A new lower bound on the independence number of a graph and applications

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Abstract

The independence number of a graph G, denoted $\alpha(G)$, is the maximum cardinality of an independent set of vertices in G. The independence number is one of the most fundamental and well-studied graph parameters. In this paper, we strengthen a result of Fajtlowicz [Combinatorica 4 (1984), 35–38] on the independence of a graph given its maximum degree and maximum clique size. As a consequence of our result we give bounds on the independence number and transversal number of 6-uniform hypergraphs with maximum degree three. This gives support for a conjecture due to Tuza and Vestergaard [Discussiones Math. Graph Theory 22 (2002), 199–210] that if H is a 3-regular 6-uniform hypergraph of order n, then $\tau(H) \leqslant n/4$.

Keywords: independence; clique; transversal

1 Introduction

In this paper we study independence in graphs. Our main aim is to strengthen a result of Fajtlowicz [3, 4] on the independence of a graph given its maximum degree and maximum clique size. As a consequence of our result we give bounds on the independence number and transversal number of 6-uniform hypergraphs with maximum degree three.

A *hypergraph* H consists of a finite vertex set $V(H)$ and a finite multiset $E(H)$ of edges, where each edge is a subset of $V(H)$. A hypergraph H has *rank* r if the largest size of an edge of H is size r. A hypergraph H is *k-uniform* if every edge of H has size k. Every graph without loops is a 2-uniform hypergraph. The *degree* of a vertex v in H, denoted by $d_H(v)$ or simply by $d(v)$ if H is clear from context, is the number of edges of H that contain v. The maximum degree among the vertices of H is denoted by $\Delta(H)$. Two edges in H are *overlapping* if they intersect in at least two vertices.

Two vertices x and y of H are *adjacent* if some edge of H contains both x and y. A set X of vertices in H is a *clique* if every two vertices of X are adjacent in H. A *k-clique* is a clique in H of size k. The *clique number* $\omega(H)$ is the size of a maximum clique in H.

The *neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from context, is the set of all vertices different from v that are adjacent to v. Two vertices x and y of H are *connected* if there is a sequence $v_0, \ldots, v_k$ of vertices of H with $x = v_0$ and $y = v_k$ in which v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant k$. A *connected hypergraph* is a hypergraph in which every two vertices are connected. A maximal connected subhypergraph of H is a *component* of H.

For a subset X of vertices in a hypergraph H, let $H[X]$ denote the hypergraph induced by the vertices in X, in the sense that $V(H[X]) = X$ and $E(H[X]) = \{e \cap X \mid e \in E(H) \text{ and } |e \cap X| \geqslant 1\}$; that is, $E(H[X])$ is obtained from $E(H)$ by shrinking edges $e \in E(H)$ that intersect X to the edges $e \cap X$.

If H denotes a hypergraph and X denotes a subset of vertices in H, then $H - X$ denotes that hypergraph obtained from H by removing the vertices X from H, removing all hyperedges that intersect X, and removing all resulting isolated vertices, if any. When $X = \{x\}$, we simply denote $H - X$ by $H - x$. In the literature this is sometimes called *strongly deleting* the vertices in X.

A *twin* in H is a pair of vertices that are intersected by exactly the same set of edges; that is, a pair of vertices u and v is a twin in H if every edge that contains u also contains v, and every edge that contains v also contains u. The hypergraph H is *twin-free* if it has no twin. Hence if H is twin-free, then for every pair of vertices u and v in H, there exists an edge e such that $|e \cap \{u, v\}| = 1$.

A set S of vertices in a hypergraph H is *strongly independent* if no two vertices in S belong to a common edge. The *strong independence number* of H, which we denote by $\alpha(H)$, is the maximum cardinality of a strongly independent set in H. A subset of vertices in a hypergraph H is a *weakly independent set* if it contains no edge of H.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T intersects every edge of H. Equivalently, a set of vertices S is transversal in H if and only if $V(H) \setminus S$ is a weakly independent set in H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. We note that, $n(H) = \tau(H) + \alpha(H)$. Transversals in hypergraphs are well studied in the literature (see, for example, [1, 2, 6, 7]).

