clutter; that is, we must demonstrate that Λ_0 satisfies the conditions of the set of circuits of a matroid. So let $A_1, A_2 \in \Lambda_0$ be different and let $x \in A_1 \cap A_2$. Since $\Lambda_0 = \mathcal{T}^{(1)}(\Lambda_0)$, there exists $A_3 \in \Lambda_0$ such that $A_3 \subseteq (A_1 \cup A_2) \setminus \{x\}$. Therefore, the circuit conditions are fulfilled.

Finally it is necessary to demonstrate that if Λ_0 is a matroidal clutter then $\Lambda_0 = \mathcal{T}^{(2)}(\Lambda_0)$. Recall that in the proof of Proposition 5 it was stated that a clutter Λ is a matroidal clutter if and only if $I_\Lambda(A_1 \cup A_2) = \emptyset$ for any two different $A_1, A_2 \in \Lambda$. Therefore, if the clutter Λ_0 is a matroidal clutter, then $I_{\Lambda_0}(A_1 \cup A_2) = \emptyset$ if $A_1, A_2 \in \Lambda_0$ are different, and so $\mathcal{T}^{(2)}(\Lambda_0) = \Lambda_0$. This completes the proof of the proposition. $\square$

In some way, the stable value of the above proposition indicates how far Λ is from being a matroid. For instance, from the above proposition it follows that a clutter Λ is the set of circuits of a matroid with ground set Ω if and only if there exists a sequence $\mathcal{I}$ such that $\mathcal{T}_*^{(\mathcal{I})}(\Lambda) = \Lambda$.

A matroid is said to be *connected* if for every pair of distinct elements of the ground set, there is a circuit containing both. A clutter Λ of a finite set Ω is said to be a *matroid port* if it corresponds to the set of circuits of a connected matroid containing a fixed point, that is, if there exists a connected matroid $\mathcal{N}$ with ground set $\Omega \cup \{\omega_0\}$, where $\omega_0 \notin \Omega$, such that $\Lambda = \{C \setminus \{\omega_0\} \,:\, \omega_0 \in C \in \mathcal{C}(\mathcal{N})\}$. In such a case it is said that the clutter Λ is the *port* of the connected matroid $\mathcal{N}$ at the point w_0.

Matroid ports were introduced by Lehman [2] to solve the Shannon switching game. There are several characterizations of these combinatorial objects which range from excluding minors [3, 7] to optimal information rates in secret-sharing schemes [4]. In addition, a combinatorial necessary condition for a clutter to be a matroid port was stated in [5]. Here we present a necessary condition in terms of clutter transformations.

Proposition 9. *Let Λ be a clutter on a finite set Ω. Assume that Λ is a matroid port. Then, $\mathcal{T}_*^{(\mathcal{I})}(\Lambda) = \mathcal{T}^{(i_1)}(\Lambda)$ for any sequence $\mathcal{I} = (i_1, i_2, \dots)$ with $i_\ell \in \{1, 2\}$ and $i_1 = 2$.*

Proof. By assumption, Λ is a matroid port, so there exists a connected matroid $\mathcal{N}$ with ground set $\Omega \cup \{\omega_0\}$, where $\omega_0 \notin \Omega$, such that $\Lambda = \{C \setminus \{\omega_0\} : \omega_0 \in C \in \mathcal{C}(\mathcal{N})\}$. Let us denote $\Lambda = \{A_1, \dots, A_r\}$. On one hand, since the clutter Λ is the port of the matroid $\mathcal{N}$ at the point w_0, the circuits of $\mathcal{N}$ containing the point w_0 are $\{A_1 \cup \{w_0\}, \dots, A_r \cup \{w_0\}\}$. On the other hand, by applying [6, Theorem 4.3.2] it follows that the circuits of $\mathcal{N}$ not containing w_0 are the minimal elements of the form $(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2)$, where $A_1, A_2 \in \Lambda$ are distinct. Therefore we have that $\mathcal{C}(\mathcal{N}) = \{A_1 \cup \{w_0\}, \dots, A_r \cup \{w_0\}\} \cup \min\{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2)$, where $A_1, A_2 \in \Lambda$ are distinct$\}$.

