Endomorphism Breaking in Graphs

Wilfried Imrich*

Montanuniversität Leoben, A-8700 Leoben, Austria

imrich@unileoben.ac.at

Rafał Kalinowski†

AGH University of Science and Technology,
al. Mickiewicza 30, 30-059 Krakow, Poland

kalinows@agh.edu.pl

Florian Lehner‡

Institut für Geometrie, Technische Universität Graz
Kopernikusgasse 24/IV, A-8010 Graz, Austria

f.lehner@tugraz.at

Monika Pilśniak§

AGH University of Science and Technology,
al. Mickiewicza 30, 30-059 Krakow, Poland

pilsniak@agh.edu.pl

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Abstract

We introduce the *endomorphism distinguishing number* $D_e(G)$ of a graph G as the least cardinal d such that G has a vertex coloring with d colors that is only preserved by the trivial endomorphism. This generalizes the notion of the distinguishing number $D(G)$ of a graph G, which is defined for automorphisms instead of endomorphisms.

As the number of endomorphisms can vastly exceed the number of automorphisms, the new concept opens challenging problems, several of which are presented here. In particular, we investigate relationships between $D_e(G)$ and the endomorphism motion of a graph G, that is, the least possible number of vertices moved

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by a nontrivial endomorphism of G. Moreover, we extend numerous results about the distinguishing number of finite and infinite graphs to the endomorphism distinguishing number.

Keywords: distinguishing number; endomorphisms; infinite graphs;

1 Introduction

Albertson and Collins [1] introduced the *distinguishing number* $D(G)$ of a graph G as the least cardinal d such that G has a vertex labeling with d labels that is only preserved by the trivial automorphism.

This concept has spawned numerous papers, mostly on finite graphs. But countable infinite graphs have also been investigated with respect to the distinguishing number; see [8], [14], [15], and [16]. For graphs of higher cardinality, see [9].

The aim of this paper is the presentation of fundamental results for the endomorphism distinguishing number, and of open problems. In particular, we extend the Motion Lemma of Russell and Sundaram [13] to endomorphisms, present endomorphism motion conjectures that generalize the Infinite Motion Conjecture of Tom Tucker [15] and the Motion Conjecture of [4], prove the validity of special cases, and support the conjectures by examples.

2 Definitions and Basic Results

We consider only simple graphs, that is, graphs without loops and multiple edges. As the distinguishing number has already been defined, let us note that $D(G) = 1$ for all asymmetric graphs. This means that almost all finite graphs have distinguishing number one, because almost all graphs are asymmetric, see Erdős and Rényi [5]. Clearly $D(G) \geqslant 2$ for all other graphs. Again, it is natural to conjecture that almost all of them have distinguishing number two. This is supported by the observations of Conder and Tucker [3].

However, for the complete graph K_n, and the complete bipartite graph $K_{n,n}$ we have $D(K_n) = n$, and $D(K_{n,n}) = n + 1$. Furthermore, the distinguishing number of the cycle of length 5 is 3, but cycles C_n of length $n \geqslant 6$ have distinguishing number 2. This compares with a more general result of Klavžar, Wong and Zhu [11] and of Collins and Trenk [2], which asserts that $D(G) \leqslant \Delta(G) + 1$, where Δ denotes the maximum degree of G. Equality holds if and only if G is a K_n, $K_{n,n}$ or C_5.

Now to the endomorphism distinguishing number. Before defining it, let us recall that an *endomorphism* of a graph $G = (V, E)$ is a mapping $\varphi : V \to V$ such that for every edge $uv \in E$ its image $\varphi(u)\varphi(v)$ is an edge, too.

Definition 1. *The endomorphism distinguishing number $D_e(G)$ of a graph G is the least cardinal d such that G has a vertex labeling with d labels that is preserved only by the identity endomorphism of G.*

Let us add that we also say *colors* instead of *labels*. If a vertex labeling c is not preserved by an endomorphism φ, we say that c *breaks* φ.

Clearly $D(G) \leqslant D_e(G)$. For graphs G with $\mathrm{Aut}(G) = \mathrm{End}(G)$ equality holds. Such graphs are called *core graphs*. Notice that complete graphs and odd cycles are core graphs, see [6]. Hence $D_e(K_n) = n$, $D_e(C_5) = 3$, and $D_e(C_{2k+1}) = 2$ for $k \geqslant 3$.

