Doubly even orientable closed 2-cell embeddings of the complete graph

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Abstract

For all $m \geqslant 1$ and $k \geqslant 2$, we construct closed 2-cell embeddings of the complete graph $K_{8km+4k+1}$ with faces of size $4k$ in orientable surfaces. Moreover, we show that when $k \geqslant 3$ there are at least $(2m-1)!/2(2m+1) = 2^{2m\log_2 m - \mathrm{O}(m)}$ nonisomorphic embeddings of this type. We also show that when $k = 2$ there are at least $\frac{1}{4}\pi^{\frac{1}{2}}m^{-\frac{5}{4}}\left(\frac{4m}{e^2}\right)^{\sqrt{m}}(1-\mathrm{o}(1))$ nonisomorphic embeddings of this type.

1 Introduction and motivation

Consider an embedding of a simple graph G in an orientable surface. If each of the faces of the embedding is homeomorphic to an open disk, then the embedding is said to be a *2-cell embedding* of G. Moreover, if no vertex appears more than once in the facial walk of any given face (that is, all the facial walks are cycles of G), then the embedding is said to be a *closed 2-cell embedding*. Hence, if the faces of a 2-cell embedding of a simple graph G with no degree one vertices are all of size less than six, then the embedding is necessarily closed. Unless otherwise stated all the embeddings we discuss will be orientable.

In this paper we will be interested in closed 2-cell embeddings of the complete graph K_n in which all the faces have the same size. We will denote such an embedding, in which all the faces have size s say, as an s-2CS(n) embedding (the etymology of this notation being that the faces of such an embedding correspond to a twofold s-cycle system of order n, an s-2CS(n)).

To date the only known results on s-2CS(n) embeddings either fix s and vary n, or vary s but have $n = f(s)$, where f is some fixed linear function. The aim of this paper is to provide s-2CS(n) embeddings in which both s and n can vary independently.

In solving the Heawood Map Colouring Conjecture for orientable surfaces of genus greater or equal to one, Youngs [14] and Ringel [13] provide triangulations of the complete graph, K_n, whenever the obvious congruence conditions ($n \equiv 0, 3, 4, 7 \pmod{12}$) are satisfied. In 2000, for $n = 12s + 7$ where $s \equiv 0, 1 \pmod 3$, Bonnington et. al. [2], showed that there are at least $2^{n^2/54-\mathrm{O}(n)}$ nonisomorphic orientable triangulations of K_n. Since 2000 there have been over half a dozen research papers on constructing nonisomorphic triangulations of the complete graph. Currently the best known lower bounds on the number of nonisomorphic triangulations of K_n in either orientable or nonorientable surfaces are of the form n^{an^2} for suitable constants $a > 0$, although these bounds have only been established for a sparse set of values of n [5, 7].

In [12], Korzhik and Voss constructed 2^{4m-1} nonisomorphic 2-cell orientable embeddings, where all the faces have size four, of the complete graph K_{8m+5}, i.e. *quadrangular embeddings*. Recently Korzhik [11] improved this result by constructing $2^{2m\log_2 m-\mathrm{O}(m)}$ nonisomorphic orientable quadrangular embeddings of K_{8m+5}.

As mentioned earlier, if an embedding of a complete graph is a triangulation or a quadrangulation (that is, all the faces either have size 3 or size 4), the embedding is trivially closed 2-cell and hence is a s-2CS(n) embedding for either $s = 3$ or $s = 4$.

Ellingham and Stephens, in [3] and [4], constructed n-2CS(n) embeddings for all $n > 5$ in nonorientable surfaces and n-2CS(n) embeddings in orientable surfaces for $n = 2^p + 2$ and $p \geqslant 3$. In [8], Griggs and McCourt constructed n-2CS($2n$+1) embeddings in orientable surfaces for all odd $n \geqslant 3$ and in nonorientable surfaces for all $n \geqslant 4$. By applying Theorem 3.1 of [6] to the results of [5] and [7], a lower bound of the form n^{an^2} on the number of nonisomorphic n-2CS(n) embeddings may be obtained for certain values of n. For both n-2CS(n) and n-2CS($2n + 1$) embeddings with $n \geqslant 6$, it is non-trivial to ensure that the embeddings are *closed* 2-cell embeddings.

For an s-2CS(n) embedding to exist, a necessary condition is that an s-2CS(n) should exist, so $s \leqslant n$ and s must divide $n(n-1)$ [1]. We will construct embeddings for $s = 4k$ where $k \geqslant 2$ and where the corresponding $4k$-2CS(n) has a cyclic automorphism of order n so that the faces of the embedding appear in orbits of length n; a necessary condition for this is that $4k$ divides $(n-1)$. As we wish to construct orientable embeddings, a quick calculation with Euler's formula yields the condition that $n = 8km + 4k + 1$, for some $m \geqslant 0$.

In Section 3, for all $m, k \geqslant 1$ we construct closed 2-cell embeddings of $K_{8km+4k+1}$ with faces of size $4k$ in orientable surfaces. In Section 4 we show that, if $k = 2$, there are at least $\frac{1}{4}\pi^{\frac{1}{2}}m^{-\frac{5}{4}}\left(\frac{4m}{e^2}\right)^{\sqrt{m}}(1 - \mathrm{o}(1))$ such embeddings and, if $k \geqslant 3$, there are at least

$(2m-1)!/2(2m+1) = 2^{2m\log_2 m - \mathrm{O}(m)}$ such embeddings. First however, in Section 2, we will discuss current graphs, the machinery of which we will use to prove the results in Section 3.

2 Current graphs

We will construct the embeddings from index 1 current graphs and we assume that the reader is familiar with these techniques. A good overview of current graphs can be found in [10], and we will follow the notation established in that book.

Our construction makes extensive use of the following Theorem due to Gross and Alpert [9] (we state a simpler version of the theorem in which all the edges in question are orientation preserving).

