On the Positive Moments of Ranks of Partitions

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Abstract

By introducing k-marked Durfee symbols, Andrews found a combinatorial interpretation of the $2k$-th symmetrized moment $\eta_{2k}(n)$ of ranks of partitions of n in terms of $(k+1)$-marked Durfee symbols of n. In this paper, we consider the k-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks of partitions of n which is defined as the truncated sum over positive ranks of partitions of n. As combinatorial interpretations of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$, we show that for given k and i with $1 \leqslant i \leqslant k+1$, $\bar{\eta}_{2k-1}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being zero and $\bar{\eta}_{2k}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being positive. The interpretations of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ are independent of i, and they imply the interpretation of $\eta_{2k}(n)$ given by Andrews since $\eta_{2k}(n)$ equals $\bar{\eta}_{2k-1}(n)$ plus twice of $\bar{\eta}_{2k}(n)$. Moreover, we obtain the generating functions of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$.

Keywords: rank of a partition; k-marked Durfee symbol; moment of ranks

1 Introduction

This paper is concerned with a combinatorial study of the symmetrized positive moments of ranks of partitions. The notion of symmetrized moments was introduced by Andrews [1]. Any odd symmetrized moment is zero because of the symmetry of ranks. For an even symmetrized moment, Andrews found a combinatorial interpretation by introducing k-marked Durfee symbols. It is natural to investigate the combinatorial interpretation of an odd symmetrized moment which is defined as a truncated sum over positive ranks of partitions of n. We give combinatorial interpretations of both the even and the odd positive moments in terms of k-marked Durfee symbols, which also lead to the combinatorial interpretation of an even symmetrized moment of ranks given by Andrews.

The rank of a partition λ introduced by Dyson [6] is defined as the largest part minus the number of parts. Let $N(m,n)$ denote the number of partitions of n with rank m. The following generating function of $N(m,n)$ was conjectured by Dyson [6] in 1944 and proved by Atkin and Swinnerton-Dyer [3] in 1954. A combinatorial proof was found by Dyson [7] in 1969.

Theorem 1.1. *For given integer m, we have*

$$\sum_{n=0}^{+\infty} N(m,n)q^n = \frac{1}{(q;q)_\infty} \sum_{n=1}^{+\infty} (-1)^{n-1} q^{n(3n-1)/2+|m|n}(1-q^n). \tag{1.1}$$

Recently, Andrews [1] introduced the k-th symmetrized moment $\eta_k(n)$ of ranks of partitions of n as given by

$$\eta_k(n) = \sum_{m=-\infty}^{+\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m,n). \tag{1.2}$$

It can be easily seen that for any k, $\eta_k(n)$ is a linear combination of the moments $N_j(n)$ of ranks given by Atkin and Garvan [4]

$$N_j(n) = \sum_{m=-\infty}^{\infty} m^j N(m,n).$$

For example,

$$\eta_6(n) = \frac{1}{720} N_6(n) - \frac{1}{144} N_4(n) + \frac{1}{180} N_2(n).$$

In view of the symmetry $N(-m,n) = N(m,n)$, we have $\eta_{2k+1}(n) = 0$. As for an even symmetrized moment $\eta_{2k}(n)$, Andrews gave the following combinatorial interpretation by introducing k-marked Durfee symbols. For the definition of k-marked Durfee symbols, see Section 2.

Theorem 1.2 (Andrews [1])**.** *For any $k \geqslant 1$, $\eta_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of n.*

Andrews [1] proved the above theorem by using the k-fold generalization of Watson's q-analog of Whipple's theorem. Ji [9] found a combinatorial proof of Theorem 1.2 by establishing a map from k-marked Durfee symbols to ordinary partitions. Kursungoz [10] gave another proof of Theorem 1.2 by using an alternative representation of k-marked Durfee symbols.

In this paper, we introduce the k-th symmetrized positive moment $\bar{\eta}_k(n)$ of ranks as given by

$$\bar{\eta}_k(n) = \sum_{m=1}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m,n),$$

or equivalently,

$$\bar{\eta}_{2k-1}(n)=\sum_{m=1}^{\infty}\binom{m+k-1}{2k-1}N(m,n) \tag{1.3}$$

and

$$\bar{\eta}_{2k}(n)=\sum_{m=1}^{\infty}\binom{m+k-1}{2k}N(m,n). \tag{1.4}$$

Furthermore, it is easy to see that for any k, $\bar{\eta}_k(n)$ is a linear combination of the positive moments $\overline{N}_j(n)$ of ranks introduced by Andrews, Chan and Kim [2] as given by