Let G be a graph, and let X and Y be disjoint subsets of $V(G)$. The set $E(X,Y)$ is the set of all the edges xy, with $x \in X$ and $y \in Y$. For each vertex $u \in V(G)$ let $w_G(u)$ denote the size of a largest clique in G containing u. We will omit the subscript when G is clear from the context.

2 Independence in Graphs

We shall prove the following result. The proof we present is similar to that of Fajtlowicz [3]. However in our proof we carefully choose a maximum independent set S in the graph G such that the number of edges from S to vertices outside S is minimized. With this choice of S, we establish a property on the graph G by considering the operation of replacing a vertex in S with a vertex outside S in order to get a smaller number of edges between S and vertices outside S.

Theorem 1. *If G is a graph of order n and p is an integer, such that (A) below holds, then $\alpha(G) \geqslant 2n/p$.*

(A): For every clique X in G there exists a vertex $x \in X$, such that $d(x) < p - |X|$.

Proof. Let $G = (V,E)$ be a graph of order n and let p be an integer such that (A) is satisfied. Let S be a maximum independent set in G, such that $|E(S, V \setminus S)|$ $(= \sum_{s\in S} d(s))$ is minimized. Let $\alpha_i(S)$ denote the number of vertices in $V \setminus S$ with exactly i neighbors in S. Since S is a maximum independent set we note that $\alpha_0(S) = 0$ and therefore the following holds.

$$n - |S| = \alpha_1(S) + \alpha_2(S) + \cdots + \alpha_{|S|}(S). \tag{1}$$

Furthermore counting the number of edges in $E(S, V \setminus S)$ we obtain the following.

$$\sum_{s\in S} d(s) = \alpha_1(S) + 2\alpha_2(S) + 3\alpha_3(S) + \cdots + |S|\alpha_{|S|}(S) \tag{2}$$

Multiplying Equation (1) by 2 and subtracting Equation (2) we obtain the following.

$$2n - 2|S| - \sum_{s\in S} d(s) = \alpha_1(S) - \alpha_3(S) - 2\alpha_4(S) - \cdots - (|S|-2)\alpha_{|S|}(S) \leqslant \alpha_1(S). \tag{3}$$

For each vertex $s \in S$, let Y_s be the set of all vertices in $V \setminus S$ adjacent to s but to no other vertex of S, and so every vertex in Y_s has no neighbor in $S \setminus \{s\}$. If Y_s does not induce a clique, then let $y_1, y_2 \in Y_s$ be non-adjacent vertices and note that $S \cup \{y_1, y_2\} \setminus \{s\}$ is an independent set in G of size greater than $|S|$, a contradiction. Therefore, $Y_s \cup \{s\}$ induces a clique in G.

Suppose that $d(s)+|Y_s|+1 \geqslant p$. If there is a vertex $y \in Y_s$ such that $d(y) < d(s)$, then $(S \cup \{y\}) \setminus \{s\}$ contradicts the minimality of $|E(S, V \setminus S)|$. Therefore, for all $y \in Y_s \cup \{s\}$ we have $d(y) + |Y_s \cup \{s\}| \geqslant d(s) + |Y_s| + 1 \geqslant p$, a contradiction to (A). This implies that $d(s) + |Y_s| + 1 \leqslant p - 1$, as $d(s)$, $|Y_s|$ and p are all integers. We now obtain the following, by Inequality (3),

$$2n \leqslant \alpha_1(S) + \sum_{s \in S} d(s) + 2|S| = \sum_{s \in S} (|Y_s| + d(s) + 2) \leqslant |S|p,$$

implying that $\alpha(G) = |S| \geqslant 2n/p$ as desired. □

As an immediate consequence of Theorem 1 we can prove the following result due to Fajtlowicz [3] on the independence of graph given its maximum degree and maximum clique size. We remark that in [4], Fajtlowicz studies some classes of graphs that achieve equality in the bound of Theorem 2.