Theorem 1 ([9]). *Let $e_1 \dots e_d$ be the rotation at vertex v of the current graph $\langle G \to S, \mathcal{B}\rangle$, and let c_i be the current carried by the direction of the edge e_i that has v as its initial vertex. Let $c = c_1 \dots c_d$, and let r be the order of c in the current group $\mathcal{B}$. Then the derived embedding has $|\mathcal{B}|/r$ faces corresponding to vertex v, each of size rd, and each of the form*

$$(e_1, b), (e_2, bc_1), (e_3, bc_1c_2), \dots, (e_d, bc_1c_2 \dots c_{d-1}), (e_1, bc), (e_2, bcc_1), (e_3, bcc_1c_2), \dots$$

$$\dots, (e_1, bc^r) = (e_1, b).$$

In order for the derived embedding to be that of a complete graph it is sufficient for the current graph to have $(|\mathcal{B}|-1)/2$ edges and be a *one-face embedding* in which each edge is labelled with an element of $\mathcal{B} \setminus \{0\}$, such that for each element $i \in \mathcal{B} \setminus \{0\}$ either exactly one edge is labelled i or exactly one edge is labelled i^{-1}, i.e. $\langle G \to S, \mathcal{B}\rangle$ is an *index 1 current graph*; moreover, if S is an orientable surface, then the derived embedding is also orientable; see [10]. In this paper our current graphs will only have type 0 (untwisted) edges.

Set $n := 8km + 4k + 1$. For the current graphs that are constructed in this paper we set $\mathcal{B} := \mathbb{Z}_n$, so, from here on, we will write the group operation additively. We make the following observation on index 1 current graphs, which is a direct consequence of Theorem 1.

Observation 2.1. *Suppose that the vertex v has degree $4k$, rotation $e_1 \dots e_{4k}$, where edge e_i carries current c_i directed away from v, and that $\sum_{i=1}^{4k} c_i = 0$. Then the corresponding faces in the derived graph have $4k$ edges and there are n such faces. Moreover, if the set $\{\sum_{1 \leqslant j \leqslant i} c_j : 1 \leqslant i \leqslant 4k\}$ has cardinality $4k$, then the boundary walk of each of these faces is a cycle of length $4k$.*

Thus, in order to construct our desired embedding the construction of a current graph $\langle G \to S, \mathbb{Z}_n\rangle$ with the following properties would suffice.

(i) The graph G has $2m+1$ vertices and is regular of degree $4k$.

(ii) There is a circuit of the embedding $G \to S$ in which every edge is traversed exactly once in each direction, i.e. the embedding is an orientable one-face embedding.

(iii) Each edge in $G \to S$ is assigned a direction and a distinct current from $\mathbb{Z}_n \setminus \{0\}$, moreover, if some edge is labelled i, then there does not exist an edge labelled $-i$.

(iv) For every vertex v, if $e_1 \dots e_{4k}$ is the rotation of the vertex, where edge e_i carries current c_i directed away from v, then $\sum_{i=1}^{4k} c_i = 0$ and the set $\{\sum_{1 \leqslant j \leqslant i} c_j : 1 \leqslant i \leqslant 4k\}$ has cardinality $4k$.

The graph G that we will employ is the (multi-)graph $2kC_{2m+1}$, this graph clearly satisfies Property (i). See Figure 1 for an example with $k = m = 2$.

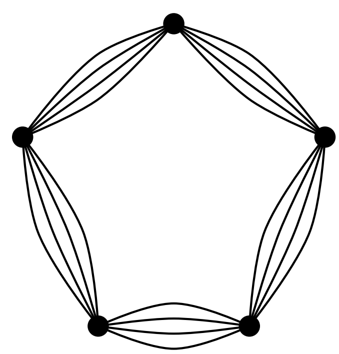


Figure 1: The graph $4C_5$.

3 Constructing doubly even embeddings

We will construct a rotation of $2kC_{2m+1}$ that induces an orientable one-face embedding (such an embedding satisfies Properties (i) and (ii)). Then we give a current assignment of $2kC_{2m+1}$ satisfying Properties (iii) and (iv).

Lemma 3.1. *For $m \geqslant 1$, there exists an upper-embedding of $2C_{2m+1}$ in an orientable surface.*

Proof. Let $V(2C_{2m+1}) = \{v_0, v_1, \dots, v_{2m}\}$. Denote the two edges between vertices v_i and v_{i+1} (where subscripts are taken modulo $2m+1$) as e_i and e_{2m+1+i}. Now, we assign the following edge rotations at each vertex

$$
\begin{array}{lcccc}
v_0: & e_0 & e_{4m+1} & e_{2m} & e_{2m+1} \\
v_1: & e_0 & e_1 & e_{2m+1} & e_{2m+2} \\
v_2: & e_1 & e_2 & e_{2m+2} & e_{2m+3} \\
\vdots & & & & \\
v_i: & e_{i-1} & e_i & e_{2m+i} & e_{2m+1+i} \\
\vdots & & & & \\
v_{2m}: & e_{2m-1} & e_{2m} & e_{4m} & e_{4m+1}
\end{array}
$$

It is easy to check that the embedding described has precisely one facial walk, namely

$$e_{2m+1}, e_{2m+2}, e_{2m+3}, \ldots, e_{4m}, e_{4m+1}, e_{2m}, e_{4m}, e_{2m-2}, e_{4m-2}, \ldots, e_2, e_{2m+2}, e_0,$$

$$e_{4m+1}, e_{2m-1}, e_{4m-1}, e_{2m-3}, \ldots, e_1, e_{2m+1}, e_0, e_1, e_2, e_3, \ldots, e_{2m-1}, e_{2m},$$

and that each edge is traversed precisely once in each direction in this walk. (We have suppressed the vertices from this sequence as they are implicit from the edges.)

Figure 2 shows the one-face embedding when $m = 2$ (in the figure the vertices have an anticlockwise orientation). □

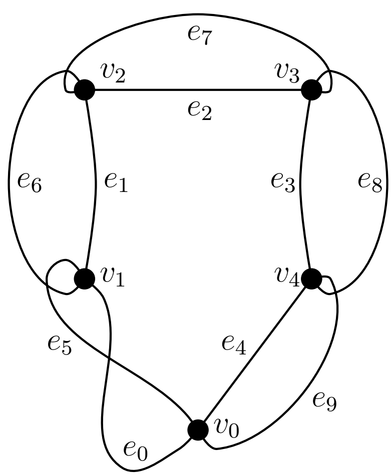


Figure 2: The embedding from Lemma 3.1 when $m = 2$.

