On a sumset problem for integers

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Abstract

Let A be a finite set of integers. We show that if k is a prime power or a product of two distinct primes then

$$|A + k \cdot A| \geqslant (k+1)|A| - \lceil k(k+2)/4 \rceil$$

provided $|A| \geqslant (k-1)^2 k!$, where $A + k \cdot A = \{a + kb : \ a, b \in A\}$. We also establish the inequality $|A + 4 \cdot A| \geqslant 5|A| - 6$ for $|A| \geqslant 5$.

Keywords: Additive combinatorics, sumsets

1 Introduction

For finite subsets $A_1, \ldots, A_k$ of $\mathbb{Z}$, their *sumset* is given by

$$A_1 + \cdots + A_k = \{a_1 + \cdots + a_k : \ a_1 \in A_1, \ldots, a_k \in A_k\},$$

which is simply denoted by kA if $A_1 = \cdots = A_k = A$. It is known that

$$|A_1 + \cdots + A_k| \geqslant |A_1| + \cdots + |A_k| - k + 1,$$

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and equality holds when $A_1, \ldots, A_k$ are arithmetic progressions with the same common difference (see, e.g., Nathanson [7, p.11]).

Let A be a finite set of integers. For $k \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we define

$$k \cdot A := \{ka : a \in A\}$$

which is called a *dilate* of A. Let $k_1, k_2, \ldots, k_l \in \mathbb{Z}^+$. Recently lower bounds for $|k_1 \cdot A + k_2 \cdot A + \cdots + k_l \cdot A|$ were investigated by various authors [1, 2, 8, 9]. In the case $(k_1, k_2, \ldots, k_l) = 1$ (where $(k_1, \ldots, k_l)$ refers to the greatest common divisor of $k_1, \ldots, k_l$), Bukh [2] obtained the following inequality:

$$|k_1 \cdot A + k_2 \cdot A + \cdots + k_l \cdot A| \geqslant (k_1 + k_2 + \cdots + k_l)|A| - o(|A|).$$

For $l = 2$ there are better quantitative results in this direction, see [3, 4, 5, 7, 9]. It was conjectured in [4] that for any $k \in \mathbb{Z}^+$ if $|A|$ is sufficiently large then

$$|A + k \cdot A| \geqslant (k+1)|A| - \lceil k(k+2)/4 \rceil.$$

This was proved in [3] with k prime. In this paper we confirm the conjecture for $k = p^\alpha$ as well as $k = p_1 p_2$, where p, p_1, p_2 are prime numbers and $\alpha \in \mathbb{Z}^+$. Motivated by the preprint form of our paper posted to arXiv, Ljujić [6] obtained similar results for $|2 \cdot A + k \cdot A|$ with k a prime power or a product of two distinct primes.

We remark that there are also some researches on sums of dilates in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ with p a prime, see Plagne [10] and Pontiveros [11].

Now we state our main theorems.

Theorem 1. *Let $k = p^\alpha$ with p a prime and $\alpha \in \mathbb{Z}^+$. Let A be a finite subset of $\mathbb{Z}$ with $|A| \geqslant (k-1)^2 k!$. Then*

$$|A + k \cdot A| \geqslant (k+1)|A| - \lceil k(k+2)/4 \rceil. \tag{1}$$

Theorem 2. *Let p_1 and p_2 be distinct primes and $k = p_1 p_2$. And let A be a finite subset of $\mathbb{Z}$ with $|A| \geqslant (k-1)^2 k!$. Then*

$$|A + k \cdot A| \geqslant (k+1)|A| - \lceil k(k+2)/4 \rceil. \tag{2}$$

By Theorem 1, if $k = 4$ then (1) holds when $|A| \geqslant 216$. In fact, we have the following refinement.

Theorem 3. *For any finite set $A \subseteq \mathbb{Z}$ with $|A| \geqslant 5$, we have*

$$|A + 4 \cdot A| \geqslant 5|A| - 6. \tag{3}$$

We remark that the lower bound given in (1) is optimal when $|A|$ is large enough. Moreover, equality holds if A has the form $k \cdot \{0, 1, \ldots, n\} + \{0, 1, \ldots, h\}$, where

$$h = \begin{cases} k/2 \text{ or } (k+2)/2 & \text{if } k \text{ is even,} \\ (k+1)/2 & \text{if } k \text{ is odd.} \end{cases}$$

Our proofs of Theorems 1-3 are based on the technical approach of [3]. Our key new idea is to employ Chowla's theorem to handle the case when k is a prime power, and use a lemma similar to Chowla's theorem to handle the case when k is a product of two distinct primes.

2 Preliminaries

Throughout this paper we use the following notations. For a finite set $A \subseteq \mathbb{Z}$ with $|A| > 1$ and a positive integer k, we define

$$\hat{A} = \{\bar{a} = a + k\mathbb{Z} : a \in A\}.$$

Let $h = |\hat{A}|$ and let $A_1, A_2, \ldots, A_h$ be the distinct classes of A modulo k. Write $A_i = k \cdot X_i + r_i$ with $0 \leqslant r_i < k$ for $i = 1, 2, \ldots, h$. Clearly $|A_i| = |X_i|$ and

$$A = \bigcup_{i=1}^{h} A_i = \bigcup_{i=1}^{h} (k \cdot X_i + r_i).$$

Define

$$F = \{1 \leqslant i \leqslant h : |\hat{X}_i| = k\}, \qquad E = \{1 \leqslant i \leqslant h : 0 < |\hat{X}_i| < k\}$$

and

$$\triangle_{rs} = (A_r + k \cdot A) \backslash (A_r + k \cdot A_s) \quad \text{for} \quad r, s = 1, 2, \ldots, h.$$

Without loss of generality, we make the following assumptions:

(I) $\gcd(A) = \gcd(\{a : a \in A\}) = 1$.
If $d = \gcd(A) > 1$, then replace A by $A' = \{a/d : a \in A\}$. Obviously $|A'| = |A|$ and

$$|A' + k \cdot A'| = |A + k \cdot A|.$$

(II) $r_1 = 0$ and $|A_1| \geqslant |A_2| \geqslant \cdots \geqslant |A_h|$.
In fact, for $A' = A - r_1$ we have $|A'| = |A|$ and $|A' + k \cdot A'| = |A + k \cdot A|$.

(III) $h = |\hat{A}| \geqslant 2$.
When $h = 1$ we have $A = A_1 = k \cdot X_1 + r_1$ and $|A + k \cdot A| = |X_1 + k \cdot X_1|$. So we may replace A by X_1, and continue this process until $|\hat{X}_1| > 1$.

Lemma 4 (cf. [4]). *For arbitrary nonempty sets B and $A = \bigcup_{i=1}^{h} (k \cdot X_i + r_i)$, we have*
(i) $|A + k \cdot B| = \sum_{i=1}^{h} |X_i + B|$.
(ii) $|A + k \cdot B| \geqslant |A| + h(|B| - 1)$.
(iii) Furthermore, if equality holds in (ii), then either $|B| = 1$ or $|X_i| = 1$ for all $i = 1, \ldots, h$ or B and all the sets X_i with more than one element are arithmetic progressions with the same difference.

Lemma 5 (I. Chowla, see [7]). *For $n \geqslant 2$, let A and B be nonempty subsets of $\mathbb{Z}/n\mathbb{Z}$. If $0 \in B$ and $(b, n) = 1$ for all $b \in B \backslash \{0\}$, then*

$$|A + B| \geqslant \min\{n, |A| + |B| - 1\}.$$

Lemma 6 (cf. [3]). *For each subset $I \subseteq \{1, 2, \ldots, h\}$, we have*

$$\sum_{i \in I} |\triangle_{ii}| \geqslant |I|(|I| - 1).$$

3 Proof of Theorem 1

In this section, we fix $k = p^\alpha$ where p is a prime and $\alpha \in \mathbb{Z}^+$. Let A be a nonempty finite subset of integers. Note that the set $\{1 \leqslant i \leqslant h : p \nmid r_i\}$ is nonempty since $\gcd(A) = 1$. Define $m = \min\{1 \leqslant i \leqslant h : p \nmid r_i\}$.

