On Divisibility of Convolutions of Central Binomial Coefficients

Mark R. Sepanski

Department of Mathematics, Baylor University
One Bear Blace #97328
Waco, TX 76798-7328, U.S.A.

Mark_Sepanski@baylor.edu

Submitted: May 5, 2013; Accepted: Jan 11, 2014; Published: Feb 13, 2014
Mathematics Subject Classifications: 11B65, 05A10

Abstract

Recently, Z. Sun proved that

$$2\,(2m+1)\binom{2m}{m} \mid \binom{6m}{3m}\binom{3m}{m}$$

for $m \in \mathbb{Z}_{>0}$. In this paper, we consider a generalization of this result by defining

$$b_{n,k} = \frac{2^k\ (n+2k-2)!!}{(n-2)!!\,k!}.$$

In this notation, Sun's result may be expressed as $2\,(2m+1) \mid b_{(2m+1),(2m+1)-1}$ for $m \in \mathbb{Z}_{>0}$. In this paper, we prove that

$$2n \mid b_{n,un\pm 2^r}$$

for $n \in \mathbb{Z}_{>0}$ and $u, r \in \mathbb{Z}_{\geqslant 0}$ with $un \pm 2^r > 0$. In addition, we prove a type of converse. Namely, fix $k \in \mathbb{Z}$ and $u \in \mathbb{Z}_{\geqslant 0}$ with $u > 0$ if $k < 0$. If

$$2n \mid b_{n,un+k}$$

for all $n \in \mathbb{Z}_{>0}$ with $un + k > 0$, then there exists a unique $r \in \mathbb{Z}_{\geqslant 0}$ so that either

$$k = 2^r \quad \text{or} \quad k = -2^r.$$

Keywords: central binomial coefficients

1 Introduction

There has been much recent work on topics relating to central binomial coefficients ([1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12]). In particular, Z. Sun in [9] proved interesting results on congruences of sums of products of central binomial coefficients. One such result is that $2(2m+1)\binom{2m}{m} \mid \binom{6m}{3m}\binom{3m}{m}$ for $m \in \mathbb{Z}_{>0}$. The new integer sequence $m \to \binom{6m}{3m}\binom{3m}{m}/[2(2m+1)\binom{2m}{m}]$ is given by A176898 in the OIES. Using this sequence, Sun proposed a number of open conjectures ([9, Conjectures 2, 4, 6 and 8]) on certain divisibility properties of this and related sequences.

In this paper, we consider a generalization of Sun's new sequence. Recognizing $\binom{6m}{3m}\binom{3m}{m}/\binom{2m}{m}$ as the coefficient of x^{2m} in the $(2m+1)$-fold convolution of the central binomial sequence with itself, we define $b_{n,k}$ to be the k^{th} term of the n-fold convolution of the central binomial sequence with itself–which turns out to be

$$b_{n,k} = \frac{2^k\ (n+2k-2)!!}{(n-2)!!\,k!}.$$

In this notation, Sun's result is that $2(2m+1) \mid b_{2m+1,(2m+1)-1}$ for $m \in \mathbb{Z}_{>0}$.

In this paper (Theorem 1), we prove that

$$2n \mid b_{n,un\pm 2^r}$$

for $n \in \mathbb{Z}_{>0}$ and $u, r \in \mathbb{Z}_{\geqslant 0}$ with $un \pm 2^r > 0$. In particular, Sun's result is a special case of the above theorem in which $n = 2m+1$, $u = 1$, $r = 0$, and the $-$ sign is chosen. In addition, we prove a type of converse (Theorem 2). Namely, fix $k \in \mathbb{Z}$ and $u \in \mathbb{Z}_{\geqslant 0}$ with $u > 0$ if $k < 0$. If

$$2n \mid b_{n,un+k}$$

for all $n \in \mathbb{Z}_{>0}$ with $un + k > 0$, then there exists a unique $r \in \mathbb{Z}_{\geqslant 0}$ so that either

$$k = 2^r \quad \text{or} \quad k = -2^r.$$

2 Definition

Write b for the sequence of *central binomial coefficients*, $b_j = \binom{2j}{j}$, with $j \in \mathbb{Z}_{\geqslant 0}$. For $n, k \in \mathbb{Z}_{\geqslant 0}$ with $n \geqslant 1$, we define the doubly indexed sequence $b_{n,k} \in \mathbb{Z}_{>0}$ to be the k^{th} term of the n-fold convolution of b with itself, $b_{n,k} = \left(b^{*n}\right)_k$. In the degenerate case of $n = 0$, we define $b_{0,0} = 1$ and $b_{0,k} = 0$ for $k > 0$. It follows that the generating function for the sequence $k \to b_{n,k}$ is $(1-4x)^{-n/2}$ and a trivial calculation shows that