Corollary 2. ([3]) *If G is a graph of order n containing no clique of size q, then $\alpha(G) \geqslant 2n/(\Delta(G) + q)$.*

Proof. Let G be a graph of order n containing no clique of size q and let $p = \Delta(G)+q$. For every clique X in G and for all vertices $x \in X$, we have $d(x) < \Delta(G)+1 \leqslant \Delta(G)+q-|X| = p - |X|$, and therefore condition (A) holds in Theorem 1. By Theorem 1 we have that $\alpha(G) \geqslant 2n/p = 2n/(\Delta(G) + q)$. □

3 Independence in hypergraphs of rank at most 6

In this section we apply our main result, namely Theorem 1, to 3-regular, 6-uniform hypergraphs as there is a very interesting conjecture for this case, namely the Tuza-Vestergaard Conjecture which we state later. The application illustrates the power of our main result. However we remark that Theorem 1 can be used in the cases where the regularity is less than the uniformity.

We will prove the following theorem. We remark that the twin-free condition in Theorem 3 is necessary, since otherwise the result is not true. Consider, for example, the Fano-plane, where each vertex gets duplicated. The resulting hypergraph, H, is a 3-regular 6-uniform hypergraph on $n = 14$ vertices, with strong independence number $\alpha(H) = 1 < 2n/23$.

Theorem 3. *If H is a twin-free 3-regular hypergraph of order n and rank at most 6, then $\alpha(H) \geqslant 2n/23$.*

Before giving a proof of Theorem 3 we need a number of preliminary results. Let H be a hypergraph of rank at most 6. For a set X of vertices in the hypergraph H, let

$$\theta_X(H) = \max |X \cap e \cap f|,$$

where the maximum is taken over all distinct edges e and f in H. Let e_1 and e_2 be two (fixed) edges in $H[X]$ such that $\theta_X(H) = |X \cap e_1 \cap e_2|$ and let Y and Z be defined by

$$Y = X \setminus (e_1 \cup e_2) \quad \text{and} \quad Z = X \cap e_1 \cap e_2.$$

We note that, $\theta_X(H) = |Z|$. We proceed further with a series of five lemmas that will prove useful when proving our main result.

Lemma 4. *Let H be a 3-regular hypergraph of rank at most 6 and let X be a clique of size at least 8 in H. Then the following hold.*

(a) $|Z| \geqslant 2$.

(b) *If H is twin-free and $|Y| \geqslant 2$, then $|Z| = 2$.*

(c) *If H is twin-free, then $|Y| \leqslant 2$.*

Proof. (a) Let $x \in X$ and let f_1, f_2, f_3 be the three edges incident with x in $H[X]$. If any two of these edges overlap, then $|Z| \geqslant 2$, as desired. Hence we may assume that $f_i \cap f_j = \{x\}$ for $1 \leqslant i < j \leqslant 3$. Since H has rank at most 6, $|f_i| \leqslant 6$ for $1 \leqslant i \leqslant 3$. Renaming edges if necessary, we may assume that $|f_1| \geqslant |f_2| \geqslant |f_3|$. Since $|X| \geqslant 8$, we note that $|f_1| \geqslant 4$ and $|f_2| \geqslant 2$. Let $v \in f_2 \setminus \{x\}$. Since X is a clique, the vertex v is adjacent to every vertex in f_1. However, $d_H(v) = 3$ and the edge f_2 does not intersect $f_1 \setminus \{x\}$. Hence one of the two remaining edges, g_1 say, containing v in $H[X]$ must contain at least two vertices of $f_1 \setminus \{x\}$. But then $|f_1 \cap g_1| \geqslant 2$, and so $|Z| \geqslant 2$.