Interestingly, almost all graphs are core graphs, as shown by Koubek and Rödl [12]. Because almost all graphs are asymmetric, this implies that almost all graphs have trivial endomorphism monoid, that is, $\mathrm{End}(G) = \{\mathrm{id}\}$. Graphs with trivial endomorphism monoid are called *rigid*. Clearly $D_e(G) = 1$ for any rigid graph G, and thus $D_e(G) = 1$ for almost all graphs G.

$D_e(G)$ can be equal to $D(G)$ even when $\mathrm{Aut}(G) \subsetneq \mathrm{End}(G)$. For example, this is the case for even cycles. We formulate this as a lemma.

Lemma 2. *The automorphism group of even cycles is properly contained in their endomorphism monoid, but* $D(C_{2k}) = D_e(C_{2k})$ *for all* $k \geqslant 2$.

Proof. It is easily seen that every even cycle admits proper endomorphisms, that is, endomorphisms that are not automorphisms. Furthermore, it is readily verified that $|\,\mathrm{End}(C_4)| = 14$ and $D(C_4) = D_e(C_4) = 3$.

Hence, let $k \geqslant 3$. Color the vertices v_1, v_2 and v_4 black and all other vertices white, see Figure 1. We wish to show that this coloring is endomorphism distinguishing. Clearly this coloring distinguishes all automorphisms.

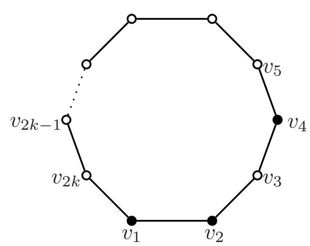

Figure 1: Distinguishing an even cycle

Let φ be a proper endomorphism. It has to map the cycle into a proper connected subgraph of itself. Thus, $\varphi(C_{2k})$ must be a path, say P.

Furthermore, the color of the endpoints of an edge must be preserved under φ. Hence v_1v_2 is mapped into itself. Because $v_{2k-1}v_{2k}$ is the only edge with two white endpoints that is adjacent to v_1v_2, it must also be mapped into itself. This fixes v_{2k-1}, v_{2k}, v_1 and v_2. But then v_3 and v_4 are also fixed.

Now we observe that the path $v_4v_5 \cdots v_{2k}v_1$ in C_{2k} has only white interior vertices and that it it has to be mapped into a walk in P from v_4 to v_1 that contains only white interior vertices. Clearly this is not possible. $\square$

To show that $D(G)$ can be smaller than $D_e(G)$, we consider graphs G with trivial automorphism group but nontrivial endomorphisms monoid. For such graphs $D(G) = 1$, but $D_e(G) > 1$. Easy examples are asymmetric, nontrivial trees T. For, every such tree has at least 7 vertices and at least three vertices of degree 1. Let a be a vertex of degree 1 and b its neighbor. Because T has at least 7 vertices and since it is connected, there must be a neighbor c of b that is different from a. Then the mapping

$$\varphi : v \mapsto \begin{cases} c \text{ if } v = a \\ v \text{ otherwise} \end{cases}$$

is a nontrivial endomorphism.

3 The Endomorphism Motion Lemma

Russel and Sundaram [13] proved that the distinguishing number of a graph is small when every automorphism of G moves many elements. We generalize this result to endomorphisms and begin with the definition of motion.

The *motion* $m(\varphi)$ of a nontrivial endomorphism φ of a graph G, is the number of elements it moves:

$$m(\varphi) = |\{v \in V(G) \mid \varphi(v) \neq v\}|.$$

The *endomorphism motion* of a graph G is

$$m_e(G) = \min_{\varphi \in \mathrm{End}(G) \setminus \{\mathrm{id}\}} m(\varphi)$$

For example, $m_e(C_4) = 1, m_e(C_5) = 4, m_e(C_{100}) = 49, m_e(K_{100}) = 2$.

In the sequel we will prove the following generalization of Theorem 1 of Russell and Sundaram [13].

Lemma 3 (Endomorphism Motion Lemma)**.** *For any finite graph G,*

$$d^{\frac{m_e(G)}{2}} \geqslant |\,\mathrm{End}(G)| \tag{1}$$

implies $D_e(G) \leqslant d$.