At this point let us consider the matroid $\mathcal{N}/\{\omega_0\}$ obtained by the contraction of the subset $\{w_0\}$ from the matroid $\mathcal{N}$ (see [6, page 104] for the definition of contraction). From [6, Proposition 3.1.11] we know that the circuits of the matroid $\mathcal{N}/\{\omega_0\}$ are the minimal non-empty members of $\{C \setminus \{w_0\}$, where $C \in \mathcal{C}(\mathcal{N})\}$. Hence it follows that $\mathcal{C}(\mathcal{N}/\{\omega_0\}) = \min\big(\Lambda \cup \big\{(A_1 \cup A_2) \setminus I_\Lambda(A_1 \cup A_2)$, where $A_1, A_2 \in \Lambda$ are different$\big\}\big)$; that is, we get that $\mathcal{C}(\mathcal{N}/\{\omega_0\}) = \mathcal{T}^{(2)}(\Lambda)$. In particular, this equality implies that the clutter $\mathcal{T}^{(2)}(\Lambda)$ is a matroidal clutter. So, by applying Proposition 8 to the clutter $\mathcal{T}^{(2)}(\Lambda)$ it follows that $\mathcal{T}_*^{(\mathcal{J})}\big(\mathcal{T}^{(2)}(\Lambda)\big) = \mathcal{T}^{(2)}(\Lambda)$ for any sequence $\mathcal{J} = (j_1, j_2, \dots)$ with $j_\ell \in \{1, 2\}$. Therefore we conclude that $\mathcal{T}_*^{(\mathcal{I})}(\Lambda) = \mathcal{T}^{(i_1)}(\Lambda)$ for any sequence $\mathcal{I} = (i_1, i_2, \dots)$ with $i_\ell \in \{1, 2\}$ and $i_1 = 2$, as we wished to prove. □

The clutters in the following example show that the necessary condition of the above proposition is not sufficient, and that there is no analogous result if we use the first elementary transformation instead of the second one.

Example 10. Let us consider the clutter $\Lambda = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}$ of the finite set $\Omega = \{1, 2, 3, 4\}$. As shown in [7], this clutter is not a matroid port. However, it is not hard to check that $\mathcal{T}_*^{(\mathcal{I})}(\Lambda) = \mathcal{T}^{(2)}(\Lambda) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ for any sequence $\mathcal{I} = (i_1, i_2, \dots)$. Therefore, the necessary condition of Proposition 9 is not sufficient. Now, in order to prove that the proposition does not work with the first elementary $\mathcal{T}$-transformation, we consider the clutter $\Lambda = \{\{1, 2, 3\}, \{2, 3, 4, 5\}, \{2, 3, 5, 6\}\}$ of the finite set $\Omega = \{1, 2, 3, 4, 5, 6\}$. Observe that $\Lambda = \{C \setminus \{7\} : 7 \in C \in \mathcal{C}(\mathcal{N})\}$, where $\mathcal{N}$ is the connected matroid with ground set $\Omega \cup \{7\}$ and set of circuits $\mathcal{C}(\mathcal{N}) = \{\{1, 2, 3, 7\}, \{2, 3, 4, 5, 7\}, \{2, 3, 5, 6, 7\}, \{1, 4, 5\}, \{1, 5, 6\}, \{4, 6\}\}$. Therefore, the clutter Λ is a matroid port. However, it is straightforward to check that the first elementary transformation of the clutter Λ is $\mathcal{T}^{(1)}(\Lambda) = \{\{1, 2, 3\}, \{1, 2, 4, 5\}, \{1, 2, 5, 6\}, \{1, 3, 4, 5\}, \{1, 3, 5, 6\}, \{2, 3, 4, 5\}, \{2, 3, 4, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}\}$. Observe that $\mathcal{T}^{(1)}(\Lambda)$ is not a matroidal-clutter because $(\{1, 2, 3\} \cap \{2, 3, 4, 6\}) \setminus \{3\} = \{1, 2, 4, 6\} \notin \mathcal{T}^{(1)}(\Lambda)^+$. Therefore, by applying Proposition 8 it follows that $\mathcal{T}_*^{(\mathcal{I})}(\Lambda) \neq \mathcal{T}^{(i_1)}(\Lambda)$ for any sequence $\mathcal{I} = (i_1, i_2, \dots)$ with $i_\ell \in \{1, 2\}$ and $i_1 = 1$.