(b) Suppose to the contrary that H is twin-free and $|Y| \geqslant 2$, but $|Z| \geqslant 3$. Let $\{z_1, z_2, z_3\} \subseteq Z$. For $i = 1, 2, 3$, let g_i be the edge in $H[X]$ containing z_i that is different from e_1 and e_2. Since H is twin-free, we note that $z_i \notin g_j$ for $1 \leqslant i, j \leqslant 3$ and $i \neq j$. Let $\{y_1, y_2\} \subset Y$. Since X is a clique, every vertex in Y is adjacent to every vertex in Z. Thus, $\{y_1, y_2\} \subset g_i$ for $i = 1, 2, 3$. But then y_1 and y_2 are twins, a contradiction. Therefore, $|Z| \leqslant 2$. By part (a), $|Z| \geqslant 2$. Consequently, $|Z| = 2$.

(c) Suppose to the contrary that H is twin-free, but $|Y| \geqslant 3$. By Part (b), $|Z| = 2$. Let $Z_1 = \{z_1, z_2\}$. For $i = 1, 2$, let g_i be the edge in $H[X]$ containing z_i that is different from e_1 and e_2. Since H is twin-free, we note that $z_1 \notin g_2$ and $z_2 \notin g_1$. Since X is a clique, every vertex in Y is adjacent to every vertex in Z. Thus, $Y \subset g_i$ for $i = 1, 2$. But then $|Z| = \theta_X(H) \geqslant |X \cap g_1 \cap g_2| \geqslant |Y| \geqslant 3$, contradicting Part (b). $\square$

Lemma 5. *If H is a twin-free 3-regular hypergraph of rank at most 6, then $\omega(H) \leqslant 10$.*

Proof. Suppose to the contrary that $\omega(H) \geqslant 11$. Let X be a clique of size 11 in H, and let $H[X]$, $\theta_X(H)$, e_1, e_2, Y and Z be as defined earlier. Then, $11 = |X| = |e_1 \cup e_2| + |Y| = |e_1| + |e_2| - |Z| + |Y|$. By Lemma 4(a), $|Z| = \theta_X(H) \geqslant 2$. Since H has rank at most 6, $|e_1| + |e_2| \leqslant 6 + 6 = 12$. If $|Z| \geqslant 3$, then $11 \leqslant 12 - 3 + |Y|$, and so $|Y| \geqslant 2$, contradicting Lemma 4(b). Therefore, $|Z| = 2$, implying that $11 \leqslant 12 - 2 + |Y|$, or, equivalently, $|Y| \geqslant 1$.

Let $y \in Y$ and let $Z = \{z_1, z_2\}$. For $i = 1, 2$, let g_i be the edge in $H[X]$ containing z_i that is different from e_1 and e_2. Since H is twin-free, we note that $z_1 \notin g_2$ and $z_2 \notin g_1$. Further since X is a clique, we have that $y \in g_1$ and $y \in g_2$. Let g_3 denote the remaining edge containing y in $H[X]$.

For $i = 1, 2$, let $e'_i = e_i \setminus Z$. Renaming the edges e_1 and e_2, if necessary, we may assume that $|e'_1| \geqslant |e'_2|$. If $|e'_1| \leqslant 3$, then $|e_1 \cup e_2| \leqslant 8$, implying that $|Y| \geqslant 3$, contradicting Lemma 4(c). Hence, $|e'_1| \geqslant 4$. However, $|e'_1| = |e_1| - |Z| \leqslant 6 - 2 = 4$. Consequently, $|e'_1| = 4$. We note therefore that either $|Y| = 1$, in which case $|e'_2| = 4$, or $|Y| = 2$, in which case $|e'_2| = 3$. Hence, $|e'_2| \geqslant 3$.