The proof will be an easy consequence of Lemma 5, the Orbit Norm Lemma. We first define orbits of endomorphisms.

Definition 4. *An orbit of an endomorphism φ of a graph G is an equivalence class with respect to the equivalence relation $\sim$ on $V(G)$, where $u \sim v$ if there exist nonnegative integers i and j such that $\varphi^i(u) = \varphi^j(v)$.*

The orbits form a (finite or infinite) partition $V(G) = I_1 \cup I_2 \cup \dots,\ I_i \cap I_j = \emptyset$ for $1 \leqslant i < j$, of $V(G)$. For finite graphs it can be characterized as the unique partition with the maximal number of sets that are invariant under the preimage φ^{-1}. For infinite graphs we characterize it as the finest partition that is invariant under φ^{-1}. For automorphisms it coincides with the cycle decomposition.

The *orbit norm* of an endomorphism φ with the orbits $I_1, I_2, \dots, I_k$ is

$$o(\varphi) = \sum_{i=1}^{k} (|I_i| - 1)\,,$$

and the *endomorphism orbit norm* of a finite graph G is

$$o(G) = \min_{\varphi \in \operatorname{End}(G) \setminus \{\mathrm{id}\}} o(\varphi)\,.$$

Notice that φ may not move all elements of a nontrivial orbit, whereas automorphisms move all elements in a nontrivial cycle of the cycle decomposition. To see this, consider an orbit $I = \{a, b\}$, where $\varphi(a) = b$, and $\varphi(b) = b$. Only one element of the orbit is moved, and the contribution of I to the orbit norm of φ is 1. Clearly $o(\varphi) \geqslant m(\varphi)/2$, and thus $o(G) \geqslant m_e(G)/2$.

Lemma 5 (Orbit Norm Lemma)**.** *A finite graph G is endomorphism d-distinguishable if*

$$\sum_{\varphi \in \operatorname{End}(G) \setminus \{\mathrm{id}\}} d^{-o(\varphi)} < 1.$$

Proof. We study the behavior of a random d-coloring c of G, the probability distribution given by selecting the color of each vertex independently and uniformly in the set $\{1, \dots, d\}$. Fix an endomorphism $\varphi \neq \mathrm{id}$ and consider the event that the random coloring c is preserved by φ, that is, $c(v) = c(\varphi(v))$ for each vertex v of G. Then it is easily seen that

$$\mathrm{Prob}[\forall v :\, c(v) = c(\varphi(v))] = \left(\frac{1}{d}\right)^{o(\varphi)} \leqslant \left(\frac{1}{d}\right)^{o(G)}.$$

Collecting together these events, we have

$$\mathrm{Prob}[\exists \varphi \neq \mathrm{id}\ \forall v :\, c(v) = c(\varphi(v))] \leqslant \sum_{\varphi \in \operatorname{End}(G) \setminus \{\mathrm{id}\}} \left(\frac{1}{d}\right)^{o(\varphi)}.$$

If this sum is strictly less than one, then there exists a coloring c such that for all nontrivial φ there is a v, such that $c(v) \neq c(\varphi(v))$, as desired. $\square$

Proof of Lemma 3. From $o(G) \geqslant m_e(G)/2$ we infer that

$$\sum_{\varphi \in \operatorname{End}(G) \setminus \{\mathrm{id}\}} d^{-o(\varphi)} \;\leqslant\; (|\operatorname{End}(G)| - 1)\, d^{-o(G)} \;\leqslant\; (|\operatorname{End}(G)| - 1)\, d^{-\frac{m_e(G)}{2}}. \qquad (2)$$

Hence, if

$$d^{\frac{m_e(G)}{2}} \geqslant |\operatorname{End}(G)|,$$

then the right side of Equation 2 is strictly less than 1, and therefore so too is the sum $\sum_{\varphi \in \operatorname{End}(G) \setminus \{\mathrm{id}\}} d^{-o(\varphi)}$. Now an application of the Orbit Norm Lemma shows that G is d-distinguishable. $\square$

Lemma 3 is similar to the *Motion Lemma* of Russell and Sundaram [13, Theorem 1], which asserts that G is 2-distinguishable if

$$m(G) > 2\log_2 |\mathrm{Aut}(G)|\,,$$

where

$$m(G) = \min_{\varphi\in\mathrm{Aut}(G)\setminus\{\mathrm{id}\}} m(\varphi)\,.$$

Actually, a short look at the proof of Russell and Sundaram shows that G is d-distinguishable under the weaker assumption

$$m(G) \geqslant 2\log_d |\mathrm{Aut}(G)|. \tag{3}$$

Thus, our Endomorphism Motion Lemma is a direct generalization of the Motion Lemma of Russell and Sundaram.