To conclude this subsection, we focus our attention on the matroid implicit on Proposition 8. Let us denote by $\mathcal{M}^{(\mathcal{I})}(\Lambda)$ the unique matroid with ground set Ω and set of circuits $\mathcal{C}(\mathcal{M}^{(\mathcal{I})}(\Lambda)) = \mathcal{T}_*^{(\mathcal{I})}(\Lambda)$. Since $\Lambda \leqslant \mathcal{T}_*^{(\mathcal{I})}(\Lambda)$, the matroid $\mathcal{M}^{(\mathcal{I})}(\Lambda)$ is a matroidal completion of Λ; that is, $\mathcal{M}^{(\mathcal{I})}(\Lambda) \in \mathrm{Mat}(\Lambda)$. The matroids obtained in this way will be called *$\mathcal{T}$-matroidal completions of* Λ.

Let us show that there exists no general result concerning the comparison between two different $\mathcal{T}$-matroidal completions of a clutter. The three clutters in the following example illustrate this fact.

Example 11. On the finite set $\Omega = \{1, 2, 3, 4, 5\}$ let us consider the clutters Λ_1, Λ_2 and Λ_3 where $\Lambda_1 = \{\{1, 2, 3\}, \{2, 3, 4, 5\}\}$, $\Lambda_2 = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}\}$ and $\Lambda_3 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}\}$. It is a straightforward calculation to check that the clutter Λ_1 has only two $\mathcal{T}$-matroidal completions $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$, while for $i = 2, 3$ the clutter Λ_i has exactely three $\mathcal{T}$-matroidal completions $\mathcal{M}_{i,1}$, $\mathcal{M}_{i,2}$ and $\mathcal{M}_{i,3}$. Namely, if $\mathcal{I} = (i_1, i_2, i_3, \dots)$ is a sequence with $i_\ell \in \{1, 2\}$, then:

- the $\mathcal{T}$-matroidal completions of the clutter Λ_1 are $\mathcal{M}^{(\mathcal{I})}(\Lambda_1) = \mathcal{M}_{1,1}$ if $i_1 = 1$ while $\mathcal{M}^{(\mathcal{I})}(\Lambda_1) = \mathcal{M}_{1,2}$ if $i_1 \neq 1$, where $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ are the matroids with ground set Ω and circuits $\mathcal{C}(\mathcal{M}_{1,1}) = \{\{1, 2, 3\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$ and $\mathcal{C}(\mathcal{M}_{1,2}) = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 4, 5\}\}$.

- the $\mathcal{T}$-matroidal completions of Λ_2 are $\mathcal{M}^{(\mathcal{I})}(\Lambda_2) = \mathcal{M}_{2,1} = \mathcal{U}_{2,5}$ the uniform matroid if $\mathcal{I} = (1, 1, i_3, \dots)$; while if $\mathcal{I} = (1, 2, i_3, \dots)$ then $\mathcal{M}^{(\mathcal{I})}(\Lambda_2) = \mathcal{M}_{2,2}$ is the matroid with set of circuits $\mathcal{C}(\mathcal{M}_{2,2}) = \{\{1, 5\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$; whereas if $\mathcal{I} = (2, i_2, i_3, \dots)$ then $\mathcal{M}^{(\mathcal{I})}(\Lambda_2) = \mathcal{M}_{2,3}$ is the matroid with set of circuits $\mathcal{C}(\mathcal{M}_{2,3}) = \{\{1, 4\}, \{2, 5\}, \{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}\}$.

- the $\mathcal{T}$-matroidal completions of Λ_3 are $\mathcal{M}^{(\mathcal{I})}(\Lambda_3) = \mathcal{M}_{3,1} = \mathcal{U}_{2,5}$ the uniform matroid if $\mathcal{I} = (1, 1, i_3, \dots)$; while if $\mathcal{I} = (1, 2, i_3, \dots)$ then $\mathcal{M}^{(\mathcal{I})}(\Lambda_3) = \mathcal{M}_{3,2}$ is the matroid with circuits $\mathcal{C}(\mathcal{M}_{3,2}) = \{\{1, 2\}, \{1, 5\}, \{2, 5\}, \{1, 3, 4\}, \{2, 3, 4\}, \{3, 4, 5\}\}$; whereas if $\mathcal{I} = (2, i_2, i_3, \dots)$ then $\mathcal{M}^{(\mathcal{I})}(\Lambda_3) = \mathcal{M}_{3,3}$ is the matroid with set of circuits $\mathcal{C}(\mathcal{M}_{3,3}) = \{\{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$.