$$b_{n,k} = \frac{2^k\ (n+2k-2)!!}{(n-2)!!\,k!}$$

when $n \geqslant 2$. For use in Theorem 2 and in order to compare with [9], we note that it is straightforward to verify that

$$b_{2m,k} = 2^{2k}\binom{m+k-1}{k}, \quad b_{2m+1,k} = \frac{\binom{2m+2k}{m+k}\binom{m+k}{m}}{\binom{2m}{m}}$$

for $m \in \mathbb{Z}_{>0}$ in the first formula above and $m \in \mathbb{Z}_{\geqslant 0}$ in the second.

3 Divisibility

Here we present the main result on the divisibility of the sequence $b_{n,k}$.

Theorem 1. *For $n \in \mathbb{Z}_{>0}$ and $u, r \in \mathbb{Z}_{\geqslant 0}$,*

$$2n \mid b_{n,un+2^r}$$

and, if also $un - 2^r > 0$,

$$2n \mid b_{n,un-2^r}.$$

Proof. Begin with the convolution definition

$$b_{n,un\pm 2^r} = \sum_{\substack{a_1,\ldots,a_n \in \mathbb{Z}_{\geqslant 0} \\ a_1+\cdots+a_n = un\pm 2^r}} \prod_{j=1}^{n} \binom{2a_j}{a_j}$$

and let $X = \left\{(a_1, \ldots, a_n) \in \mathbb{Z}_{\geqslant 0}^n \mid a_1 + \cdots + a_n = un \pm 2^r\right\}$. Observe that $0 \notin X$ since $un + 2^r, un - 2^r \geqslant 1$.

The set X carries a natural action of the symmetric group, S_n, acting by permuting the coordinates. Write $\mathcal{O}_1, \ldots, \mathcal{O}_N$ for the orbits of X under the action of S_n. Clearly each $\mathcal{O}_k$, $1 \leqslant k \leqslant N$, has a unique representative of the form

$$x_k = (\overbrace{c_{1k}, \ldots c_{1k}}^{d_{1k}}, \overbrace{c_{2k}, \ldots, c_{2k}}^{d_{2k}}, \ldots, \overbrace{c_{m_k k}, \ldots c_{m_k k}}^{d_{m_k k}})$$

with $0 \leqslant c_{1k} < c_{2k} < \cdots < c_{m_k k}$ and $d_{jk}, m_k \in \mathbb{Z}_{>0}$. Since the stabilizer of x_k in S_n is clearly isomorphic to $S_{d_{1k}} \times S_{d_{2k}} \times \cdots \times S_{d_{m_k k}}$, if follows that

$$|\mathcal{O}_k| = \frac{n!}{d_{1k}! d_{2k}! \cdots d_{m_k k}!} = \binom{n}{d_k}$$

where we use multinomial notation above and write $d_k = (d_{1k}, d_{2k}, \ldots, d_{m_k k})$. Using this, we may rewrite the formula for $b_{n,un\pm 2^r}$ as

$$b_{n,un\pm 2^r} = \sum_{k=1}^{N} \binom{n}{d_k} \prod_{j=1}^{m_k} \binom{2c_{jk}}{c_{jk}}^{d_{jk}}.$$

We will prove the theorem by demonstrating that $2n \mid \binom{n}{d_k} \prod_{j=1}^{m_k} \binom{2c_{jk}}{c_{jk}}^{d_{jk}}$ for each k.

As $x_k \in X$, it follows that

$$\sum_{j=1}^{m_k} d_{jk} = n$$
$$\sum_{j=1}^{m_k} d_{jk} c_{jk} = un \pm 2^r.$$

Therefore,

$$\sum_{j=1}^{m_k} d_{jk} \left(c_{jk} - u\right) = \pm 2^r$$

and we may write the greatest common divisor of $d_{1k}, d_{2k}, \ldots, d_{m_k k}$ as 2^{q_k} for some q_k, $0 \leqslant q_k \leqslant r$. Since it follows that $2^{q_k} \mid n$, we also see that $2^{q_k} \leqslant n$.