Lemma 7. *Suppose that A is a nonempty finite subset of integers.*
(i) In the case $i \in E \backslash \{m\}$, we have $| \vartriangle_{ii} | \geqslant |A_m|$.
(ii) If $|\hat{X_m}| + m - 1 \leqslant k$, then

$$| \vartriangle_{mm} | \geqslant |A_1| + \ldots + |A_{m-1}|. \tag{4}$$

Else if $|\hat{X_m}| + m - 1 > k$, then we have

$$|A_m + A| \geqslant (k+1)|A_m| + m(|A_1| - |A_m|) - k. \tag{5}$$

Proof. (i) Suppose $i \in E \backslash \{m\}$. Noting that $p \nmid r_m$, we have $(r_m - r_i, k) = 1$ when $p \mid r_i$. Applying Lemma 5, we get

$$|\hat{X}_i + \{0, r_m - r_i\}| \geqslant \min\{k, |\hat{X}_i| + 2 - 1\} = |\hat{X}_i| + 1$$

since $i \in E$. It follows that

$$|(\hat{X}_i + \hat{A_m}) \backslash (\hat{X}_i + \hat{A}_i)| \geqslant 1,$$

and hence,

$$\begin{aligned} |\vartriangle_{ii}| &= |(A_i + k \cdot A) \backslash (A_i + k \cdot A_i)| \\ &= |(X_i + A) \backslash (X_i + A_i)| \\ &\geqslant |(X_i + A_m) \backslash (X_i + A_i)| \\ &\geqslant |(\hat{X}_i + \hat{A_m}) \backslash (\hat{X}_i + \hat{A}_i)| \cdot |A_m| \\ &\geqslant |A_m| \quad (\text{since } |\hat{A}_i| = |\hat{A_m}| = 1). \end{aligned}$$

In the case $p \nmid r_i$, using Lemma 5 we obtain

$$|\hat{X}_i + \{0, r_i\}| \geqslant \min\{k, |\hat{X}_i| + 2 - 1\} = |\hat{X}_i| + 1,$$

hence $|(\hat{X}_i + \hat{A}_1) \backslash (\hat{X}_i + \hat{A}_i)| \geqslant 1$ and

$$|\vartriangle_{ii}| = |(A_i + k \cdot A) \backslash (A_i + k \cdot A_i)| \geqslant |(X_i + A_1) \backslash (X_i + A_i)| \geqslant |A_1| \geqslant |A_m|.$$

(ii) Recall that $p \mid r_1, \ldots, \ p \mid r_{m-1}$ and $p \nmid r_m$. Thus

$$(r_1 - r_m, k) = \cdots = (r_{m-1} - r_m, k) = 1.$$

It follows from Lemma 5 that

$$|\hat{X_m} + \{0, r_1 - r_m, r_2 - r_m, \ldots, r_t - r_m\}| \ \geqslant \min\{k, \ |\hat{X_m}| + (t+1) - 1|\}.$$

So we have

$$|\hat{X_m} + (\hat{A}_1 \cup \cdots \cup \hat{A}_t \cup \hat{A_m})| \geqslant \min\{k,\ |\hat{X_m}| + t\} \ \text{ for } \ t = 1, 2, \ldots, m-1.$$

If $|\hat{X_m}| + m - 1 \leqslant k$, then by induction on t we deduce that

$$|(X_m + A_1 \cup \cdots \cup A_t \cup A_m) \backslash (X_m + A_m)| \geqslant |A_1| + \cdots + |A_t|$$

for $t = 1, 2, \ldots, m-1$. Consequently,

$$|\triangle_{mm}| = |(X_m + A) \backslash (X_m + A_m)| \geqslant |A_1| + \ldots + |A_{m-1}|.$$

If $|\hat{X_m}| + m - 1 > k$, then $|\hat{X_m} + (\hat{A}_1 \cup \ldots \cup \hat{A_m})| = k$. With the help of Lemmas 4 and 5, we get

$$\begin{aligned}
|X_m + A| &\geqslant |X_m + A_1| + |(X_m + A_1 \cup \cdots \cup A_m) \backslash (X_m + A_1)| \\
&\geqslant |X_m + k \cdot X_1| + |(\hat{X_m} + \hat{A}_1 \cup \cdots \cup \hat{A_m}) \backslash (\hat{X_m} + \hat{A_1})| \cdot |A_m| \\
&\geqslant |X_m| + |\hat{X_m}|(|A_1| - 1) + (k - |\hat{X_m}|)|A_m| \\
&\geqslant (k+1)|A_m| + |\hat{X_m}|(|A_1| - |A_m|) - k.
\end{aligned}$$

The definition of m implies that $m \leqslant p^{\alpha-1} + 1$ and hence

$$|\hat{X_m}| > k + 1 - m \geqslant p^{\alpha} + 1 - p^{\alpha-1} - 1 \geqslant p^{\alpha-1} \geqslant m - 1.$$

Thus,

$$|X_m + A| \geqslant (k+1)|A_m| + m(|A_1| - |A_m|) - k.$$

□

Lemma 8. *Let A be a nonempty finite subset of $\mathbb{Z}$. Then*

$$|A + k \cdot A| \geqslant (k+1)|A| - k!.$$

Proof. It suffices to prove by induction that

$$|A + k \cdot A| \geqslant (t+1)|A| - (t-1)!k. \tag{6}$$

holds for every $t = 1, \ldots, k$.

Clearly (6) is true for $t = 1$ since it is known that

$$|A + k \cdot A| \geqslant 2|A| - 1 \geqslant 2|A| - k.$$

Now suppose that (6) holds for some $1 \leqslant t < k$. We want to deduce (6) with t replaced by $t + 1$, i.e., the inequality

$$|A + k \cdot A| \geqslant (t+2)|A| - t!k.$$

If $h > t$, then applying Lemma 4 we immediately get

$$|A + k \cdot A| \geqslant |A| + h(|A| - 1) \geqslant |A| + (t+1)(|A| - 1) \geqslant (t+2)|A| - t!k.$$

Below we assume $h \leqslant t$. By Lemma 4, for $i \in F$ we have

$$\begin{aligned}|A_i + k \cdot A| &\geqslant |X_i + k \cdot X_1| \\ &\geqslant |X_i| + k|X_1| - k \\ &\geqslant |A_i| + (t+1)|A_1| - k \\ &\geqslant (t+1)|A_i| + |A_1| - k.\end{aligned}$$

By the induction hypothesis and Lemma 7, for $i \in E \setminus \{m\}$ we get

$$|A_i + k \cdot A| = |A_i + k \cdot A_i| + |\triangle_{ii}| \geqslant (t+1)|A_i| - (t-1)!k + |A_m|.$$

Therefore,

$$\begin{aligned}\sum_{i \neq m} |A_i + k \cdot A| &= \sum_{i \in F \setminus \{m\}} |A_i + k \cdot A| + \sum_{i \in E \setminus \{m\}} |A_i + k \cdot A| \\ &\geqslant (t+1) \sum_{i \neq m} |A_i| + |F \setminus \{m\}||A_1| + |E \setminus \{m\}||A_m| \\ &\quad - \Big(k|F \setminus \{m\}| + |E \setminus \{m\}|(t-1)!k\Big).\end{aligned}$$

We divide the following discussion into two cases.

Case 1. $|\hat{X_m}| + m - 1 \leqslant k$.

In this case, by (4) and the induction hypothesis, we have

$$\begin{aligned}|A_m + k \cdot A| &= |A_m + k \cdot A_m| + |\triangle_{mm}| \\ &\geqslant (t+1)|A_m| - (t-1)!k + |A_1| + \cdots + |A_{m-1}|.\end{aligned}$$

It follows that

$$\begin{aligned}|A + k \cdot A| &= \sum_{i \neq m} |A_i + k \cdot A| + |A_m + k \cdot A| \\ &\geqslant (t+1)|A| + |F \setminus \{m\}||A_1| + |E \setminus \{m\}||A_m| + |A_1| + \ldots + |A_{m-1}| \\ &\quad - \Big(k|F \setminus \{m\}| + |E \setminus \{m\}|(t-1)!k + (t-1)!k\Big).\end{aligned}$$

Clearly,

$$|F \setminus \{m\}||A_1| + |E \setminus \{m\}||A_m| + |A_1| + \ldots + |A_{m-1}| \geqslant |A|$$

and

$$k|F \setminus \{m\}| + |E \setminus \{m\}|(t-1)!k + (t-1)!k \leqslant (t-1)!kh \leqslant t!k.$$

Hence

$$|A + k \cdot A| \geqslant (t+2)|A| - t!k.$$

Case 2. $|\hat{X_m}| + m - 1 > k$.

We obtain from (5) that

$$\begin{aligned}|A + k\cdot A| &= \sum_{i\neq m} |A_i + k\cdot A| + |A_m + k\cdot A| \\ &\geqslant \sum_{i\neq m} |A_i + k\cdot A| + (k+1)|A_m| + m(|A_1| - |A_m|) - k \\ &\geqslant (t+1)|A| + |F\backslash\{m\}||A_1| + |E\backslash\{m\}||A_m| + |A_m| + m(|A_1| - |A_m|) \\ &\quad - \Big(k|F\backslash\{m\}| + |E\backslash\{m\}|(t-1)!k + k\Big).\end{aligned}$$

As $|A_1| \geqslant |A_2| \geqslant \cdots \geqslant |A_h|$, we have

$$\begin{aligned}&|F\backslash\{m\}||A_1| + |E\backslash\{m\}||A_m| + |A_m| + m(|A_1| - |A_m|) \\ \geqslant&(|F| + |E|)|A_m| + m(|A_1| - |A_m|) \\ =&h|A_m| + m(|A_1| - |A_m|) = m|A_1| + (h-m)|A_m| \\ \geqslant&|A_1| + |A_2| + \cdots + |A_m| + |A_{m+1}| + \cdots + |A_h| = |A|\end{aligned}$$

and

$$k|F\backslash\{m\}| + |E\backslash\{m\}|(t-1)!k + k \leqslant (t-1)!kh \leqslant t!k.$$

Consequently,

$$|A + k\cdot A| \geqslant (t+2)|A| - t!k$$

as desired. This concludes the induction step. □

Proof of Theorem 1. Now suppose $|A| \geqslant (k-1)^2 k!$. When $h = k$, Lemma 4 shows $|A + k\cdot A| \geqslant (k+1)|A| - k$, which means that (1) is valid. Below we assume $h < k$, and thus $|A| \geqslant (k-1)^2 k! \geqslant h^2 k!$, from which we have $|A_1| \geqslant |A|/h \geqslant hk!$.