Then we have that the $\mathcal{T}$-matroidal completions of Λ_1 form a chain $\mathcal{M}_{1,1} \leqslant \mathcal{M}_{1,2}$, while there are $\mathcal{T}$-matroidal completions of the clutters Λ_2 and Λ_3 that are not comparable. To be precise, for the clutter Λ_2 we have $\mathcal{M}_{2,1} \leqslant \mathcal{M}_{2,2}$ and $\mathcal{M}_{2,1} \leqslant \mathcal{M}_{2,3}$, but $\mathcal{M}_{2,2} \nleqslant \mathcal{M}_{2,3}$ and $\mathcal{M}_{2,3} \nleqslant \mathcal{M}_{2,2}$, while for the clutter Λ_3 we have $\mathcal{M}_{3,1} \leqslant \mathcal{M}_{3,2}$, but $\mathcal{M}_{3,i} \nleqslant \mathcal{M}_{3,3}$ and $\mathcal{M}_{3,3} \nleqslant \mathcal{M}_{3,i}$ for $i = 1, 2$. Observe that the clutters Λ_1 and Λ_2 have only one minimal $\mathcal{T}$-matroidal completion, but the clutter Λ_3 has two minimal $\mathcal{T}$-matroidal completions.

4 Minimal matroidal completions of a clutter

The set $\mathrm{Mat}(\Lambda)$ of all the matroidal completions of a clutter Λ is a non-empty partially ordered set, the *poset of matroids of the clutter* Λ. Therefore, the minimal elements of this poset will be the *minimal matroidal completions* of the clutter. In this section we present two results concerning minimal matroidal completions. The first one (Theorem 12) deals with the number of minimal matroidal completions of a clutter, while in the second (Theorem 13) we focus our attention on how the minimal matroidal completions can be obtained.

In general, the poset $\mathrm{Mat}(\Lambda)$ is not a totally ordered set (see Example 11). Therefore, we do not know how many minimal elements this poset has. Our first result states that the non-matroidal clutters have at least two minimal matroidal completions.

Theorem 12. *Let Λ be a clutter. Then, the poset $\big(\mathrm{Mat}(\Lambda), \leqslant\big)$ has a unique minimal element if and only if Λ is a matroidal clutter.*

Proof. If Λ is a matroidal clutter, then there exists a matroid $\mathcal{M}_0$ such that $\mathcal{C}(\mathcal{M}_0) = \Lambda$, and hence $\min\big(\mathrm{Mat}(\Lambda)\big) = \{\mathcal{M}_0\}$. Let us show that the converse is true. So let Λ be a clutter and assume that there exists a matroid $\mathcal{M}$ such that $\min\big(\mathrm{Mat}(\Lambda)\big) = \{\mathcal{M}\}$. In such a case, it is necessary to demonstrate that Λ is a matroidal clutter.

To do this we consider the blocker $b(\Lambda)$ of the clutter Λ. The blocker of the clutter Λ is defined as the clutter $b(\Lambda) = \min\{B \subseteq \Omega \,:\, B \cap A \neq \emptyset \text{ for all } A \in \Lambda\}$. It is well known that $b(b(\Lambda)) = \Lambda$ (see for instance [6, Proposition 2.1.12]). Thus, if X is a subset of Ω such that $X \cap B \neq \emptyset$ for all $B \in b(\Lambda)$, then $X \in \Lambda^+$.

Let us denote $b(\Lambda) = \{B_1, \ldots, B_s\}$. For $1 \leqslant i \leqslant s$ let us consider the matroid $\mathcal{M}_{B_i}$ with ground set Ω and set of circuits $\mathcal{C}(\mathcal{M}_{B_i}) = \{\{x\} \,:\, x \in B_i\}$. Since $B_i \in b(\Lambda)$, then $A \cap B_i \neq \emptyset$ for all $A \in \Lambda$. Thus, $\Lambda \leqslant \mathcal{M}_{B_i}$, and therefore $\mathcal{M} \leqslant \mathcal{M}_{B_i}$ because we are assuming $\min\big(\mathrm{Mat}(\Lambda)\big) = \{\mathcal{M}\}$. Let $C \in \mathcal{C}(\mathcal{M})$ be a circuit of the matroid $\mathcal{M}$. Since $\mathcal{M} \leqslant \mathcal{M}_{B_i}$, there exists a circuit $C_i \in \mathcal{C}(\mathcal{M}_{B_i})$ such that $C_i \subseteq C$, and so $C \cap B_i \neq \emptyset$. Therefore, if $C \in \mathcal{C}(\mathcal{M})$ then $C \cap B_i \neq \emptyset$ for $i = 1, \ldots, s$. Hence, it follows that $C \in \Lambda^+$ because $b(\Lambda) = \{B_1, \ldots, B_s\}$. Therefore we have that $\mathcal{C}(\mathcal{M}) \subseteq \Lambda^+$, and thus $\mathcal{C}(\mathcal{M}) \leqslant \Lambda$ (see Lemma 1). But the matroid $\mathcal{M}$ is a matroidal completion of Λ, so $\Lambda \leqslant \mathcal{C}(\mathcal{M})$. Therefore $\Lambda = \mathcal{C}(\mathcal{M})$, as we wished to prove. □