$$\overline{N}_j(n)=\sum_{m=1}^{\infty}m^j N(m,n).$$

For example,

$$\begin{aligned}\bar{\eta}_4(n)&=\frac{1}{24}\overline{N}_4(n)-\frac{1}{12}\overline{N}_3(n)-\frac{1}{24}\overline{N}_2(n)+\frac{1}{12}\overline{N}_1(n),\\ \bar{\eta}_5(n)&=\frac{1}{120}\overline{N}_5(n)-\frac{1}{24}\overline{N}_3(n)+\frac{1}{30}\overline{N}_1(n).\end{aligned}$$

By the symmetry $N(-m,n)=N(m,n)$, it is readily seen that

$$\eta_{2k}(n)=2\overline{\eta}_{2k}(n)+\overline{\eta}_{2k-1}(n). \tag{1.5}$$

The main objective of this paper is to give combinatorial interpretations of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$. We show that for given k and i with $1\leqslant i\leqslant k+1$, $\bar{\eta}_{2k-1}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being zero and $\bar{\eta}_{2k}(n)$ equals the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being positive. It should be noted that $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ are independent of i since the ranks of k-marked Durfee symbols are symmetric, see Andrews [1, Corollary 12].

With the aid of Theorem 2.1 and Theorem 2.2 together with the generating function (1.1) of $N(m,n)$, we obtain the generating functions of $\bar{\eta}_{2k}(n)$ and $\bar{\eta}_{2k-1}(n)$.

2 Combinatorial interpretations

In this section, we give combinatorial interpretations of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ in terms of k-marked Durfee symbols. For a partition λ of n, we write $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_s)$ with the entries λ_i in nonincreasing order such that $\lambda_1+\lambda_2+\cdots+\lambda_s=n$. We assume that all the parts of λ are positive. The number of parts of λ is called the length of λ, denoted by $\ell(\lambda)$. The weight of λ is the sum of parts, denoted $|\lambda|$.

Recall that a k-marked Durfee symbol of n introduced by Andrews [1] is a two-line array composed of k pairs of partitions (α^1,β^1), $(\alpha^2,\beta^2),\ldots,(\alpha^k,\beta^k)$ along with a positive integer D which is represented in the following form:

$$\tau=\begin{pmatrix}\alpha^k, & \alpha^{k-1}, & \ldots, & \alpha^1\\ \beta^k, & \beta^{k-1}, & \ldots, & \beta^1\end{pmatrix}_D,$$

where the partitions $\alpha^i = (\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_s)$ and $\beta^i = (\beta^i_1, \beta^i_2, \ldots, \beta^i_s)$ satisfy the following four conditions:

(1) The partitions α^i $(1 \leqslant i < k)$ are nonempty, while α^k and β^i $(1 \leqslant i \leqslant k)$ are allowed to be empty;

(2) $\beta^{i-1}_1 \leqslant \alpha^{i-1}_1 \leqslant \min\{\alpha^i_s, \beta^i_s\}$ for $2 \leqslant i \leqslant k$;

(3) α^k_1, $\beta^k_1 \leqslant D$;

(4) $\sum_{i=1}^k (|\alpha^i| + |\beta^i|) + D^2 = n$.

Let

$$\tau = \begin{pmatrix} \alpha^k, & \alpha^{k-1}, & \ldots, & \alpha^1 \\ \beta^k, & \beta^{k-1}, & \ldots, & \beta^1 \end{pmatrix}_D$$

be a k-marked Durfee symbol. The pair (α^i, β^i) of partitions is called the i-th vector of τ. Andrews defined the i-th rank $\rho_i(\tau)$ of τ as follows

$$\rho_i(\tau) = \begin{cases} \ell(\alpha^i) - \ell(\beta^i) - 1, & \text{for } \ 1 \leqslant i < k, \\ \ell(\alpha^k) - \ell(\beta^k), & \text{for } \ i = k. \end{cases}$$

For example, consider the following 3-marked Durfee symbol

$$\tau = \begin{pmatrix} \overbrace{5_3, 4_3}^{\alpha^3}, & \overbrace{4_2, 3_2, 3_2, 2_2}^{\alpha^2}, & \overbrace{2_1}^{\alpha^1} \\ \underbrace{4_3}_{\beta^3}, & \underbrace{3_2, 2_2, 2_2}_{\beta^2}, & \underbrace{2_1, 2_1}_{\beta^1} \end{pmatrix}_5 .$$

We have $\rho_1(\tau) = -2$, $\rho_2(\tau) = 0$, and $\rho_3(\tau) = 1$.