Choose $w_{jk} \in \mathbb{Z}$, $1 \leqslant j \leqslant m_k$, so that $\sum_{j=1}^{m_k} w_{jk} d_{jk} = 2^{q_k}$. Write e_j for the j^{th} standard basis vector, $e_j = (\overbrace{0, \ldots 0, 1}^{j}, 0, \ldots 0) \in \mathbb{Z}^{m_k}$ (suppressing the m_k dependence). Then

$$2^{q_k} \binom{n}{d_k} = \sum_{j=1}^{m_k} w_{jk} d_{jk} \binom{n}{d_k} = \sum_{\substack{j=1 \\ d_{jk} \geqslant 1}}^{m_k} w_{jk} d_{jk} \binom{n}{d_k}$$
$$= \sum_{\substack{j=1 \\ d_{jk} \geqslant 1}}^{m_k} w_{jk} \binom{n-1}{d_k - e_j} n$$

so that $n \mid 2^{q_k} \binom{n}{d_k}$ and

$$2n \mid \binom{n}{d_k} 2^{q_k+1}.$$

As $\frac{1}{2}\binom{2j}{j} = \binom{2j-1}{j}$ for $j \geqslant 1$, it is well known that $2 \mid \binom{2c_{jk}}{c_{jk}}$ whenever $c_{jk} \neq 0$. If $c_{1k} \neq 0$, then $2^n \mid \prod_{j=1}^{m_k} \binom{2c_{jk}}{c_{jk}}^{d_{jk}}$. As $2^{q_k+1} \leqslant 2n \leqslant 2^n$, it follows that $2n \mid \binom{n}{d_k} \prod_{j=1}^{m_k} \binom{2c_{jk}}{c_{jk}}^{d_{jk}}$ and we are done.

Suppose, therefore, that we are in the case of $c_{1k} = 0$ (so $m_k \geqslant 2$). Let $s_k = n - d_{1k}$. Now all that we get is that $2^{s_k} \mid \prod_{j=1}^{m_k} \binom{2c_{jk}}{c_{jk}}^{d_{jk}}$. If $q_k + 1 \leqslant s_k$, then a similar argument as in the above paragraph shows $2n \mid \binom{n}{d_k} \prod_{j=1}^{m_k} \binom{2c_{jk}}{c_{jk}}^{d_{jk}}$ and we are done. It remains only to show that $q_k + 1 \leqslant s_k$. So suppose that $q_k + 1 > s_k$ and write $d_{jk} = 2^{q_k} t_{jk}$ for $t_{jk} \in \mathbb{Z}_{>0}$. Since we must have $t_{jk} \geqslant j - 1 \geqslant 1$ when $j \geqslant 2$, we get

$$q_k + 1 > s_k = \sum_{j=2}^{m_k} d_{jk} = \sum_{j=2}^{m_k} 2^{q_k} t_{jk} \geqslant \sum_{j=2}^{m_k} 2^{q_k} = 2^{q_k} \left(m_k - 1\right) \geqslant 2^{q_k}.$$

Since it is impossible to obtain $q + 1 > 2^q$, we arrive at the desired contradiction. $\square$

For the case of $2n \mid b_{n,un+2^r}$ in the above theorem, $n = 0$ is ruled out in order to make sure that division by $2n$ is well defined (note $b_{0,0} = 1$ and $b_{0,k} = 0$ for $k > 0$). For the case of $2n \mid b_{n,un-2^r}$, we require $un > 2^r$ since $b_{n,0} = 1$.

We also note that Sun's result

$$2\,(2m+1)\binom{2m}{m} \mid \binom{6m}{3m}\binom{3m}{m}$$

for $m \in \mathbb{Z}_{>0}$ is equivalent to

$$2\,(2m+1) \mid \frac{\binom{6m}{3m}\binom{3m}{m}}{\binom{2m}{m}}.$$

This is then a special case of our equation $2n \mid b_{n,un-2^r}$ in which $n = 2m+1$, $u = 1$, and $r = 0$. This gives the same statement as the above equation, but written as

$$2\,(2m+1) \mid b_{2m+1,2m}.$$

4 A Type of Converse

The next result is a type of converse to Theorem 1.