The following result concerns non-matroidal clutters; namely, it states that any minimal matroidal completion of the clutter can be obtained by combining the transformations of the previous section.

Theorem 13. *Let Λ be a non-matroidal clutter on a finite set Ω and let $\mathcal{M}$ be a minimal element of the poset of matroids $\big(\mathrm{Mat}(\Lambda), \leqslant\big)$. Then there is a monotone increasing sequence of clutters $\Lambda = \Lambda_0 \lneqq \Lambda_1 \lneqq \cdots \lneqq \Lambda_r = \mathcal{C}(\mathcal{M})$ such that for $i \geqslant 1$, either Λ_i is an I-transformation of Λ_{i-1} or Λ_i is a $\mathcal{T}$-transformation of Λ_{i-1}.*

Proof. It suffices to prove that if Λ' is a non-matroidal clutter on Ω, and if $\mathcal{N}$ is a matroidal completion of Λ', then either there exists an I-transformation Λ'_1 of Λ such that $\Lambda'_1 \leqslant \mathcal{N}$, or there exists a $\mathcal{T}$-transformation Λ'_1 of Λ' such that $\Lambda'_1 \leqslant \mathcal{N}$.

So, let Λ' be a non-matroidal clutter on Ω and let $\mathcal{N}$ be a matroidal completion of Λ'. Let us assume that there exists no $\mathcal{T}$-transformation Λ'_1 of Λ' with $\Lambda'_1 \leqslant \mathcal{N}$. In such a case, we must demonstrate that there exists an I-transformation Λ'_1 of Λ' such that $\Lambda'_1 \leqslant \mathcal{N}$.

By assumption, there exists no $\mathcal{T}$-transformation Λ'_1 of Λ' with $\Lambda'_1 \leqslant \mathcal{N}$. In particular, we get that $\mathcal{T}^{(1)}(\Lambda') \nleqslant \mathcal{N}$. Therefore, we have that $\mathcal{T}^{(1)}(\Lambda') \nleqslant \mathcal{N}$ and $\Lambda' \leqslant \mathcal{N}$, and hence it follows that there exists $X \in \mathcal{T}^{(1)}(\Lambda') \setminus \Lambda'$ such that $C \nsubseteq X$ if $C \in \mathcal{C}(\mathcal{N})$. Since $X \in \mathcal{T}^{(1)}(\Lambda') \setminus \Lambda'$, then there are two different subsets $A_1, A_2 \in \Lambda'$ and there exists an element $x \in A_1 \cap A_2$ such that $X = (A_1 \cup A_2) \setminus \{x\} \in \mathcal{T}^{(1)}(\Lambda')$. On one hand, we have that $A_i \in \Lambda' \leqslant \mathcal{N}$ for $i = 1, 2$. On the other, $C \nsubseteq X = (A_1 \cup A_2) \setminus \{x\}$ if $C \in \mathcal{C}(\mathcal{N})$. Therefore, for $i = 1, 2$ there exists a circuit $C_i \in \mathcal{C}(\mathcal{N})$ with $C_i \subseteq A_i$ and such that $x \in C_i$. At this point, notice that if $C_1 \neq C_2$, then there exists $C \in \mathcal{C}(\mathcal{N})$ such that $C \subseteq (C_1 \cup C_2) \setminus \{x\}$, and so there exists $C \in \mathcal{C}(\mathcal{N})$ such that $C \nsubseteq (A_1 \cup A_2) \setminus \{x\} = X$, a contradiction. Therefore, we conclude that $C_1 = C_2$. Let us denote $C_0 = C_1 = C_2$. Then we have that $C_0 \subseteq A_1 \cap A_2$. Therefore, $\Lambda'_1 = \min\big(\Lambda \cup \{A_1 \cap A_2\}\big) \leqslant \mathcal{N}$. Now the proof of our claim will be completed by showing that $I_{\Lambda'}(A_1 \cup A_2) \neq \emptyset$. If $I_{\Lambda'}(A_1 \cup A_2) = \emptyset$, then there exists $A_3 \in \Lambda'$ with $A_3 \subseteq A_1 \cup A_2$ and such that $x \notin A_3$. Thus, $A_3 \subseteq (A_1 \cup A_2) \setminus \{x\} = X$, which is a contradiction because $A_3 \in \Lambda'$ and $X \in \mathcal{T}^{(1)}(\Lambda') \setminus \Lambda'$. This completes the proof of our claim, and thereby the proof of the theorem. □