Case 1. $i \in F$ for all $1 \leqslant i \leqslant h$.

Due to Lemmas 4 and 6 we conclude that

$$\begin{aligned}|A + k\cdot A| &= \sum_{i=1}^{h} |A_i + k\cdot A| = \sum_{i=1}^{h} \Big(|X_i + k\cdot X_i| + |\triangle_{ii}|\Big) \\ &\geqslant \sum_{i=1}^{h} \Big(|X_i| + k(|X_i| - 1) + |\triangle_{ii}|\Big) \\ &\geqslant (k+1)|A| - hk + h(h-1) \\ &= (k+1)|A| - h(k+1-h).\end{aligned}$$

If k is odd then

$$h(k+1-h) \leqslant \frac{k+1}{2}\Big(k+1-\frac{k+1}{2}\Big) = \frac{(k+1)^2}{4} = \Big\lceil \frac{k(k+2)}{4} \Big\rceil;$$

if k is even then

$$h(k+1-h) \leqslant \frac{k}{2}\Big(k+1-\frac{k}{2}\Big) = \frac{k(k+2)}{4} = \Big\lceil \frac{k(k+2)}{4} \Big\rceil.$$

Therefore,

$$|A + k \cdot A| \geqslant (k+1)|A| - \lceil k(k+2)/4 \rceil.$$

Case 2. $m \in E$.

We have $|\hat{X_m} + \hat{A}_1 \cup \hat{A_m}| \geqslant |\hat{X_m}| + 1$ from Lemme 5 and hence

$$|\triangle_{mm}| = |(A_m + k \cdot A) \setminus (A_m + k \cdot A_m)| \geqslant |(X_m + A_1) \setminus (X_m + A_m)| \geqslant |A_1|.$$

Then using Lemma 8 we conclude that

$$\begin{aligned}
|A + k \cdot A| &= |A_m + k \cdot A| + \sum_{i \neq m} |A_i + k \cdot A| \\
&= |A_m + k \cdot A_m| + |\triangle_{mm}| + \sum_{i \neq m} |A_i + k \cdot A| \\
&\geqslant (k+1)|A_m| - k! + |A_1| + \sum_{i \neq m} \Big((k+1)|A_i| - k!\Big) \\
&\geqslant (k+1)|A| - hk! + |A_1|.
\end{aligned}$$

By the fact $|A_1| \geqslant hk!$, we have

$$|A + k \cdot A| \geqslant (k+1)|A|.$$

Case 3. $m \in F$ and there exists $s \neq m$ such that $s \in E$.

In this case, Lemma 7 implies $|\triangle_{ss}| \geqslant |A_m|$. Then applying Lemmas 4 and 8, we see that

$$\begin{aligned}
|A + k \cdot A| &= |A_m + k \cdot A| + \sum_{i \neq m} |A_i + k \cdot A| \\
&\geqslant |A_m + k \cdot A_1| + \sum_{i \neq m} |A_i + k \cdot A_i| + |\triangle_{ss}| \\
&\geqslant |A_m| + k|A_1| - k + \sum_{i \neq m} \Big((k+1)|A_i| - k!\Big) + |A_m| \\
&\geqslant (k+1)|A| - hk! + |A_1| \\
&\geqslant (k+1)|A|.
\end{aligned}$$

In view of the above discussions we have completed the proof of Theorem 1. $\square$

4 Proof of Theorem 2

Lemma 9. *Let k be a positive integer and let A be a nonempty subset of $\mathbb{Z}/k\mathbb{Z} = \{\bar{0}, \bar{1}, \dots \overline{k-1}\}$. For $\alpha \in \{1, 2, \dots, k-1\}$, we have $A + \bar{\alpha} = A$ if and only if*

$$A = \bigcup_{\beta \in I} \left((k,\alpha) \cdot \left\{ \bar{0}, \bar{1}, \cdots, \frac{\bar{k}}{(k,\alpha)} - \bar{1} \right\} + \bar{\beta} \right)$$

for some nonempty set $I \subseteq \{0, 1, \dots, (k,\alpha) - 1\}$.

Proof. In the case $(k,\alpha) = 1$, it is easy to get that $A + \bar{\alpha} = A$ holds if and only if $A = \mathbb{Z}/k\mathbb{Z}$, which yields the Lemma. Below we assume $(k,\alpha) > 1$.

Let $A = A^0 \cup A^1 \cup \dots \cup A^{(k,\alpha)-1}$ with

$$A^i = \{\gamma \in A : \gamma \equiv \bar{i} \pmod{(\mathrm{k}, \alpha)}\}.$$

Note that $A + \bar{\alpha} = A$ implies $A^i + \bar{\alpha} = A^i$. Then by the fact $\left(\frac{k}{(k,\alpha)}, \frac{\alpha}{(k,\alpha)} \right) = 1$, we obtain

$$A^i = (k,\alpha) \cdot \left\{ \bar{0}, \bar{1}, \cdots, \frac{\bar{k}}{(k,\alpha)} - \bar{1} \right\} + \bar{i} \quad or \quad A^i = \emptyset.$$

Thus

$$A = \bigcup_{\beta \in I} \left((k,\alpha) \cdot \left\{ \bar{0}, \bar{1}, \cdots, \frac{\bar{k}}{(k,\alpha)} - \bar{1} \right\} + \bar{\beta} \right)$$

for some nonempty set $I \subseteq \{0, 1, \dots, (k,\alpha) - 1\}$.

The sufficiency is obvious, and the claim follows. □

Lemma 10 (cf. [7])**.** *Let A and B be nonempty subsets of the abelian group G, and let g be any element of G. Let $(A(g), B(g))$ be the e-transform of the pair (A,B), defined by $A(g) = A \cup (B + g)$ and $B(g) = B \cap (A - g)$. Then*

$$A(g) + B(g) \subseteq A + B$$

and

$$A(g) \setminus A = g + (B \setminus B(g)).$$

If A and B are finite sets, then

$$|A(g)| + |B(g)| = |A| + |B|.$$

If $g \in A$ and $0 \in B$, then $g \in A(g)$ and $0 \in B(g)$.

The following lemma is a variation of Lemma 5.

Lemma 11. *Let $k > 2$ be a composite integer. And let A,B be nonempty subsets of $\mathbb{Z}/k\mathbb{Z}$ with $A \neq \mathbb{Z}/k\mathbb{Z}$. Assume $\bar{0} \in B$ and $\bar{0} \neq \bar{q} \in B$ with $(q,k) \neq 1$. If $(b,k) = 1$ for all $\bar{b} \in B \setminus \{\bar{0}, \bar{q}\}$, then*

$$|A + \{\bar{0}, \bar{q}\}| \geqslant |A| + 1 \Rightarrow |A + B| \geqslant \min\{k, |A| + |B| - 1\}.$$

Proof. Obviously it is true in the case $|A|+|B|>k$. Now we suppose $|A|+|B|\leqslant k$. It is easy to deduce that the lemma holds for $|A|=1$ or $|B|\leqslant 2$. Next suppose $|A|\geqslant 2$ and $|B|\geqslant 3$. If the claim fails, then there exist sets A, B such that $|A+B|<|A|+|B|-1$. Choose the pair (A,B) such that $|B|$ is the smallest. Since $|B|\geqslant 3$, we have $\bar{b^*}\in B\backslash\{\bar{0},\bar{q}\}$. Then $(b^*,k)=1$. Due to $A\neq\mathbb{Z}/k\mathbb{Z}$, there exists $\bar{g}\in A$ such that $\bar{g}+\bar{b^*}\notin A$ by Lemma 9. Applying the g-transform to the pair (A,B) we have

$$|A(\bar{g})+B(\bar{g})|<|A(\bar{g})|+|B(\bar{g})|-1$$

and

$$|B(\bar{g})|<|B|.$$

If $\bar{q}\in B(\bar{g})$, then it contradicts the minimality of $|B|$. If $\bar{q}\notin B(\bar{g})$, then we have $|A(\bar{g})+B(\bar{g})|\geqslant|A(\bar{g})|+|B(\bar{g})|-1$ from Lemmas 5 and 10, which is also a contradiction.

This completes the proof. $\square$

From now on we fix $k=p_1p_2$ in this section with p_1,p_2 distinct prime numbers. Suppose that A is a nonempty finite subset of $\mathbb{Z}$. In the case $(r_2,k)>1$, we may suppose $(r_2,k)=p_1$ without loss of generality. Then denote

$$n=\min\{\ 1\leqslant i\leqslant h : p_1\nmid r_i\}.$$

Lemma 12. *Let A be a nonempty finite subset of $\mathbb{Z}$.*

(i) If $(r_2,k)=1$, then $|\triangle_{22}|\geqslant|A_1|$ for $2\in E$ and $|\triangle_{ii}|\geqslant|A_2|$ for $i\in E\backslash\{2\}$.