If neither the edge g_1 nor the edge g_2 intersects e'_2, then $e'_2 \subset g_3$. But then $\theta_X(H) \geqslant |e_2 \cap g_3| \geqslant |e'_2| \geqslant 3$, a contradiction. Therefore renaming the vertices z_1 and z_2, if necessary, we may assume that g_1 intersects e'_2. Let $w \in e'_2 \cap g_1$. Since $2 = \theta_X(H) \geqslant |e_1 \cap g_1|$, we note that the edge g_1 contains z_1 and at most one vertex of e'_1. But then the edge, e_w say, that contains w and is different from e_2 and g_1, contains at least three vertices of e'_1, implying that $\theta_X(H) \geqslant |e_1 \cap e_w| \geqslant |e'_1| - 1 = 3$, a contradiction. $\square$

Lemma 6. *If H is a twin-free 3-regular hypergraph of rank at most 6 and X is a 10-clique in H, then there exists a vertex $x \in X$ with $|N(x)| \leqslant 12$.*

Proof. Let X be a 10-clique in H, and let $H[X]$, $\theta_X(H)$, e_1, e_2, Y and Z be as defined earlier. Then, $10 = |X| = |e_1 \cup e_2| + |Y| = |e_1| + |e_2| - |Z| + |Y|$. By Lemma 4, $|Y| \leqslant 2$ and $|Z| = \theta_X(H) \geqslant 2$.

We first consider the case when $|Z| \geqslant 3$. By Lemma 4 we note that $|Y| \leqslant 1$. Since H has rank at most 6, $10 = |e_1| + |e_2| - |Z| + |Y| \leqslant 12 - 3 + 1 = 10$. Since we must have equality throughout this inequality chain, this implies that $|Z| = 3$ and $|Y| = 1$. Let $Z = \{z_1, z_2, z_3\}$ and for $i = 1, 2, 3$, let g_i be the edge in $H[X]$ containing z_i that is different from e_1 and e_2. Since H is twin-free, we note that $z_i \notin g_j$ for $1 \leqslant i, j \leqslant 3$ and $i \neq j$. Suppose that g_i contains a vertex from $(e_1 \cup e_2) \setminus Z$ for some i, $1 \leqslant i \leqslant 3$. Then there are at least three vertices that belong to overlapping edges with z_i, implying that $|N_H(z_i)| \leqslant 12$ and the desired result follows. Hence we may assume that no vertex from g_i belongs to $(e_1 \cup e_2) \setminus Z$ for $i = 1, 2, 3$. Since X is a clique, we note that $y \in g_i$ for $i = 1, 2, 3$. However this implies that y is not adjacent to any vertex in $(e_1 \cup e_2) \setminus Z$, a contradiction. Therefore, $|Z| = 2$.

As before, let $Z = \{z_1, z_2\}$ and for $i = 1, 2$, let g_i be the edge in $H[X]$ containing z_i that is different from e_1 and e_2. Let $W = (e_1 \cup e_2) \setminus Z$. Then, $10 = |X| = |W| + |Y| + |Z|$, which as $|Z| = 2$ and $0 \leqslant |Y| \leqslant 2$ implies that $6 \leqslant |W| \leqslant 8$. If $|g_1 \cap W| \geqslant 2$, then $|N_H(z_1)| \leqslant 15 - 3 = 12$, and we are done. Hence we may assume that $|g_1 \cap W| \leqslant 1$. Analogously, we may assume that $|g_2 \cap W| \leqslant 1$. Let W' be the vertices in W not covered by $g_1 \cup g_2$. Let $W'_1 = W' \cap e_1$ and let $W'_2 = W' \cap e_2$, and so $|W'| = |W'_1| + |W'_2|$.