Lemma 3.2. *Suppose we have a graph G and a rotation D of G inducing an orientable one-face embedding of G. Suppose that u and v are distinct vertices of G and the edge rotation at u is $D_u = \ldots, a, b, \ldots$ and the edge rotation at v is $D_v = \ldots, c, d, \ldots$. Then if we add two parallel (type 0) edges f_1 and f_2 joining u and v, and define the rotation D' of the obtained graph as $D'_u = \ldots, a, f_1, f_2, b, \ldots$, $D'_v = \ldots, c, f_1, f_2, d, \ldots$, and $D'_w = D_w$ for all $w \notin \{u, v\}$, then D' induces an orientable one-face embedding.*

See Figure 3 for an illustration of Lemma 3.2.

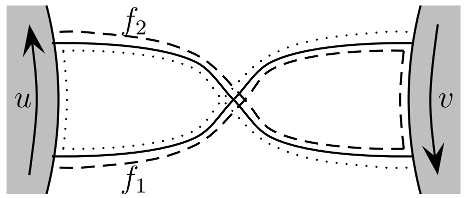


Figure 3: Illustration of Lemma 3.2.

Proof. Without loss of generality the facial walk is of the form

$$a, u, b, \dots c, v, d, \dots$$

The addition of the edges f_1 and f_2 as described in the statement of the lemma yields an embedding with the facial walk

$$a, u, f_1, v, f_2, u, b, \dots c, v, f_1, u, f_2, v, d, \dots$$

Hence, we have an orientable upper-embedding of $G = (V(G), E(G) \cup \{f_1, f_2\})$. $\square$

Theorem 2. *For $k, m \geqslant 1$, there exists an orientable $4k$-2CS($8km + 4k + 1$) embedding.*

Proof. When $k = 1$ the result follows from [12]. So we need only consider $k \geqslant 2$.

For each of the cases below we first construct an upper-embedding of $2kC_{2m+1}$ using Lemmas 3.1 and 3.2. Starting with the embedding of $2C_{2m+1}$ from Lemma 3.1, for each $0 \leqslant i \leqslant 2m$, we add the pairs of edges f_i^{2j-1}, f_i^{2j}, where $1 \leqslant j \leqslant k-1$, between vertices v_i and v_{i+1} (with subscripts taken modulo $2m+1$) using Lemma 3.2 to yield the following edge rotations (the sign of the edge indicates the orientation of the edge, positive for an edge directed into the vertex and negative otherwise).

$$\begin{array}{lcccccccccccc}
v_0: & -e_0 & e_{4m+1} & f_0^1 & f_0^2 & \dots & f_0^{2(k-1)} & e_{2m} & -e_{2m+1} & -f_{2m}^1 & -f_{2m}^2 & \dots & -f_{2m}^{2(k-1)} \\
v_1: & e_0 & -e_1 & f_1^1 & f_1^2 & \dots & f_1^{2(k-1)} & e_{2m+1} & -e_{2m+2} & -f_0^1 & -f_0^2 & \dots & -f_0^{2(k-1)} \\
v_2: & e_1 & -e_2 & f_2^1 & f_2^2 & \dots & f_2^{2(k-1)} & e_{2m+2} & -e_{2m+3} & -f_1^1 & -f_1^2 & \dots & -f_1^{2(k-1)} \\
\vdots & & & & & & & & & & & & \\
v_i: & e_{i-1} & -e_i & f_i^1 & f_i^2 & \dots & f_i^{2(k-1)} & e_{2m+i} & -e_{2m+1+i} & -f_{i-1}^1 & -f_{i-1}^2 & \dots & -f_{i-1}^{2(k-1)} \\
\vdots & & & & & & & & & & & & \\
v_{2m}: & e_{2m-1} & -e_{2m} & f_{2m}^1 & f_{2m}^2 & \dots & f_{2m}^{2(k-1)} & e_{4m} & -e_{4m+1} & -f_{2m-1}^1 & -f_{2m-1}^2 & \dots & -f_{2m-1}^{2(k-1)}
\end{array}$$

We denote this embedding as $2kC_{2m+1} \to S$. By Lemmas 3.1 and 3.2, $2kC_{2m+1} \to S$ is an orientable upper embedding and so Property (ii) is satisfied.

We begin with the case where $k \geqslant 3$. The edges of $2kC_{2m+1} \to S$ are labelled as follows:

$e_i : 4km - 2ki + 2k$, for $0 \leqslant i \leqslant 2m$;

$e_i : 2ki + 2$, for $2m + 1 \leqslant i \leqslant 4m + 1$;

$f_{2i}^{j}:(-1)^{j-1}(2ki+j)$, for $0 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant 2(k-1)$; and

$f_{2i-1}^{j}:(-1)^{j-1}(4km-2ki+2k+j)$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant 2(k-1)$.

Note that these currents (and their negatives) provide each of the elements of $\mathbb{Z}_n \setminus \{0\}$ exactly once; hence, Property (iii) is satisfied. This labelling, together with the above edge rotations at the vertices, yields the following series;

$$\begin{aligned} v_0: \quad & (-(4km+2k)) + (8km+2k+2) + \left[\sum_{j=1}^{2(k-1)} (-1)^{j-1} j\right] \\ & + (2k) + (-(4km+2k+2)) + \left[\sum_{j=1}^{2(k-1)} (-1)^{j}(2km+j)\right] \end{aligned}$$

and, for $1 \leqslant i \leqslant m$;

$$\begin{aligned} v_{2i-1}: \quad & (4km+6k-4ki) + (-(4km+4k-4ki)) \\ & + \left[\sum_{j=1}^{2(k-1)} (-1)^{j-1}(4km+2k-2ki+j)\right] + (4km+4ki-2k+2) \\ & + (-(4km+4ki+2)) + \left[\sum_{j=1}^{2(k-1)} (-1)^{j}(2ki-2k+j)\right] \\ v_{2i}: \quad & (4km+4k-4ki) + (-(4km+2k-4ki)) + \left[\sum_{j=1}^{2(k-1)} (-1)^{j-1}(2ki+j)\right] \\ & + (4km+4ki+2) + (-(4km+2k+4ki+2)) \\ & + \left[\sum_{j=1}^{2(k-1)} (-1)^{j}(4km+2k-2ki+j)\right] \end{aligned}$$