(ii)Suppose $(r_2,k)=p_1$. Then

$$|\triangle_{11}|\geqslant|A_2| \quad or \quad p_2|A_n| \qquad if\ \ 1\in E,$$

$$|\triangle_{ii}|\geqslant|A_1| \quad or \quad p_2|A_n| \qquad for\ \ i\in E\cap\{2,3,\ldots,n-1\}$$

and

$$|\triangle_{ii}|\geqslant|A_n| \qquad for\ \ i\in E\cap\{n+1,\ldots,h\}.$$

When $n\in E$, we have $|\triangle_{nn}|\geqslant|A_2|$. Moreover,

$$|X_n+A|\geqslant\begin{cases}|A_n|+p_1|A_1|-k & if\ |\hat{X}_n|\geqslant p_1>p_2,\\ |A_n|+p_2|A_1|-k & if\ |\hat{X}_n|\geqslant p_2>p_1,\\ |A_n|+|\hat{X}_n|\cdot|A_1|+|A_2|+\ldots+|A_l|-k & if\ p_1\leqslant|\hat{X}_n|<p_2,\end{cases}$$

where $l=\min\{n-1,p_2+1-|\hat{X}_n|\}$, and

$$|\triangle_{nn}|\geqslant|A_1|+|A_2|+\ldots+|A_{n-1}| \quad if\ |\hat{X}_n|<p_1.$$

Proof. (i) Note that $r_1=0$ and $(r_2,k)=1$. Applying Lemma 5 we get

$$|\hat{X}_i+\{0,r_2\}|\geqslant\min\{k,|\hat{X}_i|+2-1\}$$

and hence
$$\hat{X}_i + \hat{A}_1 \neq \hat{X}_i + \hat{A}_2 \qquad \text{for all} \qquad i \in E.$$
So we have that $|\triangle_{11}| \geqslant |A_2|$ for $1 \in E$ and that $|\triangle_{22}| \geqslant |A_1|$ for $2 \in E$.
For $i \in E\backslash\{1, 2\}$, if $(r_i, k) = 1$ then by Lemma 5 we have

$$\hat{X}_i + \hat{A}_i \neq \hat{X}_i + \hat{A}_1.$$

Now suppose $(r_i, k) \neq 1$. In the case $(r_i - r_2, k) = 1$, we have

$$\hat{X}_i + \hat{A}_i \neq \hat{X}_i + \hat{A}_2.$$

If $(r_i - r_2, k) \neq 1$, then $(r_i, k) \neq (r_i - r_2, k)$, and hence by Lemma 9 we obtain

$$\hat{X}_i + \hat{A}_i \neq \hat{X}_i + \hat{A}_1 \quad \text{or} \quad \hat{X}_i + \hat{A}_i \neq \hat{X}_i + \hat{A}_2.$$

Consequently,
$$|\triangle_{ii}| = |(X_i + A) \setminus (X_i + A_i)| \geqslant |A_2|.$$

(ii) Suppose $1 \in E$. If $\hat{X}_1 + \hat{A}_1 \neq \hat{X}_1 + \hat{A}_2$ then

$$|\triangle_{11}| \geqslant |(X_i + A_2) \setminus (X_i + A_1)| \geqslant |A_2|.$$

In the case $\hat{X}_1 = \hat{X}_1 + \hat{A}_1 = \hat{X}_1 + \hat{A}_2$, by Lemma 9 there is a proper subset I of $\{0, 1, \ldots, p_1 - 1\}$ such that

$$\hat{X}_1 = \bigcup_{\beta \in I} \left(p_1 \cdot \{\bar{0}, \bar{1}, \cdots, \overline{p_2} - \bar{1}\} + \bar{\beta} \right)$$

since $p_1 = (r_2, k)$ and $1 \in E$. Recall that $p_1 \nmid r_n$, and thus we have

$$|(\hat{X}_1 + \hat{A}_n) \backslash (\hat{X}_1 + \hat{A}_1)| \geqslant p_2$$

because of $I \neq \{0, 1, \ldots, p_1 - 1\}$, from which we get

$$|\triangle_{11}| \geqslant p_2 |A_n|.$$

Similarly, for $i \in E \cap \{2, \ldots, n - 1\}$, we have

$$|\triangle_{ii}| \geqslant |A_1| \quad \text{or} \quad p_2 |A_n|.$$

If $i > n$ and $i \in E$, then we also have

$$|\triangle_{ii}| \geqslant |A_1| \quad \text{or} \quad p_2 |A_n| \geqslant |A_n|$$

when $(r_i, k) = 1$ or p_1. In the case $p_2 | r_i$, we have $(r_i - r_2, k) = 1$, and hence $\hat{X}_i + \hat{A}_2 \neq \hat{X}_i + \hat{A}_i$ by Lemma 9. So $|\triangle_{ii}| \geqslant |A_2| \geqslant |A_n|$.

Below we discuss $|\triangle_{nn}|$ and $|X_n+A|$ for $n\in E$. Since $p_1\nmid r_n$ and $k=p_1p_2$, we have $(r_n,k)=1$ or $(r_n-r_2,k)=1$. Therefore

$$\hat{X}_n+\hat{A}_n\neq\hat{X}_n+\hat{A}_1\quad\text{or}\quad\hat{X}_n+\hat{A}_n\neq\hat{X}_n+\hat{A}_2,$$

which states

$$|\triangle_{nn}|=|(X_n+A)\backslash(X_n+A_n)|\geqslant|A_2|.$$

Moreover with the help of Lemma 4, we get

$$|X_n+A|\geqslant|X_n+A_1|\geqslant|X_n|+|\hat{X}_n|(|A_1|-1)\geqslant|X_n|+|\hat{X}_n||A_1|-k,$$

and hence the claim holds for the case $|\hat{X}_n|\geqslant p_1>p_2$ or $|\hat{X}_n|\geqslant p_2>p_1$.

Now we turn to the last two cases.

Case 1. $p_1\leqslant|\hat{X}_n|<p_2$.

Since $|\hat{X}_n|<p_2$, we have $|\{x \pmod{p_2}: x\in X_n\}|\leqslant|\hat{X}_n|<p_2$. Observing that

$$(p_2,r_2)=(p_2,r_3)=\ldots=(p_2,r_{n-1})=1$$

and that

$$|\{r_i \pmod{p_2}: 2\leqslant i\leqslant n-1\}|=n-2,$$

in light of Lemma 5 we get

$$|\hat{X}_n+(\hat{A}_1\cup\cdots\cup\hat{A}_t)|\geqslant\min\{p_2,|\hat{X}_n|+t-1\}\quad\text{for}\quad t=1,2,\ldots,n-1.$$

Hence

$$\begin{aligned}|(X_n+A)\setminus(X_n+A_1)|&\geqslant|(X_n+A_1\cup\cdots\cup A_{n-1})\setminus(X_n+A_1)|\\&\geqslant|A_2|+\cdots+|A_l|,\end{aligned}$$

where $l=\min\{n-1,p_2+1-|\hat{X}_n|\}$. Consequently,

$$\begin{aligned}|X_n+A|&\geqslant|X_n+A_1|+|A_2|+\ldots+|A_l|\\&\geqslant|A_n|+|\hat{X}_n||A_1|+|A_2|+\cdots+|A_l|-k.\end{aligned}$$

Case 2. $|\hat{X}_n|<p_1$.

By the definition of n, we have $n\leqslant p_2+1$ and hence

$$|\hat{X}_n|+n-1<p_1+p_2\leqslant p_1p_2=k.$$

Recall that $p_1\mid r_i$ for $1\leqslant i\leqslant n-1$ and that $p_1\nmid r_n$. If there exists $1\leqslant s\leqslant n-1$ with $(r_s-r_n,k)\neq1$, then we have $p_2\mid(r_s-r_n)$. It follows that

$$|\{1\leqslant i\leqslant n-1:(r_i-r_n,k)\neq1\}|\leqslant1.$$

Since $|\hat{X}_n|<p_1$, in view of Lemma 9, we have

$$|(\hat{X}_n+(r_s-r_n))\setminus\hat{X}_n|\geqslant1.$$

Then using Lemma 11, we get

$$|\hat{X}_n + (\hat{A}_1 \cup \ldots \cup \hat{A}_t \cup \hat{A}_n)| \geqslant |\hat{X}_n| + t \quad \text{for} \quad 1 \leqslant t \leqslant n-1,$$

and consequently,

$$|\triangle_{nn}| = |(X_n + A) \backslash (X_n + A_n)| \geqslant |A_1| + \ldots + |A_{n-1}|.$$

Combining the above we have completed the proof. □

Lemma 13. *Let A be a nonempty finite subset of $\mathbb{Z}$. Then*

$$|A + k \cdot A| \geqslant (k+1)|A| - k!.$$

Proof. We use induction to show that

$$|A + k \cdot A| \geqslant (t+1)|A| - (t-1)!k. \tag{7}$$

holds for every $t = 1, \ldots, k$.

Clearly (7) is true for $t = 1$.