The Motion Lemma allows the computation of the distinguishing number of many classes of finite graphs. We know of no such applications for the Endomorphism Motion Lemma, but will show the applicability of its generalization to infinite graphs.

4 Infinite graphs

Suppose we are given an infinite graph G with infinite endomorphism motion $m_e(G)$ and wish to generalize Equation 1 to this case for finite d. Notice that

$$d^{m_e(G)/2} = d^{m_e(G)} = 2^{m_e(G)}$$

in this situation. Thus the natural generalization would be that

$$2^{m_e(G)} \geqslant |\,\mathrm{End}(G)| \tag{4}$$

implies endomorphism 2-distinguishability. We formulate this as a conjecture.

Endomorphism Motion Conjecture. *Let G be a connected, infinite graph with endomorphism motion $m_e(G)$. If $2^{m_e(G)} \geqslant |\mathrm{End}(G)|$, then $D_e(G) = 2$.*

This is a generalization of the Motion Conjecture of [4] for automorphisms of graphs. Notice that we assume connectedness now, which we did not do before. The reason is, that we not only have to break all endomorphisms of every connected component if the graph is disconnected, but that we also have to worry about breaking mappings between possibly infinitely many different connected components, which requires extra effort.

A special case are countable graphs. Let G be an infinite, connected countable graph with infinite endomorphism motion $m_e(G)$. Then $m_e(G) = \aleph_0$ and $2^{m_e(G)} = 2^{\aleph_0} = \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the continuum.

Notice, for countable graphs, $|\,\mathrm{End}(G)| \leqslant \aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. This means that Equation 4 is always satisfied for countably infinite graphs with infinite motion. This motivates the following conjecture:

Endomorphism Motion Conjecture for Countable Graphs. *Let G be a countable connected graph with infinite endomorphism motion. Then G is endomorphism 2-distinguishable.*

In the last section we will verify this conjecture for countable trees with infinite endomorphism motion. Their endomorphism monoids are uncountable and we will show that they have endomorphism distinguishing number 2.

We now prove the conjecture for countable endomorphism monoids. In fact, we show that almost every coloring is distinguishing if the endomorphism monoid is countable.

Theorem 6. *Let G be a graph with infinite endomorphism motion whose endomorphism monoid is countable. Let c be a random 2-coloring where all vertices have been colored independently and assume that there is an $\varepsilon > 0$ such that, for every vertex v, the probability that it is assigned a color $x \in \{black, white\}$ satisfies*

$$\varepsilon \leqslant \operatorname{Prob}[c(v) = x] \leqslant 1 - \varepsilon.$$

Then c is almost surely distinguishing.

Proof. First, let φ be a fixed, non-trivial endomorphism of G. Since the motion of φ is infinite we can find infinitely many disjoint pairs $\{v_i, \varphi(v_i)\}$. Clearly the colorings of these pairs are independent and the probability that φ preserves the coloring in any of the pairs is bounded from above by some constant $\varepsilon' < 1$. Now

$$\operatorname{Prob}[\varphi \text{ preserves } c] \leqslant \operatorname{Prob}[\forall i\colon c(v_i) = c(\varphi(v_i))] = 0.$$

Since there are only countably many endomorphisms we can use σ-subadditivity of the probability measure to conclude that

$$\operatorname{Prob}[\exists \varphi \in \operatorname{End}(G) \setminus \{id\}\colon \varphi \text{ preserves c}] \leqslant \sum_{\varphi \in \operatorname{End}(G)\setminus\{\mathrm{id}\}} \operatorname{Prob}[\varphi \text{ preserves } c] = 0,$$

which completes the proof. □ □

We will usually only use the following Corollary of Theorem 6.