Hence, these labellings yield the following sequences of partial sums;

$$\begin{aligned} v_0: \quad & 4km+2k+1, 4km+2, (4km+2+j, 4km+2-j)_{j=1}^{k-1}, \\ & 4km+k+3, 8km+3k+2, (6km+3k+2-j, 8km+3k+2+j)_{j=1}^{k-1} \end{aligned}$$

and, for $1 \leqslant i \leqslant m$;

$$\begin{aligned} v_{2i-1}: \quad & 4km+6k-4ki, 2k, (4km+4k-2ki+j, 2k-j)_{j=1}^{k-1}, 4km-k+4ki+3, \\ & 8km+3k+2, (8km+5k-2ki+2-j, 8km+3k+2+j)_{j=1}^{k-1} \\ v_{2i}: \quad & 4km+4k-4ki, 2k, (2k+2ki+j, 2k-j)_{j=1}^{k-1}, 4km+k+4ki+3, \\ & 8km+3k+2, (4km+k+2ki+2-j, 8km+3k+2+j)_{j=1}^{k-1}. \end{aligned}$$

It is easy to check that, as $k \geqslant 3$, for vertex v_l, where $0 \leqslant l \leqslant 2m$, the terms in the sequence of partial sums are all distinct and that the final term is 0; hence Property (iv) holds.

For $k = 2$ the labellings given above do not satisfy Property (iv); for example, the partial sums for v_0 given by $4km + 2k + 1$ and $4km + k + 3$ are equal. Instead, when $k = 2$, the edges of $2kC_{2m+1} \to S$ may be labelled as follows:

$e_i : 8m - 4i + 4$, for $0 \leqslant i \leqslant 2m$;

$e_i : 4i + 2$, for $2m + 1 \leqslant i \leqslant 4m + 1$;

$f^j_{2i} : (-1)^j(4i + j)$, for $0 \leqslant i \leqslant m$ and $j = 1, 2$; and

$f^j_{2i-1} : (-1)^j(8m - 4i + 4 + j)$, for $1 \leqslant i \leqslant m$ and $j = 1, 2$.

Note that this labelling satisfies Property (iii). This time the labelling yields the following series;

$$v_0 : \quad (-(8m+4)) + (16m+6) + (-1) + (2) + (4) + (-(8m+6)) + (4m+1) + (-(4m+2))$$

and, for $1 \leqslant i \leqslant m$;

$$\begin{aligned} v_{2i-1} : \quad & (8m + 12 - 8i) + (-(8m + 8 - 8i)) + (-(8m - 4i + 5)) + (8m - 4i + 6) \\ & + (8m + 8i - 2) + (-(8m + 8i + 2)) + (4i - 3) + (-(4i - 2)) \\ v_{2i} : \quad & (8m + 8 - 8i) + (-(8m + 4 - 8i)) + (-(4i + 1)) + (4i + 2) \\ & + (8m + 8i + 2) + (-(8m + 8i + 6)) + (8m + 5 - 4i) + (-(8m + 6 - 4i)). \end{aligned}$$

Once again a modicum of checking shows that for each vertex in $2kC_{2m+1} \to S$ the sequence of partial sums is made up of 8 distinct elements and so Property (iv) is satisfied. □

4 Constructing nonisomorphic embeddings

A pair of graphs, G_1 and G_2, are *isomorphic* if there exists a pair of bijections, (θ_V, θ_E) say, where $\theta_V : V(G_1) \to V(G_2)$ and $\theta_E : E(G_1) \to E(G_2)$ such that an edge $e \in E(G_1)$ has end vertices $x, y \in V(G_1)$, if and only if, $\theta_E(e)$ has end vertices $\theta_V(x), \theta_V(y) \in V(G_2)$. Two 2-cell embeddings of the complete graph K_n, say $K_n \to S$ and $K_n \to S'$, are *isomorphic* if there is a bijection $\theta : V(K_n) \to V(K_n)$ such that $v_1, v_2, \ldots, v_k$ is a facial walk in $K_n \to S$, if and only if, $\theta(v_1), \theta(v_2), \ldots, \theta(v_k)$ is a facial walk in $K_n \to S'$.

Now consider a pair of 2-cell embedding of the complete graph K_n that are lifts of a pair of two current graphs with edge labels from the same Abelian group. In [12] Korzhik and Voss showed that a necessary condition for such a pair of embeddings to be isomorphic is that the two underlying graphs must also be isomorphic (also see [11]). This result is a key ingredient in our proofs of Theorems 3 and 4.

Theorem 3. *For $k \geqslant 3$ and $m \geqslant 1$, there are at least $((2m-1)!)/2(2m+1) = 2^{2m\log_2 m - O(m)}$ nonisomorphic orientable $4k$-2CS($8km+4k+1$) embeddings.*

Theorem 4. *The number of nonisomorphic orientable 8-2CS($16m+9$) embeddings is at least $\sum_{s=1}^{\hat{s}} s!p(2m-5-\frac{s(s+1)}{2}, s)$, where $\hat{s} = \left\lfloor \frac{\sqrt{16m-31}-3}{2} \right\rfloor$ and $p(n,s)$ denotes the number of partitions of n into s positive integer parts.*

In order to prove Theorems 3 and 4 we first make the following observation on the proof of Theorem 2.

Observation 4.1. *Let $\psi \in S_{2m+1}$ be the permutation mapping i to $i+1$ modulo $2m+1$, for all $0 \leqslant i \leqslant 2m$. Then, for any permutation $\phi \in S_{2m+1}$ conjugate to ψ where $\phi(2m) = 0$, suppose that we modify the construction from the proof of Theorem 2 so that the edges $f_i^1, f_i^2, \ldots, f_i^{2(k-1)}$ are added between vertices v_i and $v_{\phi(i)}$ (instead of between vertices v_i and $v_{\psi(i)}$) so that the edge rotations at the vertices are now*