Now assume that (7) holds for a fixed $1 \leqslant t < k$. We want to deduce (7) with t replaced by $t+1$, i.e.,

$$|A + k \cdot A| \geqslant (t+2)|A| - t!k. \tag{8}$$

As discussed in Lemma 8, we only need to deal with the case $h \leqslant t$. By Lemma 4 and the induction hypothesis, we have

$$|A_i + k \cdot A| \geqslant |X_i + A_1| \geqslant (t+1)|A_i| + |A_1| - k \quad \text{for} \quad i \in F \tag{9}$$

and

$$|A_i + k \cdot A| = |X_i + A| \geqslant |X_i + k \cdot X_i| + |\triangle_{ii}| \geqslant (t+1)|A_i| - (t-1)!k + |\triangle_{ii}| \quad \text{for} \quad i \in E. \tag{10}$$

Case 1. $(r_2, k) = 1$.

If $2 \in F$, then $2 \notin E$. By Lemma 12, we have $|\triangle_{ii}| \geqslant |A_2|$ for $i \in E$. Combining (9) and (10), we have

$$\begin{aligned}
|A + k \cdot A| &= \sum_{i \in F} |A_i + k \cdot A| + \sum_{i \in E} |A_i + k \cdot A| \\
&\geqslant (t+1)|A| + |F||A_1| + |E||A_2| - \Big(k|F| + |E|(t-1)!k\Big) \\
&\geqslant (t+2)|A| - t!k.
\end{aligned}$$

When $2 \in E$, we have $|\triangle_{22}| \geqslant |A_1|$. Furthermore, $|\triangle_{ii}| \geqslant |A_2|$ for $i \in E \setminus \{2\}$. Hence

$$\begin{aligned}
|A + k \cdot A| &= \sum_{i \in F} |A_i + k \cdot A| + \sum_{i \in E} |A_i + k \cdot A| \\
&\geqslant (t+1)|A| + |F||A_1| + |A_1| + (|E|-1)|A_2| - \Big(k|F| + |E|(t-1)!k\Big) \\
&\geqslant (t+2)|A| - t!k.
\end{aligned}$$

Case 2. $(r_2, k) = p_1$.

Observe that

$$\begin{aligned}
|A + k\cdot A| &= |A_n + k\cdot A| + \sum_{i\in F\setminus\{n\}} |A_i + k\cdot A| + \sum_{i\in E\setminus\{n\}} |A_i + k\cdot A| \\
&\geqslant |X_n + A| + \sum_{i\in F\setminus\{n\}} (|A_i| + k|A_1| - k) + \sum_{i\in E\setminus\{n\}} |A_i + k\cdot A_i| + \sum_{i\in E\setminus\{n\}} |\triangle_{ii}| \\
&\geqslant |X_n + A| + (t+1)\sum_{i\neq n} |A_i| + |F\setminus\{n\}||A_1| + \sum_{i\in E\setminus\{n\}} |\triangle_{ii}| \\
&\quad - \Big(k|F\setminus\{n\}| + |E\setminus\{n\}|(t-1)!k\Big) \\
&\geqslant |X_n + A| + |F\setminus\{n\}||A_1| + \sum_{i\in E\setminus\{n\}} |\triangle_{ii}| + (t+1)\sum_{i\neq n} |A_i| - (t-1)(t-1)!k.
\end{aligned}$$

For convenience we denote

$$\mathcal{S} = |X_n + A| + |F\setminus\{n\}||A_1| + \sum_{i\in E\setminus\{n\}} |\triangle_{ii}|.$$

In order to get (8), it is sufficient to prove

$$\mathcal{S} \geqslant (t+1)|A_n| + |A| - (t-1)!k.$$

If $n \in F$, then

$$|X_n + A| \geqslant |X_n| + k|A_1| - k \geqslant (t+1)|A_n| + t(|A_1| - |A_n|) + |A_1| - k.$$

Notice that $|\triangle_{ii}| \geqslant |A_n|$ for every $i \in E$ and that $h \leqslant t$. Thus

$$\begin{aligned}
\mathcal{S} &\geqslant (t+1)|A_n| + t(|A_1| - |A_n|) + |F||A_1| + |E||A_n| - k \\
&\geqslant (t+1)|A_n| + |A| - (t-1)!k.
\end{aligned}$$

Below we assume $n \in E$.

When $|A_1| \leqslant p_2|A_n|$, by Lemma 12 we get

$$\begin{aligned}
\mathcal{S} &\geqslant (t+1)|A_n| - (t-1)!k + |A_2| + |F||A_1| \\
&\quad + |A_2||E\cap\{1\}| + |A_1||E\cap\{2,\cdots,n-1\}| + |A_n||E\cap\{n+1,\cdots,h\}| \\
&\geqslant (t+1)|A_n| + |A| - (t-1)!k.
\end{aligned}$$

Now suppose $|A_1| > p_2|A_n|$. If $|\hat{X}_n| < p_1$, then from Lemma 12

$$\begin{aligned}
\mathcal{S} &\geqslant (t+1)|A_n| - (t-1)!k + |A_1| + \cdots + |A_{n-1}| + |F||A_1| + (|E|-1)|A_n| \\
&\geqslant (t+1)|A_n| + |A| - (t-1)!k.
\end{aligned}$$

If $|\hat{X}_n| \geqslant p_1 > p_2$, then

$$\begin{aligned}|X_n + A| &\geqslant |A_n| + p_1|A_1| - k = (k+1)|A_n| + p_1(|A_1| - p_2|A_n|) - k \\ &\geqslant (t+1)|A_n| + |A_n| + (n-1)(|A_1| - p_2|A_n|) - k\end{aligned}$$

since $p_1 > p_2 \geqslant n-1$. When $|\hat{X}_n| \geqslant p_2 > p_1$ we also have

$$\begin{aligned}|X_n + A| &\geqslant |A_n| + p_2|A_1| - k = (k+1)|A_n| + p_2(|A_1| - p_1|A_n|) - k \\ &\geqslant (t+1)|A_n| + |A_n| + (n-1)(|A_1| - p_2|A_n|) - k.\end{aligned}$$

With the help of Lemma 12, we deduce that

$$\mathcal{S} \geqslant (t+1)|A_n| + |A| - (t-1)!k$$

when $|\hat{X}_n| \geqslant p_1 > p_2$ or $|\hat{X}_n| \geqslant p_2 > p_1$.

If $p_1 \leqslant |\hat{X}_n| < p_2$, then

$$\begin{aligned}\mathcal{S} \geqslant\ & |A_n| + |\hat{X}_n||A_1| + \sum_{i=2}^{l} |A_i| - k + |F||A_1| \\ & + p_2|A_n||E \cap \{2, \cdots, n-1\}| + |A_n||E \cap \{n+1, \cdots, h\}| \\ \geqslant\ & \sum_{i=1}^{n} |A_i| + (|\hat{X}_n| - n + l)|A_1| + (n-2)p_2|A_n| + (h-n)|A_n| - k \\ \geqslant\ & |A| + (|\hat{X}_n| + l - 2)p_2|A_n| - k.\end{aligned}$$

Observe that

$$|\hat{X}_n| + l - 2 = \min\{|\hat{X}_n| + n - 3, p_2 - 1\} \geqslant p_1.$$

Therefore

$$\mathcal{S} \geqslant |A| + k|A_n| - k \geqslant |A| + (t+1)|A_n| - (t-1)!k.$$

This concludes the proof. $\square$

Proof of Theorem 2. Suppose $|A| \geqslant (k-1)^2 k!$. As discussed in the proof of Theorem 1, (2) is valid when $h = k$ or $|\hat{X}_i| = k$ for all $1 \leqslant i \leqslant h$. Below assume $h < k$ and $E \neq \emptyset$. Then $|A_1| \geqslant |A|/h \geqslant hk!$.

Case 1. $(r_2, k) = 1$.

If $|\hat{X}_2| = k$, then there exists $s \in E \setminus \{2\}$ since $E \neq \emptyset$. From Lemma 12, $|\triangle_{ss}| \geqslant |A_2|$. Then in light of Lemmas 4 and 13

$$\begin{aligned}|A + k \cdot A| &= |A_2 + k \cdot A| + |A_s + k \cdot A| + \sum_{i \neq 2,s} |A_i + k \cdot A| \\ &\geqslant |A_2| + k|A_1| - k + (k+1)|A_s| - k! + |A_2| + \sum_{i \neq 2,s} \Big((k+1)|A_i| - k!\Big) \\ &\geqslant (k+1)|A| - hk! + |A_1| \geqslant (k+1)|A|.\end{aligned}$$

Else if $|\hat{X}_2| < k$, then $|\triangle_{22}| \geqslant |A_1|$ and

$$\begin{aligned}
|A + k\cdot A| &= |A_2 + k\cdot A| + \sum_{i\neq 2} |A_i + k\cdot A| \\
&= |A_2 + k\cdot A_2| + |\triangle_{22}| + \sum_{i\neq 2} |A_i + k\cdot A| \\
&\geqslant (k+1)|A_2| - k! + |A_1| + \sum_{i\neq 2} \Big((k+1)|A_i| - k!\Big) \\
&\geqslant (k+1)|A| - hk! + |A_1| \geqslant (k+1)|A|.
\end{aligned}$$

Case 2. $(r_2, k) = p_1$.