Corollary 7. *Let G be a graph with infinite motion whose endomorphism monoid is countable. Then*

$$D_e(G) = 2.$$

The endomorphism motion conjecture for countable graphs generalizes the

Infinite Motion Conjecture of Tucker [15]. *Let G be a connected, locally finite infinite graph with infinite motion. Then G is 2-distinguishable.*

It was shown in [4], and follows from Theorem 6, that it is true for countable $\operatorname{Aut}(G)$. There are numerous applications of this result, see [10].

For the Endomorphism Motion Conjecture for Countable Graphs we have the following generalization of [9, Theorem 3.2]:

Theorem 8. *Let Γ be a finitely generated infinite group. Then there is a 2-coloring of the elements of Γ, such that the identity endomorphism of Γ is the only endomorphism that preserves this coloring. In other words, finitely generated groups are endomorphism 2-distinguishable.*

Proof. Let S be a finite set of generators of Γ that is closed under inversion. Since every element g of Γ can be represented as a product $s_1 s_2 \cdots s_k$ of finite length in elements of S, we infer that Γ is countable.

Also, if $\varphi \in \mathrm{End}(\Gamma)$, then

$$\varphi(g) = \varphi(s_1 s_2 \cdots s_k) = \varphi(s_1)\varphi(s_2)\cdots\varphi(s_k).$$

Hence, every endomorphism φ is determined by the finite set

$$\varphi(S) = \{\varphi(s) \mid s \in S\}.$$

Because every $\varphi(s)$ is a word of finite length in elements of S there are only countably many elements in $\varphi(S)$. Hence $\mathrm{End}(\Gamma)$ is countable.

Now, let us consider the motion of the nonidentity elements of $\mathrm{End}(\Gamma)$. Let φ be such an element and consider the set

$$\mathrm{Fix}(\varphi) = \{g \in \Gamma \mid \varphi(g) = g\}.$$

It is easily seen that these elements form a subgroup of Γ. Since φ does not fix all elements of Γ it is a proper subgroup. Since its smallest index is two, the set $\Gamma \setminus \mathrm{Fix}(\varphi)$ is infinite. Thus $m(\varphi)$ is infinite. As φ was arbitrarily chosen, Γ has infinite endomorphism motion.

By Corollary 7 we conclude that Γ is 2-distinguishable. $\square$

The next theorem shows that the endomorphism motion conjecture is true if $m_e(G) = |\mathrm{End}(G)|$, even if $m_e(G)$ is not countable.

Theorem 9. *Let G be a connected graph with uncountable endomorphism motion. Then $|\mathrm{End}(G)| \leqslant m_e(G)$ implies $D_e(G) = 2$.*

Proof. Set $\mathfrak{n} = |\mathrm{End}(G)|$, and let ζ be the smallest ordinal number whose underlying set has cardinality $\mathfrak{n}$. Furthermore, choose a well ordering $\prec$ of $A = \mathrm{End}(G) \setminus \{\mathrm{id}\}$ of order type ζ, and let φ_0 be the smallest element with respect to $\prec$. Then the cardinality of the set of all elements of A between φ_0 and any other $\varphi \in A$ is smaller than $\mathfrak{n} \leqslant m_e(G)$.

Now we color all vertices of G white and use transfinite induction to break all endomorphisms by coloring selected vertices black. By the assumptions of the theorem, there exists a vertex v_0 that is not fixed by φ_0. We color it black. This coloring breaks φ_0.

For the induction step, let $\psi \in A$. Suppose we have already broken all $\varphi \prec \psi$ by pairs of vertices $(v_\varphi, \varphi(v_\varphi))$, where v_φ and $\varphi(v_\varphi)$ have distinct colors. Clearly, the cardinality of the set R of all $(v_\varphi, \varphi(v_\varphi))$, $\varphi \prec \psi$, is less than $\mathfrak{n} \geqslant m_e(G)$. By assumption, ψ moves at least $m_e(G)$ vertices. Since there are still $\mathfrak{n}$ vertices not in R, there must be a vertex v_ψ that does not meet R. If $\psi(v_\psi)$ is white, we color v_ψ black. This coloring breaks ψ. $\square$

Corollary 10. *Let G be a connected graph with uncountable endomorphism motion. If the general continuum hypothesis holds, and if* $|\mathrm{End}(G)| < 2^{m_e(G)}$, *then* $D_e(G) = 2$.