Suppose that $|Y| = 2$ and let $Y = \{y_1, y_2\}$. Then, $|W| = 6$. Since $|g_1 \cap W| \leqslant 1$ and $|g_2 \cap W| \leqslant 1$, we note that $|W'| \geqslant 4$. For $i = 1, 2$, let f_i be the edge containing y_i that is different from g_1 and g_2. Since X is a clique, we have that $W' \subset f_i$ for $i = 1, 2$. On the one hand, if $f_1 = f_2$, then $\{y_1, y_2\}$ are twins. On the other hand, if $f_1 \neq f_2$, then $\theta_X(H) \geqslant |f_1 \cap f_2 \cap W'| \geqslant |W'| \geqslant 4$. Both cases produce a contradiction. Hence, $|Y| \leqslant 1$.

Suppose that $|Y| = 1$ and let $Y = \{y\}$. Then, $|W| = 7$. Since $|g_1 \cap W| \leqslant 1$ and $|g_2 \cap W| \leqslant 1$, we note that $|W'| \geqslant 5$. Let g_3 be the edge of $H[X]$ containing y that is different from g_1 and g_2. Since X is a clique, we have that $W' \subset g_3$. Renaming z_1 and z_2 if necessary, we may assume that $|W_1'| \geqslant |W_2'|$, implying that $\theta_X(H) \geqslant |e_1 \cap g_3| \geqslant |W_1'| \geqslant 3$, a contradiction. Hence, $Y = \emptyset$.

Since $|Y| = 0$, we have that $X = e_1 \cup e_2$. Further since H has rank at most 6, this implies that $|e_1| = |e_2| = 6$. For $i = 1, 2$, let $e_i' = e_i \setminus Z$, and so $|e_i'| = 4$. Let v be an arbitrary vertex in $X \setminus Z$. Renaming z_1 and z_2 if necessary, we may assume that $v \in e_1'$. Let e_v' and e_v'' be the two edges in $H[X]$ different from e_1 that contain v. Since X is a clique, the vertex v is adjacent to every vertex in e_2. If $|e_v' \cap e_2'| \leqslant 1$, then $|e_v'' \cap e_2'| \geqslant 3$, implying that $\theta_X(H) \geqslant |e_v'' \cap e_2| \geqslant 3$, a contradiction. Hence, $|e_v' \cap e_2'| \geqslant 2$. If $|e_v' \cap e_2'| > 2$, then $\theta_X(H) \geqslant 3$, a contradiction. Hence, $|e_v' \cap e_2'| = 2$. Analogously, $|e_v'' \cap e_2'| = 2$. Since v is adjacent to every vertex in e_2, we note that $(e_v' \cap e_2') \cap (e_v'' \cap e_2') = \emptyset$. This is true for every vertex $v \in X \setminus Z$. Hence every edge in $H[X]$ different from e_1 and e_2 has size 4 in $H[X]$ and contains two vertices in e_1' and two vertices in e_2'.

Let $e_1' = \{a_1, b_1, c_1, d_1\}$ and let $e_2' = \{a_2, b_2, c_2, d_2\}$. Let h_1 be an arbitrary edge in $E(H[X]) \setminus \{e_1, e_2\}$. Renaming vertices if necessary, we may assume that $h_1 = \{a_1, b_1, a_2, b_2\}$. Let h_2 and h_3 be the edges of $H[X]$ containing a_1 and b_1, respectively, that are different from e_1 and h_1. Then, $\{a_1, c_2, d_2\} \subset h_2$ and $\{b_1, c_2, d_2\} \subset h_3$. If $h_2 = h_3$, then a_1 and b_1 are twins in H. If $h_2 \neq h_3$, then c_2 and d_2 are twins in H. In both cases we contradict the fact that H is twin-free. □

Lemma 7. *If H is a 3-regular hypergraph of rank at most 6 and X is a 9-clique in H, then there exists a vertex $x \in X$ with $|N(x)| \leqslant 13$.*