$$\begin{array}{llllllllllll}
v_0: & -e_0 & e_{4m+1} & f_0^1 & f_0^2 & \ldots & f_0^{2(k-1)} & e_{2m} & -e_{2m+1} & -f_{2m}^1 & -f_{2m}^2 & \ldots & -f_{2m}^{2(k-1)} \\
v_1: & e_0 & -e_1 & f_1^1 & f_1^2 & \ldots & f_1^{2(k-1)} & e_{2m+1} & -e_{2m+2} & -f_{\phi^{-1}(1)}^1 & -f_{\phi^{-1}(1)}^2 & \ldots & -f_{\phi^{-1}(1)}^{2(k-1)} \\
v_2: & e_1 & -e_2 & f_2^1 & f_2^2 & \ldots & f_2^{2(k-1)} & e_{2m+2} & -e_{2m+3} & -f_{\phi^{-1}(2)}^1 & -f_{\phi^{-1}(2)}^2 & \ldots & -f_{\phi^{-1}(2)}^{2(k-1)} \\
\vdots & & & & & & & & & & & & \\
v_i: & e_{i-1} & -e_i & f_i^1 & f_i^2 & \ldots & f_i^{2(k-1)} & e_{2m+i} & -e_{2m+1+i} & -f_{\phi^{-1}(i)}^1 & -f_{\phi^{-1}(i)}^2 & \ldots & -f_{\phi^{-1}(i)}^{2(k-1)} \\
\vdots & & & & & & & & & & & & \\
v_{2m}: & e_{2m-1} & -e_{2m} & f_{2m}^1 & f_{2m}^2 & \ldots & f_{2m}^{2(k-1)} & e_{4m} & -e_{4m+1} & -f_{\phi^{-1}(2m)}^1 & -f_{\phi^{-1}(2m)}^2 & \ldots & -f_{\phi^{-1}(2m)}^{2(k-1)}
\end{array}$$

Then the resulting embedding still satisfies Properties (i) to (iv).

Proof. Properties (i) to (iii) follow exactly as in the proof of Theorem 2. We just need to check that all the new construction satisfies Property (iv); i.e. the sequences of partial sums at each vertex is made up of $4k$ distinct elements. The sequence of partial sums at v_0 is unchanged so we need only to consider vertices v_i, where $1 \leqslant i \leqslant 2m$. We will consider the two cases from the proof of Theorem 2 separately, namely $k \geqslant 3$ and $k = 2$.

Suppose that $k \geqslant 3$. First note that, in the proof of Theorem 2, for $0 \leqslant i \leqslant m$, the $(2k+2)$-th term of the sequence of partial sums for both v_{2i-1} and v_{2i} is always $8km+3k+2$. Let

$$D = \{2k-t, 8km+3k+2+t : 0 \leqslant t \leqslant k-1\};$$

$$D_{\text{odd}} = \{4km+6k-4ki, 4km+4k-2ki+t, 4km-k+4ki+3 : 1 \leqslant t \leqslant k-1\}; \text{ and}$$

$$D_{\text{even}} = \{4km+4k-4ki, 2k+2ki+t, 4km+k+4ki+3 : 1 \leqslant t \leqslant k-1\}.$$

Then, for all $1 \leqslant j \leqslant m$, and $1 \leqslant \ell \leqslant k-1$ the following all hold:

$8km+5k-2kj+2-\ell, 4km+k+2kj+2-\ell \notin D$ (Claim A.1 in Appendix A);

$8km+5k-2kj+2-\ell \notin D_{\text{odd}}$ (Claim A.2 in Appendix A);

if $i \neq j-1$, then $8km+5k-2kj+2-\ell \notin D_{\mathrm{even}}$ (Claim A.3 in Appendix A);

if $i \neq j$, then $4km+k+2kj+2-\ell \notin D_{\mathrm{odd}}$ (Claim A.4 in Appendix A); and

$4km+k+2kj+2-\ell \notin D_{\mathrm{even}}$ (Claim A.5 in Appendix A).

Hence, for the case $k \geqslant 3$, Property (iv) is satisfied and the observation follows.

Now suppose that $k=2$. This time, in the proof of Theorem 2, for $0 \leqslant i \leqslant m$, the 6th term of the sequence of partial sums for both v_{2i-1} and v_{2i-1} is always 1. For all $1 \leqslant j \leqslant m$, we have the following:

$$4j-2, 8m+6-4j \notin \{8m+12-8i, 4, 8m+8+4i, 5, 8m+3+8i, 1\}$$

and

$$4j-2, 8m+6-4j \notin \{8m+8-8i, 4, 16m+12-4i, 5, 8m+7+8i, 1\}.$$

As for the previous case, the observation follows. □

Let ψ be defined as in Observation 4.1 and put

$$\mathcal{S} = \{\phi \in S_{2m+1} : \text{where } \phi \text{ is conjugate to } \psi \text{ and } \phi(2m)=0\}.$$

For each $\phi \in \mathcal{S}$, denote by G^{ϕ} the graph constructed in Observation 4.1. From such a graph G^{ϕ} a further graph G_{ϕ} may be constructed on the same vertex set by identifying the edge e_{2m+1+i} with the parallel edge e_i for $0 \leqslant i \leqslant 2m$, and by identifying together all the parallel edges f_i^j $(1 \leqslant j \leqslant 2(k-1))$ as a single edge, denoted by f_i, for $0 \leqslant i \leqslant 2m$. We will refer to the edges $e_i \in E(G_{\phi})$ as *e-edges* and the edges $f_i \in E(G_{\phi})$ as *f-edges*. Clearly if G^{ϕ_1} and G^{ϕ_2} are isomorphic, then so are G_{ϕ_1} and G_{ϕ_2}.

We now deal with the case where $k \geqslant 3$ which is somewhat simpler than the case $k=2$.

Theorem 3. *For $k \geqslant 3$ and $m \geqslant 1$, there are at least $(2m-1)!/2(2m+1) = 2^{2m\log_2 m - O(m)}$ nonisomorphic orientable $4k$-2CS$(8km+4k+1)$ embeddings.*

Proof. Fix $k \geqslant 3$ and $m \geqslant 1$. Each $\phi \in \mathcal{S}$ may be written in cycle notation as

$$\phi = (2m,\, 0,\, a_1,\, a_2,\, \ldots\, , a_{2m-1}),$$

where $\{a_1, a_2, \ldots, a_{2m-1}\} = \{1, 2, \ldots, 2m-1\}$. So the cardinality of $\mathcal{S}$ is $(2m-1)!$.

In G^{ϕ}, each pair of vertices is joined by 0, 2, $2(k-1)$ or $2k$ edges. Vertices joined by 2 or $2k$ edges must be joined by e-edges in G_{ϕ} and vertices joined by $2(k-1)$ or $2k$ edges must be joined by f-edges in G_{ϕ}. Since $k \geqslant 3$, it is therefore possible to identify the cycle (of length $2m+1$) of vertices joined by the e-edges in G_{ϕ}.