Proof. By the generalized continuum hypothesis $2^{m_e(G)}$ is the successor of $m_e(G)$. Hence, the inequality $2^{m_e(G)} > |\mathrm{End}(G)|$ is equivalent to $m_e(G) \geqslant |\mathrm{End}(G)|$. $\square$

5 Examples and outlook

So far we have only determined the endomorphism distinguishing numbers of core graphs, such as the complete graph and odd cycles, and proved that $D_e(C_{2k}) = 2$ for $k \geqslant 3$. Furthermore, it is easily seen that $D_e(K_{n,n}) = n+1$ and $D_e(K_{m,n}) = \max(m, n)$ if $m \neq n$.

In the case of infinite structures we proved Theorem 8, which shows that $D_e(\Gamma) = 2$ for finitely generated infinite groups Γ.

We will now determine the endomorphism distinguishing numbers of finite and infinite paths and we begin with the following lemma.

Lemma 11. *Let φ be an endomorphism of a (possibly infinite) tree G such that $\varphi(u) = \varphi(v)$ for two distinct vertices u, v. Then there exist two vertices x, y on the path between u and v such that $\varphi(x) = \varphi(y)$ and* $\mathrm{dist}(x, y) = 2$.

Proof. Suppose $\mathrm{dist}(u, v) \neq 2$. Hence $\mathrm{dist}(u, v) > 2$. Let P be the path connecting u and v in G, and let P' be the subgraph induced by the image $\varphi(P)$. Clearly, P' is a finite tree with at least one edge.

Because every nontrivial finite tree has at least two pendant vertices, there must be a pendant vertex w of P' that is different from $\varphi(u) = \varphi(v)$. Thus $w = \varphi(z)$ for some internal vertex z of P. If x and y are the two neighbors of z on P, then clearly $\varphi(x) = \varphi(y)$ and $\mathrm{dist}(x, y) = 2$. $\square$

The above lemma implies the following corollary for finite graphs, because any injective endomorphism of a finite graph is an automorphism.

Corollary 12. *Let G be a finite tree. Then for every $\varphi \in \mathrm{End}(G) \backslash \mathrm{Aut}(G)$ there exist two vertices x, y of distance 2 such that $\varphi(x) = \varphi(y)$.* $\square$

Lemma 13. *The endomorphism distinguishing number of all finite paths P_n of order $n \geqslant 2$ is two.*

Proof. Clearly, $D_e(P_n) \geqslant 2$ since $\mathrm{End}(P_n) \neq \mathrm{Aut}(P_n)$. To see that $D_e(P_n) = 2$ consider the following vertex labeling

$$c(P_n) = \begin{cases} (11221122\cdots1122) & \text{if } n \equiv 0 \mod 4, \\ (11221122\cdots11221) & \text{if } n \equiv 1 \mod 4, \\ (1221122\cdots22112) & \text{if } n \equiv 2 \mod 4, \\ (11221122\cdots22112) & \text{if } n \equiv 3 \mod 4. \end{cases}$$

The only nontrivial automorphism of a path (symmetry with respect to the center) does not preserve this labeling. By Corollary 12, any other nontrivial endomorphism has to identify two vertices of distance two. Then φ cannot preserve the coloring, because any two vertices of distance two have distinct labels. $\square$

Next let us consider the ray and the double ray which can be viewed as an infinite analog to finite paths. It turns out that their endomorphism distinguishing number is 2 as well.

Lemma 14. *The endomorphism distinguishing number of the infinite ray and of the infinite double ray is two.*

Later in this section Theorem 17 will show that every countable tree with at most one pendant vertex has endomorphism distinguishing number two. Clearly Lemma 14 constitutes a special case of this result. It is also worth noting that by the following theorem every double ray has infinite endomorphism motion. Hence we verify the Endomorphism Motion Conjecture for the class of countable trees.

Theorem 15. *A infinite tree has infinite endomorphism motion if and only if it has no pendant vertices.*

The proof uses the following lemma which may be of independent interest. Note that in the statement of the lemma there is no restriction on the cardinality of the tree or the motion of the endomorphism.

Lemma 16. *Let T be a tree and let φ be an endomorphism of T. Then the set of fixed points of φ induces a connected subgraph of T.*

Proof. Denote by $\mathrm{Fix}(\varphi)$ the set of fixed points of φ and assume that it does not induce a connected subgraph. Consider two vertices $v_1, v_2 \in \mathrm{Fix}(\varphi)$ lying in different components of this graph.