To conclude we provide two examples. In the first one the clutter has two minimal matroidal completions and they are obtained by I-transformations or by $\mathcal{T}$-transformations, while in the second the clutter has four minimal matroidal completions, one of which is obtained by combining both kinds of transformations.

Example 14. First let us consider the clutter $\Lambda = \{\{1,2,3\},\{1,2,4\}\}$ of the finite set $\Omega = \{1,2,3,4\}$. In this case, the I-transformation of Λ is $\Lambda_1 = \{\{1,2\}\}$, the first elementary transformation of Λ is $\Lambda_2 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$, and the second elementary transformation is $\Lambda_3 = \{\{1,2,3\},\{1,2,4\},\{3,4\}\}$. Observe that Λ_1, Λ_2 and Λ_3 are matroidal clutters. Therefore, from Theorem 13 it follows that the minimal matroidal completions of Λ are the minimal elements of $\{\Lambda_1, \Lambda_2, \Lambda_3\}$. In this case, $\Lambda_2 \leqslant \Lambda_3$, and so $\min\big(\mathrm{Mat}(\Lambda)\big) = \{\mathcal{M}_1, \mathcal{M}_2\}$ where $\mathcal{M}_i$ is the matroid with set of circuits Λ_i.

Example 15. Finally, on the finite set $\Omega = \{1,2,3,4,5\}$, we consider the clutter $\Lambda = \{\{1,2,3\},\{1,2,4\},\{1,5\},\{4,5\}\}$. In such a case, eleven matroidal clutters $\Lambda_1, \ldots, \Lambda_{11}$ can be obtained by using or by combining I-transformations and $\mathcal{T}$-transformations. Namely, by using only I-transformations we obtain the matroidal clutters $\Lambda_1 = \{\{5\},\{1,2\}\}$ and $\Lambda_2 = \{\{1\},\{4,5\}\}$. The matroidal clutters obtained by using only $\mathcal{T}$-transformations are the clutters $\Lambda_3 = \{\{1,4\},\{1,5\},\{4,5\},\{1,2,3\},\{2,3,4\},\{2,3,5\}\}$ and $\Lambda_4 = \{\{1,3\}, \{1,4\},\{1,5\},\{3,4\},\{3,5\},\{4,5\}\}$, whereas the matroidal clutters obtained by combining the I-transformations and the $\mathcal{T}$-transformations are the clutters $\Lambda_5 = \{\{1\},\{5\}\}$, $\Lambda_6 = \{\{1\},\{2,4\},\{2,5\},\{4,5\}\}$, $\Lambda_7 = \{\{5\},\{1,2\},\{1,4\},\{2,4\}\}$, $\Lambda_8 = \{\{1,2\},\{1,4\},\{1,5\}, \{2,4\},\{2,5\},\{4,5\}\}$, $\Lambda_9 = \{\{5\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$, $\Lambda_{10} = \{\{5\},\{3,4\}, \{1,2,3\},\{1,2,4\}\}$, and $\Lambda_{11} = \{\{4\},\{1,5\},\{1,2,3\},\{2,3,5\}\}$. Therefore, if we denote by $\mathcal{M}_i$ the matroid with set of circuits Λ_i, then by applying Theorem 13 we get that the set of minimal matroidal completions of Λ is $\min\big(\mathrm{Mat}(\Lambda)\big) = \min\{\mathcal{M}_1, \ldots, \mathcal{M}_{11}\} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_9\}$.

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