When $|\hat{X}_n| = k$, using Lemma 12 we have $s \in E$ with $s \neq n$ such that $|\triangle_{ss}| \geqslant |A_n|$, which states

$$\begin{aligned}
|A + k\cdot A| &= |A_n + k\cdot A| + |A_s + k\cdot A| + \sum_{i\neq n,s} |A_i + k\cdot A| \\
&\geqslant |A_n| + k|A_1| - k + (k+1)|A_s| - k! + |A_n| + \sum_{i\neq n,s} \Big((k+1)|A_i| - k!\Big) \\
&\geqslant (k+1)|A|.
\end{aligned}$$

Below suppose $|\hat{X}_n| < k$.

Subcase 1. $|\hat{X}_n| \geqslant p_1$ and $|A_1| \leqslant p_2|A_n|$.

In the case $|\hat{X}_2| = k$, by Lemma 12 we have $|\triangle_{nn}| \geqslant |A_2|$ and

$$\begin{aligned}
|A + k\cdot A| &= |A_2 + k\cdot A| + |A_n + k\cdot A| + \sum_{i\neq 2,n} |A_i + k\cdot A| \\
&\geqslant |A_2| + k|A_1| - k + (k+1)|A_n| - k! + |A_2| + \sum_{i\neq 2,n} \Big((k+1)|A_i| - k!\Big) \\
&\geqslant (k+1)|A|.
\end{aligned}$$

When $|\hat{X}_2| < k$, we obtain $|\triangle_{22}| \geqslant |A_1|$ from Lemma 12 and hence

$$\begin{aligned}
|A + k\cdot A| &= |A_2 + k\cdot A_2| + |\triangle_{22}| + \sum_{i\neq 2} |A_i + k\cdot A| \\
&\geqslant (k+1)|A_2| - k! + |A_1| + \sum_{i\neq 2} \Big((k+1)|A_i| - k!\Big) \\
&\geqslant (k+1)|A|.
\end{aligned}$$

Subcase 2. $|\hat{X}_n| \geqslant p_1$ and $|A_1| > p_2|A_n|$.

If $|\hat{X}_n| > p_1$, then

$$|X_n + A| \geqslant |A_n| + |\hat{X}_n||A_1| - k \geqslant |A_n| + p_1|A_1| + |A_2| - k.$$

For $|\hat{X}_n| = p_1$, using Lemma 9, we have $|(\hat{X}_n + \hat{A}_2)\backslash \hat{X}_n| \geqslant 1$ and then

$$|X_n + A| \geqslant |X_n + A_1| + |A_2| \geqslant |A_n| + p_1|A_1| + |A_2| - k.$$

Applying the above, if $|\hat{X}_2| = k$, then

$$\begin{aligned}|A + k \cdot A| &= |A_2 + k \cdot A| + |A_n + k \cdot A| + \sum_{i \neq 2,n} |A_i + k \cdot A| \\ &\geqslant |A_2| + k|A_1| - k + |A_n| + p_1|A_1| + |A_2| - k + \sum_{i \neq 2,n} \Big((k+1)|A_i| - k!\Big) \\ &\geqslant (k+1)|A|.\end{aligned}$$

In the case $|\hat{X}_2| < k$, clearly $|\triangle_{22}| \geqslant p_2|A_n|$ from Lemma 12 and therefore,

$$\begin{aligned}|A + k \cdot A| &= |A_2 + k \cdot A| + |A_n + k \cdot A| + \sum_{i \neq 2,n} |A_i + k \cdot A| \\ &\geqslant |\triangle_{22}| + |A_n| + p_1|A_1| + |A_2| - k + \sum_{i \neq n} \Big((k+1)|A_i| - k!\Big) \\ &\geqslant (k+1)|A|.\end{aligned}$$

Subcase 3. $|\hat{X}_n| < p_1$.
In this case we get

$$\begin{aligned}|A + k \cdot A| &= |A_n + k \cdot A_n| + |\triangle_{nn}| + \sum_{i \neq n} |A_i + k \cdot A| \\ &\geqslant |\triangle_{nn}| + \sum_{1 \leqslant i \leqslant h} \Big((k+1)|A_i| - k!\Big) \geqslant (k+1)|A|\end{aligned}$$

because of $|\triangle_{nn}| \geqslant |A_1|$ from Lemma 12.
Combining the above we complete the proof of Theorem 2. □

5 Proof of Theorem 3

Lemma 14. *Let $A \subseteq \mathbb{Z}$, then*
(i) $|A + 4 \cdot A| = 4$, *if* $|A| = 2$.
(ii) $|A + 4 \cdot A| \geqslant 8$, *if* $|A| = 3$.
(iii) $|A + 4 \cdot A| \geqslant 12$, *if* $|A| = 4$.

Proof. Lemma 14 can be proved easily by a direct analysis. □

Observing that

$$|(X_i + A_1) \setminus (X_i + A_2)| \geqslant 1 \quad \text{or} \quad |(X_i + A_2) \setminus (X_i + A_1)| \geqslant 1$$

when $|A_1| = |A_2|$, so in this case, we may suppose $|(X_i + A_2) \setminus (X_i + A_1)| \geqslant 1$ without loss of generality. Below we fix $A \subseteq \mathbb{Z}$ with $|A| \geqslant 5$, and use the notations in Section 2.

Lemma 15. *When $h = 3$, for $i = 1, 2, 3$ we have*

$$|A_i| \leqslant 4 \Rightarrow |A_i + 4 \cdot A| \geqslant |A| + 2|A_i| - 2.$$

Proof. Recall that $A_i = 4 \cdot X_i + r_i$, $|A_1| \geqslant |A_2| \geqslant |A_3|$ and $|A_i + 4 \cdot A| = |X_i + A|$.

(I) If $|\hat{X}_i| = 1$, in light of Lemma 4 we have

$$\begin{aligned} |X_i + A| &= |X_i + A_1| + |X_i + A_2| + |X_i + A_3| \\ &\geqslant |X_i| + |A_1| - 1 + |X_i| + |A_2| - 1 + |X_i| + |A_3| - 1 \\ &\geqslant |A| + 3|A_i| - 3 \\ &\geqslant |A| + 2|A_i| - 2. \end{aligned}$$

(II) When $|\hat{X}_i| = 2$, Lemma 5 implies $|\hat{X}_i + \hat{A}| = 4$.

In the case $|\hat{X}_i + \hat{A}_1 \cup \hat{A}_2| \geqslant 3$,

$$\begin{aligned} |X_i + A| &= |X_i + A_1| + |(X_i + A) \setminus (X_i + A_1)| \\ &\geqslant |A_i| + 2|A_1| - 2 + |A_2| + |A_3| \\ &\geqslant |A| + 2|A_i| - 2. \end{aligned}$$

When $|\hat{X}_i + \hat{A}_1 \cup \hat{A}_2| = 2$, we have

$$\begin{aligned} |X_i + A| &= |X_i + A_1 \cup A_2| + |X_i + A_3| \\ &= |X_i + A_1| + |(X_i + A_2) \setminus (X_i + A_1)| + |X_i + A_3| \\ &\geqslant |A_i| + 2|A_1| - 2 + \delta_{|A_1|,\ |A_2|} + |A_i| + 2|A_3| - 2 \\ &\geqslant |A| + 2|A_i| - 2, \end{aligned}$$

where the Kronecker symbol $\delta_{s,t}$ takes 1 or 0 according to $s = t$ or not.

(III) If $|\hat{X}_i| = 3$, then

$$|X_i + A_1| \geqslant 3|A_1| \text{ for } |A_i| = 3 \quad \text{and} \quad |X_i + A_1| \geqslant 3|A_1| + 1 \text{ for } |A_i| = 4.$$

Clearly $|(\hat{X}_i + \hat{A}_2) \setminus (\hat{X}_i + \hat{A}_1)| \geqslant 1$ and hence in the case $|A_1| > |A_3|$ we have

$$|X_i + A| \geqslant |X_i + A_1| + |A_2| \geqslant |A| + 2|A_i| - 2.$$

In the case $|A_1| = |A_2| = |A_3|$, the reader can get directly that

$$|X_i + A| \geqslant 13 \text{ if } |A_i| = 3 \quad \text{and} \quad |X_i + A| \geqslant 18 \text{ if } |A_i| = 4.$$

Thus we also have $|X_i + A| \geqslant |A| + 2|A_i| - 2$.

(IV) If $|\hat{X}_i| = 4$, then we have $|A_i| = 4$ since $|A_i| \leqslant 4$. Let a be the minimal number of X_i and let b be the maximal one of A. Because $|\hat{X}_i| = 4$ and $|\hat{A}| = 3$, we have $|\widehat{\{a\}} + \hat{A}| = 3$ and $|\hat{X}_i + (\widehat{A_i \setminus \{b\}})| = 4$. Then

$$|(X_i + (A_i \backslash \{b\})) \backslash ((a + A) \cup (b + X_i))| \geqslant 3.$$

It turns out that

$$\begin{aligned}|X_i + A| &\geqslant |a + A| + |b + X_i| - 1 + |(X_i + (A_i\backslash\{b\}))\backslash((a + A) \cup (b + X_i))|\\ &\geqslant |A| + 4 - 1 + 3 \geqslant |A| + 2|A_i| - 2.\end{aligned}$$

The proof is complete. □

Lemma 16. *When $h = 2$, we have for $i = 1, 2$ that*

$$|A_i| \leqslant 4 \Rightarrow |A_i + 4 \cdot A| \geqslant |A| + 3|A_i| - 3.$$

Proof. We divide the proof into four parts.