For an odd symmetrized moment $\bar{\eta}_{2k-1}(n)$, we have the following combinatorial interpretation.

Theorem 2.1. *For given positive integers k and i with $1 \leqslant i \leqslant k+1$, $\bar{\eta}_{2k-1}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of n with the i-th rank equal to zero.*

For the even case, we have the following interpretation.

Theorem 2.2. *For given positive integers k and i with $1 \leqslant i \leqslant k+1$, $\bar{\eta}_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being positive.*

The proofs of the above two interpretations are based on the following partition identity obtained by Ji [9]. We shall adopt the notation $D_k(m_1, m_2, \ldots, m_k; n)$ as used by Andrews [1] to denote the number of k-marked Durfee symbols of n with the i-th rank equal to m_i for $1 \leqslant i \leqslant k$.

Theorem 2.3. *For $k \geqslant 2$ and $n \geqslant 1$, we have*

$$D_k(m_1, \ldots, m_k; n) = \sum_{t_1,\ldots,t_{k-1}=0}^{\infty} N\left(\sum_{i=1}^{k} |m_i| + 2\sum_{i=1}^{k-1} t_i + k - 1, n\right). \tag{2.1}$$

To derive the above interpretations of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$, we also need the following symmetric property given by Andrews [1]. Boulet and Kursungoz [5] found a combinatorial proof of this fact.

Theorem 2.4. *For $k \geqslant 2$ and $n \geqslant 1$, $D_k(m_1, \ldots, m_k; n)$ is symmetric in $m_1,\ m_2, \ldots, m_k$.*

We are now in a position to prove Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. By Theorem 2.4, it suffices to show that

$$\sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n) = \bar{\eta}_{2k-1}(n). \tag{2.2}$$

Using Theorem 2.3, we get

$$\begin{aligned} &\sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} D_{k+1}(0, m_2, m_3, \ldots, m_{k+1}; n) \\ &\qquad = \sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} N\left(\sum_{i=2}^{k+1} |m_i| + 2\sum_{i=1}^{k} t_i + k, n\right). \end{aligned} \tag{2.3}$$

For $k \geqslant 1$ and $m \geqslant k$, let $c_k(m)$ denote the number of integer solutions to the equation

$$|m_2| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = m - k,$$

where m_i are integers and t_i are nonnegative integers. It is easy to see that the generating function of $c_k(m)$ is equal to

$$\begin{aligned} \sum_{m=k}^{\infty} c_k(m) q^{m-k} &= (1 + 2q + 2q^2 + 2q^3 + \cdots)^k (1 + q^2 + q^4 + q^6 + \cdots)^k \\ &= \left(\frac{1+q}{1-q}\right)^k \left(\frac{1}{1-q^2}\right)^k \\ &= \frac{1}{(1-q)^{2k}} \\ &= \sum_{m=k}^{\infty} \binom{m+k-1}{2k-1} q^{m-k}. \end{aligned} \tag{2.4}$$

Hence

$$c_k(m) = \binom{m+k-1}{2k-1},$$

and (2.3) can be written as

$$\sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty} D_{k+1}(0,m_2,m_3,\ldots,m_{k+1};n)$$
$$= \sum_{m=1}^{\infty} \binom{m+k-1}{2k-1} N(m,n),$$

which is the defining expression of $\bar{\eta}_{2k-1}(n)$. This completes the proof. $\square$

Proof of Theorem 2.2. Similarly, by Theorem 2.4, it is sufficient to show that

$$\sum_{\substack{m_1>0 \\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} D_{k+1}(m_1,m_2,\ldots,m_{k+1};n) = \bar{\eta}_{2k}(n). \tag{2.5}$$

Invoking Theorem 2.3, we get

$$\sum_{\substack{m_1>0 \\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} D_{k+1}(m_1,m_2,\ldots,m_{k+1};n)$$
$$= \sum_{\substack{m_1>0 \\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} N\left(m_1+\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k,n\right). \tag{2.6}$$