Now suppose that $\phi_1, \phi_2 \in \mathcal{S}$ and that G^{ϕ_1} and G^{ϕ_2} are isomorphic. Then there must be an isomorphism from G_{ϕ_1} to G_{ϕ_2} mapping the cycle of e-edges to itself. We can rotate this cycle or reflect it, so the size of an isomorphism class of G^{ϕ} is at most $2(2m+1)$. Hence the number of isomorphism classes of G^{ϕ} is at least $(2m-1)!/2(2m+1)$.

As mentioned at the start of this section, results from [12] show that nonisomorphic graphs G^{ϕ} produce nonisomorphic embeddings. So the number of nonisomorphic orientable $4k$-2CS$(8km+4k+1)$ embeddings is at least $(2m-1)!/2(2m+1) = 2^{2m\log_2 m - O(m)}$ as $m \to \infty$. □

We next consider the case $k = 2$. Our strategy is based on the following lemma

Lemma 4.1. *Suppose that $m \geqslant 4$. Then two different ordered partitions of $2m-5$ into distinct integer parts, each part of size at least 2, may be used to generate nonisomorphic graphs G^{ϕ}. Hence the number of nonisomorphic orientable 8-2CS($16m+9$) embeddings is at least the number of such partitions.*

Proof. Suppose that $\mathcal{R}$ is a partition of $r = 2m-5$ ($m \geqslant 4$) into s distinct integer parts $r_1, r_2, \ldots, r_s$ where $r_1 + r_2 + \ldots + r_s = r$ and $r_i \geqslant 2$ for $i = 1, 2, \ldots, s$. With each r_i associate a graph R_i having $r_i + 1$ vertices and $r_i - 1$ pairs of parallel edges having the form shown in Figure 4.

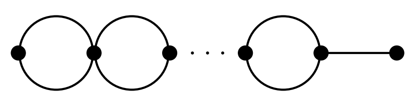


Figure 4: The graph R_i.

The graphs $R_1, R_2, \ldots, R_s$ may be joined end to end in order by identifying the vertex of degree 1 in R_i with the vertex of degree 2 in R_{i+1} for $1 \leqslant i \leqslant s-1$ to form a graph R on $r+1$ vertices as shown in Figure 5.

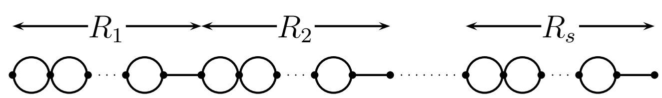


Figure 5: The graph R.

For any graph G, let $\Pi(G)$ be the subgraph induced by its parallel edges. For example, $\Pi(R_i)$ is isomorphic to $2P_{r_i}$, where P_j denotes a path with j vertices. Similarly, $\Pi(R)$ is isomorphic to a graph with s components $2P_{r_1}, 2P_{r_2}, \ldots, 2P_{r_s}$. It follows that the partition $\mathcal{R}$ may be recovered from the graph R, as may the ordering of the parts.

Further edges and five new vertices are now added to R to form a graph G which is regular of degree 4 and whose edges may be labelled to form a graph G_{ϕ}. The new edges and vertices are added in two stages and in such a way that no additional parallel edges are created, so that $\Pi(G) = \Pi(R)$, and the ordering of the partition $\mathcal{R}$ may also be recovered from G. The placement of the additional edges depends on whether s is even or odd. In both cases the first stage is to take the subgraphs R_i together in pairs, with R_s

left over if s is odd. So, for each $i = 1, 2, \ldots, \lfloor \frac{s}{2} \rfloor$, join the vertex of degree 3 in R_{2i-1} to the vertex of degree 3 in R_{2i}, and the vertex of degree 1 in R_{2i-1} to the vertex of degree 1 in R_{2i} (see Figure 6 where the additional edges are shown dashed). If s is odd, R_s receives no additional edges at this stage.

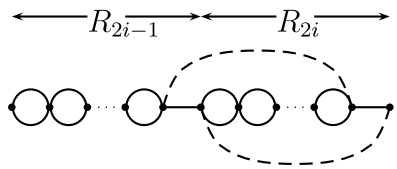
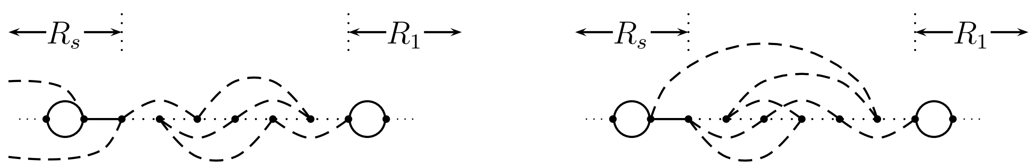


Figure 6: Additional edges between R_{2i-1} and R_{2i}.

It is not possible to recover the subgraphs R_i unambiguously from the new graph because of the additional edges. However, both the partition $\mathcal{R}$ and the ordering of this partition can still be recovered.

The second stage is to connect the end vertices of R, that is the vertex of degree 2 from R_1 and the vertex of degree 1 from R_s, by adding five new vertices and 12 new edges as shown in Figure 7, where six of the new edges are shown with dotted lines and six with dashed lines.



(a) s even. (b) s odd.

Figure 7: Completing the graph G.

The resulting graph G is regular of degree 4. The vertex of degree 2 in the subgraph R_1 lies in a triangle in G where the other two vertices of which are not in parallel edges, while the vertex of degree 3 in the subgraph R_s does not lie in such a triangle in G. (This latter vertex will lie in a triangle if $r_s = 2$ and s is even, but the other two vertices of this triangle are in parallel edges.) So the vertex, say v_0, of degree 2 in R_1 may be identified and hence the direction of the ordering of the subgraphs R_i in G may be determined (that is, $(R_1, R_2, \ldots, R_s)$ may be distinguished from $(R_s, R_{s-1}, \ldots, R_1)$).