Theorem 2. *Fix $k \in \mathbb{Z}$ and $u \in \mathbb{Z}_{\geqslant 0}$ with $u > 0$ if $k < 0$. If*

$$2n \mid b_{n,un+k}$$

for all $n \in \mathbb{Z}_{>0}$ with $un + k > 0$, then there exists a unique $r \in \mathbb{Z}_{\geqslant 0}$ so that either

$$k = 2^r \quad \text{or} \quad k = -2^r.$$

Proof. First we show that $k \neq 0$. For this choose any odd prime p and consider $b_{2p,2up}$. Using Theorem 4 of [4] and the Division Algorithm, work $\bmod p$ to see that

$$b_{2p,2up} = 2^{4up}\binom{(1+2u)\,p-1}{p-1} \equiv 2^{4u}\tbinom{2u}{0}\tbinom{p-1}{p-1} \equiv 2^{4u}.$$

Thus $p \nmid b_{2p,2up}$ and so $k \neq 0$.

Next we consider the case of $u = 0$ (so $k > 0$ here). Actually the following argument works whenever $k > 0$ so that is all we actually assume. If k has an odd prime divisor, p, write $k = pk'$ for some $k' \in \mathbb{Z}_{>0}$. Consider $b_{2p,2up+k}$. Then, working $\bmod p$ again,

$$b_{2p,2up+k} = 2^{2(2u+k')p}\binom{(1+2u+k')p-1}{p-1} \equiv 2^{2(2u+k')}\binom{2u+k'}{0}\binom{p-1}{p-1} \equiv 2^{2(2u+k')}.$$

Thus $p \nmid b_{2p,2up+k}$ and so k must be a power of 2. Note that the only reason this argument may fail for $k < 0$ is that we might have $2up + k \leqslant 0$.

Finally, consider the case of $u \neq 0$. Suppose there exists an odd prime p so $p \mid k$. Then write $k = pk'$ and fix $m_0 \in \mathbb{Z}_{\geqslant 0}$ so $2^{m_0+1}u + k' > 0$ and $2^{m_0+1}u$ is congruent $\bmod p$

to either 0 or 1 (depending on whether $p \mid u$ or $p \nmid u$). Now consider $n = 2^{m_0+1}p\left(p^N+1\right)$ for any sufficiently large N. We will show that $p \nmid b_{n,un+k}$. For this, write

$$\begin{aligned} un+k &= 2^{m_0+1}up\left(p^N+1\right)+k'p \\ &= 2^{m_0+1}up^{N+1}+\left(2^{m_0+1}u+k'\right)p. \end{aligned}$$

For sufficiently large N, $un+k$ can be expanded in base p as

$$\begin{aligned} un+k = a_r p^r + a_{r-1}p^r + \cdots + a_{N+1}p^{N+1} \\ + b_s p^s + b_{s-1}p^{s-1} + \cdots + b_1 p \end{aligned}$$

with $0 \leqslant a_i, b_j \leqslant p-1$, $a_{N+1} \leqslant 1$, and $s \leqslant N$. Now we apply Kummer's theorem to the binomial coefficient in

$$b_{n,un+k} = 2^{p\left(p^N+1\right)}\binom{p^{N+1}+p-1+un+k}{p^{N+1}+p-1}.$$

Clearly adding $un+k$ to $p^{N+1}+(p-1)$ in base p results in no carries so that $p \nmid b_{n,un+k}$. As a result, k has no odd prime divisors and we are done. □

5 Relation to Known Sequences

As a result of Theorem 1, we have the following integer sequences

$$n \to B_{n,u,r,\pm} \equiv \frac{b_{n,un\pm 2^r}}{2n}.$$

For most choices of parameters $u, r, \pm$, this sequence seems to be new. However, for a few special choices, the sequence is known. Up to a shift and a few initial terms, the sequence $B_{n,0,2,+}$ is the OEIS integer sequence A077415, $B_{n,1,0,+}$ is A085614, $B_{n,1,1,+}$ is A078531, and $B_{n,1,0,-}$ appears as every other term in A089073. In addition, the odd terms of $B_{n,0,2,+}$ are A162540, the even terms of $B_{n,0,2,+}$ are A102860 and the negative of A136264, and (by construction) the odd terms of $B_{n,1,0,-}$ are Sun's A176898.

6 Final Remarks

It would be interesting to find a combinatorial interpretation for the sequences $B_{n,u,r,\pm}$. For instance, one is given the case of $B_{n,1,0,-}$ (A089073) or $B_{n,1,1,+}$ (A078531) as the number of symmetric non-crossing connected graphs on equidistant nodes of a circle and $B_{n,1,0,+}$ (A085614) is the number of elementary arches of size n.

In addition, information on corresponding generating functions would be of interest. Some are known. For example $B_{n,1,0,+}$ (A085614) is the series reversion of $x - 3x^2 + 2x^3$.

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