(I) When $|\hat{X}_i| = 1$, we use Lemma 14 to obtain for $|A_i| \leqslant 4$ that,

$$\begin{aligned}|X_i + A| &= |X_i + A_i| + |X_i + (A\backslash A_i)|\\ &\geqslant |X_i + 4 \cdot X_i| + |A| - |A_i| + |A_i| - 1\\ &\geqslant |A| + 3|A_i| - 3.\end{aligned}$$

(II) If $|\hat{X}_i| = 2$, then we have $3 \leqslant |\hat{X}_i + \hat{A}| \leqslant 4$ from Lemma 5. Then we distinguish two cases.

Case 1. $|\hat{X}_i + \hat{A}| = 4$.

Since $|\hat{X}_i + \hat{A}_i| = 2$, by Lemma 14 and $|A| \geqslant 5$, we get

$$\begin{aligned}|X_i + A| &= |X_i + A_i| + |X_i + (A\backslash A_i)|\\ &\geqslant |X_i + 4 \cdot X_i| + 2(|A| - |A_i|)\\ &\geqslant |A| + 3|A_i| - 3.\end{aligned}$$

Case 2. $|\hat{X}_i + \hat{A}| = 3$.

Define $\eta = |\ \{x \in X_i : (\{\hat{x}\} + \widehat{A\backslash A_i}) \nsubseteqq (\hat{X}_i + \hat{A}_i)\}\ |$. When $\eta \geqslant 2$, by Lemma 14 we have

$$\begin{aligned}|X_i + A| &\geqslant |X_i + A_i| + |A| - |A_i| + 1\\ &\geqslant |A| + 3|A_i| - 3.\end{aligned}$$

Below suppose $\eta = 1$. It is easy to see that

$$|X_i + A| \geqslant |A| + |A| - |A_i| \geqslant |A| + 3|A_i| - 3$$

for $|A_i| = 2$ since $|A| \geqslant 5$.

For $|A_i| = 3$, we write $X_i = \{s_1, s_2, s_3\}$ with $s_1 \equiv s_2 \pmod 4$ and $s_1 < s_2$. Then we have $\{\hat{s_3}\} + \hat{A}_i = \{\hat{s_1}\} + \hat{A}\backslash\hat{A}_i$ because of $\eta = 1$. Now we show

$$|((s_1 \cup s_2) + A\backslash A_i)\backslash(s_3 + A_i)| \geqslant 1. \tag{11}$$

Clearly (11) holds for $|A| \geqslant 6$. If $|A| = 5$, then $|A_1| = 3$. Let $A_1 = \{a_1, a_2, a_3\}$ with $a_1 = 4s_1,\ a_2 = 4s_2$ and $a_3 = 4s_3$. And let $A_2 = \{a_4, a_5\}$ with $a_4 < a_5$. If $a_1 < a_2 < a_3$

and (11) fails, then $s_3 + a_1 = s_1 + a_4$, $s_3 + a_2 = s_1 + a_5 = s_2 + a_4$ and $s_3 + a_3 = s_2 + a_5$, and hence $a_2 - a_1 = s_2 - s_1$, which contradicts $a_1 - a_2 = 4(s_1 - s_2)$. When $a_3 < a_1 < a_2$ or $a_1 < a_3 < a_2$, we also get (11). From (11) and Lemma 14 we have

$$\begin{aligned} |X_i + A| &\geqslant |X_i + A_i| + |A| - |A_i| + 1 \\ &\geqslant 8 + |A| - 3 + 1 \geqslant |A| + 3|A_i| - 3. \end{aligned}$$

For $|A_i| = 4$, we have $|X_i + 4 \cdot X_i| \geqslant 13$ in the case $|\hat{X}_i| = 2$ and $\eta = 1$, and therefore

$$\begin{aligned} |X_i + A| &\geqslant |X_i + A_i| + |A| - |A_i| \\ &\geqslant 13 + |A| - 4 \geqslant |A| + 3|A_i| - 3. \end{aligned}$$

Now we give the reason for $|X_i + 4 \cdot X_i| \geqslant 13$. We write $X_i = X_{i1} \cup X_{i2}$ with $X_{i1} = 4 \cdot Y_1 + r_1$ and $X_{i2} = 4 \cdot Y_2 + r_2$. The fact $\eta = 1$ allows us to assume $|Y_1| = 3$ and $|Y_2| = 1$. To discuss $|X_i + 4 \cdot X_i|$, we may suppose $0 \in \hat{X}_i$ and $\gcd(X_i) = 1$ without loss of generality. Then with the help of Lemma 5 we have $|(\hat{Y}_1 + \hat{X}_{i2}) \backslash (\hat{Y}_1 + \hat{X}_{i1})| \geqslant 1$ since $|\hat{Y}_1| \leqslant 3$ and hence

$$\begin{aligned} |X_i + 4 \cdot X_i| &= |Y_1 + X_i| + |Y_2 + X_i| \\ &\geqslant |Y_1 + X_{i1}| + 1 + |X_i| \\ &= |Y_1 + 4 \cdot Y_1| + 1 + |X_i| \\ &\geqslant 8 + 1 + 4 = 13. \end{aligned}$$

(III) In the case $|\hat{X}_i|$=3, we have $|\hat{X}_i + \hat{A}| = 4$. Applying Lemma 14 we get

$$\begin{aligned} |X_i + A| &\geqslant |X_i + A_i| + |A| - |A_i| \\ &\geqslant 3|A_i| + |A_i| - 3 + |A| - |A_i| \geqslant |A| + 3|A_i| - 3. \end{aligned}$$

(IV) If $|\hat{X}_i| = 4$, then $|A_i| = 4$. For $|A| = 8$, we have $|A_1| = |A_2| = 4$ and $|(X_i + A_2) \backslash (X_i + A_1)| \geqslant 1$ by the assumption. Note that $|X_i + A_1| = 4|A_1|$ since $|\hat{X}_i| = 4$. Thus

$$\begin{aligned} |X_i + A| &\geqslant |X_i + A_1| + |(X_i + A_2) \backslash (X_i + A_1)| \\ &\geqslant 4|A_1| + \delta_{|A_1|,|A_2|} \\ &\geqslant |A| + 3|A_i| - 3. \end{aligned}$$

□

Proof of Theorem 3. If $|\hat{A}| = 4$, then $|A + 4 \cdot A| \geqslant 5|A| - 4$ and hence (3) holds. Below we assume $|\hat{A}| \leqslant 3$.

We prove (3) by induction on $|A|$. Clearly, (3) holds for $|A| = 5$ with the help of Lemmas 15 and 16. Now we let $|A| > 5$ and assume that

$$|B + 4 \cdot B| \geqslant 5|B| - 6 \quad \text{for any} \quad B \subset \mathbb{Z} \quad \text{with} \quad 5 \leqslant |B| < |A|.$$

We divide our proof of (3) into three parts.

Claim I. (3) holds when $h = 3$ and $|A_3| \leqslant 4$.

To prove Claim I, we distinguish three small cases.

Case I.1. $|A_1| \leqslant 4$.

We use Lemma 15 to obtain

$$|A + 4 \cdot A| = |X_1 + A| + |X_2 + A| + |X_3 + A| \geqslant 5|A| - 6.$$

Case I.2. $|A_1| \geqslant 5$ and $|A_2| \leqslant 4$.

For $|\hat{X}_1| = 4$, by Lemmas 4 and 15, we have

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant |X_1 + A_1| + |X_2 + A| + |X_3 + A| \\ &\geqslant 5|A_1| - 4 + |A| + 2|A_2| - 2 + |A| + 2|A_3| - 2 \\ &\geqslant 5|A| + |A_1| - |A_2| + |A_1| - |A_3| - 8 \geqslant 5|A| - 6.\end{aligned}$$

For $|\hat{X}_1| \leqslant 3$, since $|\hat{A}| = 3$, applying Lemma 5 we have $|\hat{X}_1 + \hat{A}| > |\hat{X}_1|$ and then

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant |X_1 + 4 \cdot X_1| + |A_3| + |X_2 + A| + |X_3 + A| \\ &\geqslant 5|A_1| - 6 + |A_3| + |A| + 2|A_2| - 2 + |A| + 2|A_3| - 2 \\ &\geqslant 5|A| + |A_1| - |A_2| + |A_1| - 10 \geqslant 5|A| - 6.\end{aligned}$$

Case I.3. $|A_2| \geqslant 5$ and $|A_3| \leqslant 4$.