Since $r_1 \geqslant 2$, the subgraph R_1 has a pair of parallel edges incident with v_0. Label the adjacent vertex of R_1 as v_{2m}. The graph G may be decomposed into two edge-disjoint cycles of length $2m + 1$ each having v_0 and v_{2m} as adjacent vertices. For example, one cycle may be taken as comprising all the dashed edges together with one edge from each pair of parallel edges, and the other cycle as comprising all the remaining edges. Label all the unlabelled vertices $v_1, v_2, \ldots, v_{2m-1}$ so that one of these cycles is $(v_0, v_1, \ldots, v_{2m})$. Then label the edges of this cycle $e_0, e_1, \ldots, e_{2m}$ so that e_i has end vertices v_i and v_{i+1}. The other cycle then determines a permutation $\phi \in \mathcal{S}$ such that this cycle is $(v_0, v_{2m}, v_{\phi(2m)}, v_{\phi^2(2m)}, \ldots, v_{\phi^{(2m-1)}(2m)})$, and the remaining unlabelled edges may be labelled $f_0, f_1, \ldots, f_{2m}$ so that f_i has end vertices $v_{\phi(i)}$ and v_i. The resulting labelled graph may be denoted by G_ϕ. Finally each edge e_i is replaced by parallel edges e_i and e_{2m+1+i}, and each edge f_i is replaced by parallel edges f_i^1 and f_i^2, resulting in the graph G^ϕ.

If G^{ϕ_1} and G^{ϕ_2} formed by this construction are isomorphic, then so are G_{ϕ_1} and G_{ϕ_2}, and so both correspond to the same ordered partition $\mathcal{R}$. Applying Observation 4.1, the number of nonisomorphic orientable embeddings of 8-2CS($16m+9$) systems is therefore at least as great as the number of different ordered partitions of $2m-5$ into distinct integer parts, each part of size at least 2. $\square$

We now derive an estimate for the number of different ordered partitions of n into distinct integer parts with each part of size at least 2. We will denote this number by $Q(n)$. Let $q(n,s)$ be the number of partitions of the integer n into s distinct integer parts with each part of size at least 2. The number of ordered partitions of the integer n into s distinct integer parts with each part of size at least 2 is $s!q(n,s)$, and so $Q(n) = \sum_s s!q(n,s)$.

Let $p(n,s)$ denote the number of partitions of the integer n into s positive integer parts. If $\sum_{i=1}^{s} n_i = n$ where $n_s \geqslant n_{s-1} \geqslant \ldots \geqslant n_1 \geqslant 1$ are integers (so that $n \geqslant s$), then with $n_i' = n_i + i$ we have $\sum_{i=1}^{s} n_i' = n' = n + \frac{s(s+1)}{2}$ where $n_s' > n_{s-1}' > \ldots > n_1' > 1$ (and $n' \geqslant s(s+3)/2$). Thus every partition of $n' \geqslant s(s+3)/2$ into s distinct integer parts where each part is of size at least 2 corresponds to a partition of $n = n' - s(s+1)/2$ into s positive integer parts. It follows that $q(n,s) = p\left(n - \frac{s(s+1)}{2}, s\right)$. Hence $Q(n) = \sum_{s=1}^{\hat{s}} s!p\left(n - \frac{s(s+1)}{2}, s\right)$, where $\hat{s} = \hat{s}(n) = \left\lfloor \frac{\sqrt{8n+9}-3}{2} \right\rfloor$, so that $\frac{\hat{s}(\hat{s}+3)}{2} \leqslant \frac{1}{2}\left(\frac{\sqrt{8n+9}-3}{2}\right)\left(\frac{\sqrt{8n+9}+3}{2}\right) = n < \frac{(\hat{s}+1)(\hat{s}+4)}{2}$. We can now state the following result.

Theorem 4. *The number of nonisomorphic orientable* 8-2CS($16m+9$) *embeddings is at least* $Q(2m-5) = \sum_{s=1}^{\hat{s}} s!p(2m-5-\frac{s(s+1)}{2}, s)$*, where* $\hat{s} = \hat{s}(2m-5) = \left\lfloor \frac{\sqrt{16m-31}-3}{2} \right\rfloor$ *and* $p(n,s)$ *denotes the number of partitions of* n *into* s *positive integer part.*

Despite the existence of various estimates for $p(n,s)$, it has proved difficult to find a good asymptotic estimate for $Q(2m-5)$. The following approach is very crude. Consider the partition $2+3+\ldots+\hat{s}+(\hat{s}+1)$ of $\hat{s}(\hat{s}+3)/2$, where $\hat{s} = \hat{s}(2m-5) = \left\lfloor \frac{\sqrt{16m-31}-3}{2} \right\rfloor$ is as defined previously, so that $\hat{s}(\hat{s}+3)/2 \leqslant 2m-5$. Put $r = (2m-5) - \hat{s}(\hat{s}+3)/2$ and replace the term $(\hat{s}+1)$ in the partition by $(\hat{s}+1+r)$ so that it now forms a partition of $2m-5$ into $\hat{s}$ distinct parts with each part of size at least 2. The number of ordered partitions of $2m-5$ into $\hat{s}$ distinct parts with each part of size at least 2 is at least $\hat{s}!$, so $Q(2m-5) \geqslant \hat{s}!$.

It is easy to show that if m is sufficiently large then $\hat{s} > 2\sqrt{m} - 3 = 2\sqrt{m}(1 - \frac{3}{2\sqrt{m}})$.

Then, using Stirling's Theorem, we have

$$\begin{aligned} Q(2m-5) &> \sqrt{2\pi(2\sqrt{m}-3)}\left(\frac{2\sqrt{m}(1-\frac{3}{2\sqrt{m}})}{e}\right)^{2\sqrt{m}-3} \\ &= 2\pi^{\frac{1}{2}}m^{\frac{1}{4}}\left(\tfrac{4m}{e^2}\right)^{\sqrt{m}}(1-\tfrac{3}{2\sqrt{m}})^{2\sqrt{m}}\left(\tfrac{e}{2\sqrt{m}-3}\right)^3(1-\mathrm{o}(1)) \\ &= 2\pi^{\frac{1}{2}}m^{\frac{1}{4}}\left(\tfrac{4m}{e^2}\right)^{\sqrt{m}}e^{-3}\tfrac{e^3}{8m^{\frac{3}{2}}}(1-\mathrm{o}(1)) = \tfrac{1}{4}\pi^{\frac{1}{2}}m^{-\frac{5}{4}}\left(\tfrac{4m}{e^2}\right)^{\sqrt{m}}(1-\mathrm{o}(1)). \end{aligned}$$

This is likely to be a considerable under-estimate for the number of embeddings.