For $k \geqslant 1$ and $m \geqslant k+1$, let $\bar{c}_k(m)$ denote the number of integer solutions to the equation

$$m_1+|m_2|+\cdots+|m_{k+1}|+2t_1+\cdots+2t_k = m-k,$$

where m_1 is a positive integer, m_i ($2 \leqslant i \leqslant k+1$) are integers and t_i are nonnegative integers. An easy computation shows that

$$\sum_{m=k+1}^{\infty} \bar{c}_k(m)q^{m-k} = \frac{q}{(1-q)^{2k+1}}, \tag{2.7}$$

so that

$$\bar{c}_k(m) = \binom{m+k-1}{2k}.$$

Thus, the sum on the right hand side of (2.6) becomes

$$\sum_{m=1}^{\infty} \binom{m+k-1}{2k} N(m,n),$$

which is in accordance with the definition of $\bar{\eta}_{2k}(n)$, and hence the proof is complete. $\square$

Note that the number $D_k(m_1,\ldots,m_k;n)$ has the mirror symmetry with respect to each m_i, that is, for $1 \leqslant i \leqslant k$, we have

$$D_k(m_1,\ldots,m_i,\ldots,m_k;n) = D_k(m_1,\ldots,-m_i,\ldots,m_k;n).$$

Using this symmetry property, Theorem 2.2 can be restated as follows.

Theorem 2.5. *For given positive integers k and i with $1 \leqslant i \leqslant k+1$, $\bar{\eta}_{2k}(n)$ is also equal to the number of $(k+1)$-marked Durfee symbols of n with the i-th rank being negative.*

$\overline{\eta}_1(5)$	$\overline{\eta}_2(5)$	$\overline{\eta}_2(5)$
$\begin{pmatrix} 1_2 & 1_2 & 1_2 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & & \\ 1_1 & 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_1 & 1_1 \\ 1_1 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & 1_1 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 \\ 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_2 & 1_1 \\ 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_2 & 1_1 & 1_1 \\ & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_2 & 1_1 \\ 1_1 & & \end{pmatrix}_1$
$\begin{pmatrix} 1_1 & & \\ 1_2 & 1_2 & 1_2 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 \\ 1_1 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 \\ 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_1 & 1_1 \\ 1_2 & 1_1 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 \\ 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & & \\ 1_2 & 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_1 \\ 1_2 & 1_2 \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & 1_1 \\ 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 \\ 1_2 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_1 \\ \end{pmatrix}_2$	$\begin{pmatrix} 1_1 & 1_1 \\ 1_2 & 1_2 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & & \\ 1_2 & 1_2 & 1_1 \end{pmatrix}_1$

Table 2.1: 2-Marked Durfee Symbols of 5.

For example, for $n = 5$, $k = 1$ and $i = 1$, there are twenty-one 2-marked Durfee symbols of 5 as listed in Table 2.1. The first column in Table 2.1 gives seven 2-marked Durfee symbols τ with $\rho_1(\tau) = 0$, the second column contains seven 2-marked Durfee symbols τ with $\rho_1(\tau) > 0$ and the third column contains seven 2-marked Durfee symbols τ with $\rho_1(\tau) < 0$. It can be verified that $\overline{\eta}_1(5) = 7$, $\overline{\eta}_2(5) = 7$ and $\eta_2(5) = \overline{\eta}_1(5) + 2\overline{\eta}_2(5) = 21$.

3 The generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$

In this section, we obtain the generating functions of $\bar{\eta}_{2k-1}(n)$ and $\bar{\eta}_{2k}(n)$ with the aid of Theorem 2.1 and Theorem 2.2. In doing so, we use the generating function of $N(m,n)$ to derive the generating functions of $D_{k+1}(0, m_2, \ldots, m_{k+1}; n)$ and $D_{k+1}(m_1, m_2, \ldots, m_{k+1}; n)$.

Theorem 3.1. *For $k \geqslant 1$, we have*

$$\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{n=0}^{\infty} D_{k+1}(0,m_2,\ldots,m_{k+1};n) x_1^{m_2}\cdots x_k^{m_{k+1}} q^n$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} \frac{(1-q^n)}{\prod_{j=1}^{k}(1-x_j q^n)(1-x_j^{-1}q^n)}. \tag{3.1}$$

Proof. Let

$$G_k(x_1,\ldots,x_k;q) = \sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{n=0}^{\infty} D_{k+1}(0,m_2,\ldots,m_{k+1};n) x_1^{m_2}\cdots x_k^{m_{k+1}} q^n.$$

By Theorem 2.3, we have

$$G_k(x_1,\ldots,x_k;q)$$
$$= \sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} x_1^{m_2}\cdots x_k^{m_{k+1}} \sum_{n=0}^{\infty} N\left(\sum_{i=2}^{k+1}|m_i| + 2\sum_{i=1}^{k} t_i + k, n\right) q^n. \tag{3.2}$$