We have $|\triangle_{11}| + |\triangle_{22}| \geqslant 2$ by Lemma 6 and hence

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant |X_1 + 4 \cdot X_1| + |\triangle_{11}| + |X_2 + 4 \cdot X_2| + |\triangle_{22}| + |X_3 + A| \\ &\geqslant 5|A_1| - 4 + 5|A_2| - 4 + 2 + |A| + 2|A_3| - 2 \\ &\geqslant 5|A| + |A_1| - |A_3| + |A_2| - |A_3| - 8 \geqslant 5|A| - 6\end{aligned}$$

when $|\hat{X}_1| = 4$ and $|\hat{X}_2| = 4$. In the case $|\{1 \leqslant i \leqslant 2 : \ |\hat{X}_i| = 4\}| = 1$ we obtain

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant 5(|A_1| + |A_2|) - 6 - 4 + |A_3| + |A| + 2|A_3| - 2 \\ &\geqslant 5|A| + |A_1| + |A_2| - |A_3| - 12 \geqslant 5|A| - 6\end{aligned}$$

in view of Lemma 5 and the induction hypothesis.

When $|\hat{X}_1| \leqslant 3$ and $|\hat{X}_2| \leqslant 3$, by Lemma 5 we have $|\hat{X}_1 + \hat{A}| > |\hat{X}_1|$ and $|\hat{X}_2 + \hat{A}| > |\hat{X}_2|$. Then

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant 5|A_1| - 6 + |A_3| + 5|A_2| - 6 + |A_3| + |A| + 2|A_3| - 2 \\ &\geqslant 5|A| + |A_1| + |A_2| - 14 \geqslant 5|A| - 6.\end{aligned}$$

Claim II. (3) holds when $h = 3$ and $|A_3| \geqslant 5$.

By Lemma 5, when $|\hat{X}_i| \leqslant 3$, we have $|\hat{X}_i + \hat{A}| > |\hat{X}_i|$ and hence

$$|X_i + A| \geqslant |X_i + A_i| + |A_3|.$$

If $|\hat{X}_1| \leqslant 3$, $|\hat{X}_2| \leqslant 3$ and $|\hat{X}_3| \leqslant 3$, then we have

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant 5|A_1| - 6 + |A_3| + 5|A_2| - 6 + |A_3| + 5|A_3| - 6 + |A_3| \\ &\geqslant 5|A| + 3|A_3| - 18 \geqslant 5|A| - 6.\end{aligned}$$

When $|\ \{i : |\hat{X}_i| \leqslant 3\}\ | = 2$, by Lemma 4 we get

$$|A + 4 \cdot A| \geqslant 5(|X_1| + |X_2| + |X_3|) - 6 - 6 - 4 + 2|A_3| \geqslant 5|A| - 6.$$

In the case $|\hat{X}_1| = |\hat{X}_2| = |\hat{X}_3| = 4$, we have $|\triangle_{11}| + |\triangle_{22}| + |\triangle_{33}| \geqslant 6$ in light of Lemma 6. Then by Lemma 4 we get

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant |X_1 + A_1| + |\triangle_{11}| + |X_2 + A_2| + |\triangle_{22}| + |X_3 + A_3| + |\triangle_{33}| \\ &\geqslant 5(|A_1| + |A_2| + |A_3|) - 4 - 4 - 4 + 6 \geqslant 5|A| - 6.\end{aligned}$$

It remains to handle the case $|\ \{i : |\hat{X}_i| \leqslant 3\}\ | = 1$, and we make detailed discussions.

Case II.1. $|A_1| = |A_2| = |A_3|$.

We may suppose $|\hat{X}_3| \leqslant 3$ and $|\hat{X}_1| = |\hat{X}_2| = 4$. Note that $|X_1 + A_i| \geqslant 5|A_1| - 4$ and $|X_2 + A_i| \geqslant 5|A_2| - 4$ for all i. Now we prove

$$|X_1 + A| \geqslant 5|A_1| - 2 \text{ and } |X_2 + A| \geqslant 5|A_2| - 2.$$

Note that

$$\min(X_1 + A_i) \notin X_1 + A_j \quad \text{or} \quad \max(X_1 + A_i) \notin X_1 + A_j$$

if $|X_1 + A_i| = |X_1 + A_j|$. When $|X_1 + A_i| \geqslant 5|A_1| - 3$ for some i, we have $|X_1 + A| \geqslant 5|A_1| - 2$ since $|X_1 + A \setminus A_i| \geqslant 5|A_1| - 3$. If $|X_1 + A_i| = 5|A_1| - 4$ for all i, then A_1, A_2 and A_3 must be arithmetic progressions with the same difference by Lemma 4, and therefore $|\triangle_{11}| \geqslant 2$ and

$$|X_1 + A| \geqslant |X_1 + A_1| + |\triangle_{11}| \geqslant 5|A_1| - 2.$$

Similarly, $|X_2 + A| \geqslant 5|A_2| - 2$. So

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A| \\ &\geqslant 5|A_1| - 2 + 5|A_2| - 2 + 5|A_3| - 6 + |A_3| \\ &\geqslant 5|A| + |A_3| - 10 \\ &\geqslant 5|A| - 6.\end{aligned}$$

Case II.2. $|A_1| = |A_2| = |A_3|$ fails.

Note that $|A_1| > |A_3| \geqslant 5$. If $|A_1| > |A_2|$, then

$$|X_2 + A| \geqslant |X_2 + 4 \cdot X_1| \geqslant |A_2| + 4|A_1| - 4 \geqslant 5|A_2|$$

or

$$|X_3 + A| \geqslant |X_3 + 4 \cdot X_1| \geqslant |A_3| + 4|A_1| - 4 \geqslant 5|A_3|$$

since $|\ \{i : |\hat{X}_i| \leqslant 3\}\ | = 1$. Hence

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| + |X_3 + A|\\ &\geqslant 5(|A_1| + |A_2| + |A_3|) - 6 - 4 + |A_3|\\ &\geqslant 5|A| - 6.\end{aligned}$$

Now suppose $|A_1| = |A_2| > |A_3|$. If $|\hat{X}_3| \leqslant 3$, then $|\hat{X}_1| = |\hat{X}_2| = 4$. As mentioned in case II.1 we have

$$|A + 4 \cdot A| \geqslant 5|A_1| - 4 + 1 + 5|A_2| - 4 + 1 + 5|A_3| - 6 + |A_1| \geqslant 5|A| - 6.$$

When $|\hat{X}_3| = 4$, it is clear that

$$|A + 4 \cdot A| \geqslant 5(|A_1| + |A_2|) - 4 - 6 + |A_3| + |A_3| + 4|A_1| - 4 \geqslant 5|A| - 6.$$

Claim III. (3) holds for $h = 2$.

We first note that if $|\hat{X}_i| \leqslant 3$ then $|\hat{X}_i + \hat{A}| > |\hat{X}_i + \hat{A}_i|$ by Lemma 5.

Case III.1. $|A_1| \leqslant 4$.

Applying Lemma 16 we get

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A|\\ &\geqslant |A| + 3|A_1| - 3 + |A| + 3|A_2| - 3 \geqslant 5|A| - 6.\end{aligned}$$

Case III.2. $|A_1| \geqslant 5$ and $|A_2| \leqslant 4$.

When $|\hat{X}_1| = 4$, we have

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| \geqslant |X_1 + A_1| + |X_2 + A|\\ &\geqslant 5|A_1| - 4 + |A| + 3|A_2| - 3\\ &= 5|A| + |A_1| - |A_2| - 7 \geqslant 5|A| - 6.\end{aligned}$$

If $|\hat{X}_1| \leqslant 3$, then

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A|\\ &\geqslant |X_1 + A_1| + |\triangle_{11}| + |X_2 + A|\\ &\geqslant 5|A_1| - 6 + |A_2| + |A| + 3|A_2| - 3\\ &= 5|A| + |A_1| - |A_2| + |A_2| - 9 \geqslant 5|A| - 6.\end{aligned}$$

Case III.3. $|A_2| \geqslant 5$.

For $|\hat{X}_1| = |\hat{X}_2| = 4$, if $|A_1| = |A_2|$ then we have

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| \\ &\geqslant |X_1 + A_1| + |\triangle_{11}| + |X_2 + A_2| + |\triangle_{22}| \\ &\geqslant 5|A| - 4 - 4 + 2 \geqslant 5|A| - 6\end{aligned}$$

by Lemma 6. In the case $|A_1| > |A_2|$, with the help of Lemma 4 we obtain

$$\begin{aligned}|A + 4 \cdot A| &= |X_1 + A| + |X_2 + A| \geqslant 5|A_1| - 4 + |A_2| + 4|A_1| - 4 \\ &= 5|A| + 4(|A_1| - |A_2|) - 8 \geqslant 5|A| - 6.\end{aligned}$$

When $|\ \{i : |\hat{X}_i| \leqslant 3\}\ | = 1$, it is easy to see that

$$|A + 4 \cdot A| \geqslant 5(|A_1| + |A_2|) + |A_2| - 4 - 6 \geqslant 5|A| - 6.$$

In the case $|\ \{i : |\hat{X}_i| \leqslant 3\}\ | = 2$, we have

$$|A + 4 \cdot A| \geqslant 5|A_1| - 6 + |A_2| + 5|A_2| - 6 + |A_1| \geqslant 5|A| - 6.$$

Combining the above, we have completed the proof of Theorem 3. □

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