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A Observation 4.1

Suppose that $m \geqslant 1$ and $k \geqslant 3$. For $0 \leqslant i \leqslant m$, in the proof of Observation 4.1, the following sets are defined.

$$D = \{2k - t, 8km + 3k + 2 + t : 0 \leqslant t \leqslant k - 1\};$$

$$D_{\text{odd}} = \{4km + 6k - 4ki, 4km + 4k - 2ki + t, 4km - k + 4ki + 3 : 1 \leqslant t \leqslant k - 1\}; \text{ and}$$

$$D_{\text{even}} = \{4km + 4k - 4ki, 2k + 2ki + t, 4km + k + 4ki + 3 : 1 \leqslant t \leqslant k - 1\}.$$

Claim A.1. *For all* $1 \leqslant j \leqslant m$ *and* $1 \leqslant \ell \leqslant k - 1$*;* $8km + 5k - 2kj + 2 - \ell, 4km + k + 2kj + 2 - \ell \notin D$.

Proof. Let $0 \leqslant t \leqslant k - 1$. Suppose that, for some $1 \leqslant j \leqslant m$ and $1 \leqslant \ell \leqslant k - 1$; $\{8km + 5k - 2kj + 2 - \ell, 4km + k + 2kj + 2 - \ell\} \cap D \neq \emptyset$. Then

$8km+5k-2kj+2-\ell = 2k-t$, so $8km+3k+2+t = 2kj+2k+\ell \leqslant 2km+3k-1$; or

$8km + 5k - 2kj + 2 - \ell = 8km + 3k + 2 + t$, so $2k = t + 2kj + \ell$, hence $2k|(t + \ell)$; or

$4km + k + 2kj + 2 - \ell = 2k - t$, so $4km + 2kj + 2 + t = k + \ell \leqslant 2k - 1$; or

$4km + k + 2kj + 2 - \ell = 8km + 3k + 2 + t$, so $2km \geqslant 2kj = 4km + 2k + t + \ell$.

Thus, in all four cases, we have a contradiction. □

Claim A.2. *For all* $1 \leqslant j \leqslant m$ *and* $1 \leqslant \ell \leqslant k - 1$*;* $8km + 5k - 2kj + 2 - \ell \notin D_{odd}$.

Proof. Let $0 \leqslant t \leqslant k - 1$. Suppose that, for some $1 \leqslant i, j \leqslant m$ and $1 \leqslant \ell \leqslant k - 1$; $8km + 5k - 2kj + 2 - \ell \in D_{\text{odd}}$. Then

$8km+5k-2kj+2-\ell = 4km+6k-4ki$, so $4km+4ki+2 = k+2kj+\ell \leqslant 2km+2k-1$; or

$8km + 5k - 2kj + 2 - \ell = 4km + 4k - 2ki + t$, so $4km + k + 2ki + 2 = 2kj + \ell + t \leqslant 2km + 2k - 2$; or

$8km + 5k - 2kj + 2 - \ell = 4km - k + 4ki + 3$, so $4km + 6k = 4ki + 2kj + \ell + 1$, hence $2k|(\ell + 1)$.

Thus, in all three cases, we have a contradiction. □

Claim A.3. *For all* $1 \leqslant j \leqslant m$ *and* $1 \leqslant \ell \leqslant k-1$*; if* $i \neq j-1$*, then* $8km+5k-2kj+2-\ell \notin D_{even}$.

Proof. Let $0 \leqslant t \leqslant k - 1$. Suppose that, for some $1 \leqslant i, j \leqslant m$ where $i \neq j - 1$ and $1 \leqslant \ell \leqslant k - 1$; $8km + 5k - 2kj + 2 - \ell \in D_{\text{even}}$. Then

$8km+5k-2kj+2-\ell = 4km+4k-4ki$, so $4km+k+4ki+2 = 2kj+\ell \leqslant 2km+k-1$; or

$8km + 5k - 2kj + 2 - \ell = 2k + 2ki + t$, so $8km + 3k + 2 = 2ki + 2kj + t + \ell \leqslant 4km + 2k - 2$; or

$8km+5k-2kj+2-\ell = 4km+k+4ki+3$, so $4km+4k = 4ki+2kj+\ell+1$, hence $2k|(\ell+1)$.

Thus, in all three cases, we have a contradiction. $\square$

Claim A.4. *For all* $1 \leqslant j \leqslant m$ *and* $1 \leqslant \ell \leqslant k-1$*; if* $i \neq j$*, then* $4km+k+2kj+2-\ell \notin D_{odd}$.

Proof. Let $0 \leqslant t \leqslant k-1$. Suppose that, for some $1 \leqslant i,j \leqslant m$ where $i \neq j$ and $1 \leqslant \ell \leqslant k-1$; $4km+k+2kj+2-\ell \in D_{\text{odd}}$. Then

$4km+k+2kj+2-\ell = 4km+6k-4ki$, so, as $i \neq j$, $8k+2 \leqslant 2+2kj+4ki = 5k+\ell \leqslant 6k-1$; or

$4km+k+2kj+2-\ell = 4km+4k-2ki+t$, so, as $i \neq j$, $6k+2 \leqslant 2+2kj+2ki = 3k+t+\ell \leqslant 5k-2$; or

$4km+k+2kj+2-\ell = 4km-k+4ki+3$, so $2k+2kj = 4ki+\ell+1$, hence $2k|(\ell+1)$.

Thus, in all three cases, we have a contradiction. $\square$

Claim A.5. *For all* $1 \leqslant j \leqslant m$ *and* $1 \leqslant \ell \leqslant k-1$*;* $4km+k+2kj+2-\ell \notin D_{even}$.

Proof. Let $0 \leqslant t \leqslant k-1$. Suppose that, for some $1 \leqslant i,j \leqslant m$ and $1 \leqslant \ell \leqslant k-1$; $4km+k+2kj+2-\ell \in D_{\text{even}}$. Then

$4km+k+2kj+2-\ell = 4km+4k-4ki$, so $4ki+2kj+2 = 3k+\ell \leqslant 4k-1$; or

$4km+k+2kj+2-\ell = 2k+2ki+t$, so $4km+2kj+2 = k+2ki+t+\ell \leqslant 2km+3k-2$; or

$4km+k+2kj+2-\ell = 4km+k+4ki+3$, so $2kj = 4ki+\ell+1$, hence $2k|(\ell+1)$.

Thus, in all three cases, we have a contradiction. $\square$