Using the generating function (1.1) of $N(m,n)$ with m replaced by $\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k} t_i + k$, we find that

$$\sum_{n=0}^{\infty} N\left(\sum_{i=2}^{k+1}|m_i| + 2\sum_{i=1}^{k} t_i + k, n\right) q^n$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+n(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k} t_i+k)} (1-q^n). \tag{3.3}$$

Substituting (3.3) into (3.2), we get

$$G_k(x_1,\ldots,x_k;q) = \sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} x_1^{m_2}\cdots x_k^{m_{k+1}}$$
$$\times \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+n(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k} t_i+k)} (1-q^n). \tag{3.4}$$

Write (3.4) in the following form

$$G_k(x_1,\ldots,x_k;q) = \frac{1}{(q;q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} (1-q^n)$$
$$\times \sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} x_1^{m_2}\cdots x_k^{m_{k+1}} q^{n(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k} t_i)}. \tag{3.5}$$

Notice that

$$\sum_{a=-\infty}^{+\infty}\sum_{b=0}^{+\infty} x^a q^{n(|a|+2b)} = \frac{1}{(1-xq^n)(1-x^{-1}q^n)}. \tag{3.6}$$

Applying the above formula (3.6) repeatedly to (3.5), we deduce that

$$G_k(x_1,\ldots,x_k;q) = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+kn}\frac{(1-q^n)}{\prod_{j=1}^{k}(1-x_jq^n)(1-x_j^{-1}q^n)},$$

as required. □

Setting $x_j=1$ for $1\leqslant j\leqslant k$ in Theorem 3.1 and applying Theorem 2.1, we arrive at the following generating function of $\bar{\eta}_{2k-1}(n)$.

Corollary 3.2. *For $k\geqslant 1$, we have*

$$\sum_{n=1}^{\infty}\bar{\eta}_{2k-1}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n-1)/2+kn}\frac{1}{(1-q^n)^{2k-1}}. \tag{3.7}$$

Since $\bar{\eta}_1(n)=\overline{N}_1(n)$, when taking $k=1$ in (3.7), we are led to the generating function for $\overline{N}_1(n)$ as given by Andrews, Chan and Kim in [2, Theorem 1].

The following generating function can be derived by using the same reasoning as in the proof of Theorem 3.1.

Theorem 3.3. *For $k\geqslant 1$, we have*

$$\sum_{\substack{m_1>0\\ m_2,\ldots,m_{k+1}=-\infty}}^{\infty}\sum_{n=1}^{\infty} D_{k+1}(m_1,m_2,\ldots,m_{k+1};n)x_1^{m_1}\cdots x_{k+1}^{m_{k+1}}q^n$$

$$= \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n+1)/2+kn}\frac{x_1(1-q^n)}{(1-x_1q^n)\prod_{j=2}^{k+1}(1-x_jq^n)(1-x_j^{-1}q^n)}. \tag{3.8}$$

Setting $x_j=1$ for $1\leqslant j\leqslant k+1$ in Theorem 3.3 and using Theorem 2.2, we come to the following generating function of $\bar{\eta}_{2k}(n)$.

Corollary 3.4. *For $k\geqslant 1$, we have*

$$\sum_{n=1}^{\infty}\bar{\eta}_{2k}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(3n+1)/2+kn}\frac{1}{(1-q^n)^{2k}}. \tag{3.9}$$

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References

[1] G. E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

[2] G. E. Andrews, S. H. Chan, and B. Kim, The odd moments of ranks and cranks, J. Combin. Theory Ser. A 120 (2013) 77–91.

[3] A. O. L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, Proc. London Math. Soc. III. Ser. 4 (1954) 84–106.

[4] A. O. L. Atkin and F. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. 7 (2003) 343–366.

[5] C. Boulet and K. Kursungoz, Symmetry of k-marked Durfee symbols, Int. J. Number Theory 7 (2011) 215–230.

[6] F. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

[7] F. Dyson, A new symmetry of partitions, J. Combin. Theory Ser. A 7 (1969) 56–61.

[8] F. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7, 11, Trans. Amer. Math. Soc. 305 (1988) 47–77.

[9] K. Q. Ji, The combinatorics of k-marked Durfee symbols, Trans. Amer. Math. Soc. 363 (2011) 987–1005.

[10] K. Kurşungöz, Counting k-marked Durfee symbols, Electron. J. Combin. 18(1) (2011) #P41.