Then φ maps the unique path in T from v_1 to v_2 to a v_1-v_2-walk of the same length. But the only such walk is the path connecting v_1 and v_2, so this path has to be fixed pointwise. $\square$

Proof of Theorem 15. Clearly, if an infinite tree has a pendant vertex, then there is an endomorphism which moves only this vertex and fixes everything else.

So let $T = (V, E)$ be a tree without pendant vertices and let φ be a nontrivial endomorphism of T. Assume that the motion of φ is finite. Then the set $\mathrm{Fix}(\varphi)$ of fixed points of φ contains all but finitely many vertices of T. Since T has no pendant vertices such a set does not induce a connected subgraph. This contradicts Lemma 16. $\square$

Now that we have characterized the trees with infinite endomorphism motion, we would like to show that all of them have endomorphism distinguishing number 2.

Theorem 17. *The endomorphism distinguishing number of countable trees T with at most one pendant vertex is 2.*

Proof. The proof consists of two stages. First we color part of the vertices such that every endomorphism which preserves this partial coloring has to fix all distances from a given vertex v_0. Then we color the other vertices in order to break all remaining endomorphisms.

For the first part of the proof, let v_0 be a pendant vertex of T, or any vertex if T is a tree without pendant vertices. Denote by S_n the set of vertices at distance n from v_0, that is the sphere of radius n with center v_0. Now color v_0 white and all of S_1 and S_2 black. Periodically color all subsequent spheres according to the pattern outlined in Figure 2. In other words always color two spheres white, then four spheres black, leave two spheres uncolored, color another four spheres black and proceed inductively. Furthermore, we require that adjacent uncolored vertices are assigned different colors in the second step of the proof.

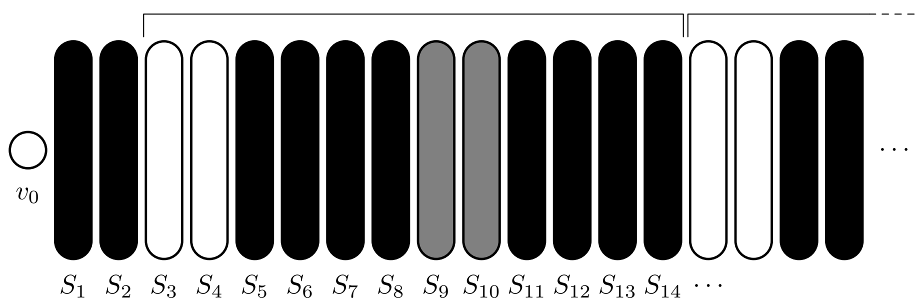

Figure 2: Coloring of the spheres in the first part of the proof of Theorem 17 with the period of the periodic part indicated at the top. Grey spheres are left uncolored for the second stage of the proof.

Now we claim that this coloring fixes v_0 in every endomorphism. To prove this consider a ray $v_0v_1v_2v_3\dots$ starting at v_0. Clearly $v_i \in S_i$ holds for every i. Assume that there is a color preserving endomorphism φ of T which does not fix v_0 and consider the image of the previously chosen ray under φ, that is, let $\tilde{v}_i = \varphi(v_i)$. Clearly $\tilde{v}_0$ has to lie either in a white sphere or in a sphere which has not yet been colored. We will look at those cases and show that all of them lead to a contradiction. So assume that $\tilde{v}_0 \in S_k$ for some $k > 0$.

- If $k = 3$, then $\tilde{v}_1$ must lie in S_2 since it must be a black neighbor of $\tilde{v}_0$. For similar reasons $\tilde{v}_2 \in S_1$ and $\tilde{v}_3 = v_0$ must hold. Now $\tilde{v}_4$ has to be a white neighbor of $\tilde{v}_3$ but v_0 only has black neighbors, a contradiction.

- If $k \in 3 + 12\mathbb{N}$ we get $\tilde{v}_1 \in S_{k-1}$ and $\tilde{v}_2 \in S_{k-2}$ by the same argument as above. Now $\tilde{v}_3$ would need to be a white neighbor of $\tilde{v}_2$ but $\tilde{v}_2$ only has black neighbors.

- If $k \in 4 + 12\mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then, for similar reasons as in the previous cases, $\tilde{v}_1 \in S_{k+1}$ and $\tilde{v}_2 \in S_{k+2}$. Again $\tilde{v}_2$ has no white neighbors.

- If $k \in 9 + 12\mathbb{N}_0$, then $\tilde{v}_2$ lies in one of S_{k-2}, S_k and S_{k+2}. In the first case $\tilde{v}_2$ clearly has no white neighbors. In the other cases it may have a white neighbor $\tilde{v}_3$ in S_{k+1}, but then $\tilde{v}_3$ has no white neighbors, because its neighbor in S_k must have a different color.

- If $k \in 10 + 12\mathbb{N}_0$, we can use an argument that is symmetric to the previous case.

Since there are no more cases left we can conclude that v_0 has to be fixed by every endomorphism which preserves this coloring.

However, we wish to prove that such an endomorphism φ preserves all distances from v_0, that is, that φ maps S_k into itself for each k.

We first show that any $v \in S_k$ for $k \in 2 + 12\mathbb{N}_0$ must have its image in S_l for some $l \in 2 + 12\mathbb{N}_0$. Since v_0 is fixed, v must be mapped to a vertex at even distance from v_0. Furthermore, this vertex must be black and have a white neighbor, which again must have a white neighbor. It is easy to check that the only vertices for which all of this holds lie in S_l for some $l \in 2 + 12\mathbb{N}_0$.

Now assume that φ does not map S_k into itself for every k and consider the smallest k such that $\varphi(S_k) \nsubseteq S_k$. Then there must be some vertex $u \in S_k$ such that $\varphi(u) \in S_{k-2}$. This immediately implies that $k \notin \{1, 2\}$ and that $k \notin \{3, 4, 5, 6\} + 12\mathbb{N}_0$, because otherwise a white vertex would be mapped to a black vertex or vice versa. In order to treat the remaining cases, consider a vertex $v \in S_l$ whose predecessor in S_k is u, where l is chosen to be minimal with respect to the properties $l > k$, $l \in 2 + 12\mathbb{N}_0$. The unique u-v-path in T must be mapped to a $\varphi(u)$-$\varphi(v)$-walk with length at most $l-k$. This implies that $\varphi(v)$ cannot lie in S_m for $m \geqslant l$. The u-v-path does not contain two consecutive white vertices, hence the $\varphi(u)$-$\varphi(v)$-walk cannot cross the two consecutive white layers S_{l-10} and S_{l-11}. So $\varphi(v)$ cannot lie in S_m for $m \leqslant l - 12$. But this contradicts the fact that $\varphi(v)$ must lie in some S_m for $m \in 2 + 12\mathbb{N}_0$.

This completes the proof of the fact that all distances from v_0 are fixed by any endomorphism which preserves such a coloring.

For the second part of the proof, consider any enumeration $(v_i)_{i \geqslant 0}$ of the vertices of T such that, for all $i \geqslant 0$, we have $v_i \in S_j$ for some $j < 12i + 9$. It is easy to see that such an enumeration is possible. Now color all vertices in S_{12i+9} whose predecessor is v_i black and color all other vertices in this sphere white. Color the vertices of S_{12i+10} whose predecessor is v_i white, and color all other vertices in this sphere black.

We claim that the so obtained coloring is not preserved by any endomorphism but the identity. We already know that every color preserving endomorphism φ maps every sphere S_k into itself. Assume that there is a vertex v_i which is not fixed by φ. Then it is easy to see that all vertices in S_{12i+9} whose predecessor is v_i will be mapped to vertices whose predecessor is $\varphi(v_i)$. Hence φ is not color preserving. □

We conjecture that this result can be extended to uncountable trees. One does need a lower bound on the minimum degree though, see [9]. As we already noted, the fact that $D_e(T) = 2$, together with the observations that $|\mathrm{End}(T)| = \mathfrak{c}$ and $m_e(T) = \aleph_0$, supports the Endomorphism Motion Conjecture. Of course, a proof of the Endomorphism Motion Conjecture is still not in sight, not even for countable structures.

Finally, the computation of $D_e(Q_k)$ seems to be an interesting problem, even for finite cubes. Similarly, the computation of $D_e(K_n^k)$, where K_n^k denotes the k-th Cartesian power[1] of K_n, looks demanding.

[1]For the definition of the Cartesian product and Cartesian powers see [7].

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