Proof. Let X be a 9-clique in H. If there are two twins in $H[X]$, then each of them have degree at most 13 and we are done. Hence we may assume that there are no twins in $H[X]$. For each edge, f, in H containing some vertex y we note that there are at most five vertices in $f \setminus \{y\}$ since H has rank at most 6. Define a graph G_X with vertex set X and with an edge between $x, x' \in X$ if and only if $\{x, x'\}$ is a subset of two distinct edges in H. Thus every neighbor, x', of a vertex $x \in X$ belongs to two edges of H that contain both x and x'. This implies that $|N_H(x)| \leqslant 3 \times 5 - d_{G_X}(x)$. Thus if $d_{G_X}(x) \geqslant 2$, then $|N_H(x)| \leqslant 13$, and we are done. Therefore we may assume that $d_{G_X}(x) \leqslant 1$ for all $x \in X$. Since $|X| = 9$ is odd, this implies that some vertex $x \in X$ is an isolated vertex in G_X. Let f_1, f_2, f_3 be the three edges in $H[X]$ containing x.

Suppose that $|f_1| = 6$. Let $v \in X \setminus f_1$. Renaming the edges f_2 and f_3, if necessary, we may assume that $v \in f_2$. Since X is a clique, the vertex v is adjacent to all five vertices in $f_1 \setminus \{x\}$. Hence one of the two edges different from f_2 that contain v must intersect $f_1 \setminus \{x\}$ in at least three vertices. But this implies that $d_{G_X}(w) \geqslant 2$ for some $w \in f_1$, a contradiction. Therefore, $|f_1| \neq 6$. Analogously, $|f_2| \neq 6$ and $|f_3| \neq 6$.

Suppose that $|f_1| = 4$. Let $U = X \setminus f_1$ and note that $|U| = 5$. Let $u \in U$. Since X is a clique, u is adjacent to all three vertices in $f_1 \setminus \{x\}$. Hence one of the two edges in $H[X]$ containing u that is different from f_2 and f_3 contains at least two vertices in $f_1 \setminus \{x\}$. Let g_u be such an edge of $H[X]$ that contains u. If the five such edges, g_u, for all $u \in U$ are identical, then $|g_u| \geqslant |U| + 2 = 7$, a contradicting the rank of H. Therefore there exist two distinct vertices u and u' in U such that $g_u \neq g_{u'}$. Since both g_u and $g_{u'}$ contain at least two vertices in $f_1 \setminus \{x\}$, and since $|f_1 \setminus \{x\}| = 3$, there exists some vertex in $f_1 \setminus \{x\}$

that belongs to both g_u and $g_{u'}$. If $|g_u \cap g_{u'} \cap f_1| \geqslant 2$, then there exists two twins in $H[X]$, a contradiction. Therefore, $g_u \cap g_{u'} \cap f_1 = \{w\}$ for some vertex $w \in f_1 \setminus \{x\}$. But then $f_1 \setminus \{x, w\} \subseteq N_{G_X}(w)$, implying that $d_{G_X}(w) \geqslant 2$, a contradiction. Therefore, $|f_1| \neq 4$. Analogously, $|f_2| \neq 4$ and $|f_3| \neq 4$.

Renaming the edges if necessary, we may assume that $|f_1| \geqslant |f_2| \geqslant |f_3|$. If $|f_1| \leqslant 3$, then $|X| \leqslant 7$, a contradiction. Hence, $|f_1| \geqslant 4$. However as shown earlier, $|f_1| \neq 4$ and $|f_1| \neq 6$. Therefore, $|f_1| = 5$ and $|f_2| \geqslant 3$. Let $f_1' = f_1 \setminus \{x\}$ and note that $|f_1'| = 4$. Further, let $Q = \{q_1, q_2, q_3, q_4\} = X \setminus f_1$. Consider the vertex $q_i \in Q$ where $1 \leqslant i \leqslant 4$. Since X is a clique, q_i is adjacent to all four vertices in f_1'. Further since no two edges of H intersect in more than two vertices, this implies that there exists two edges r_i and r_i' containing q_i, such that $|r_i \cap f_1'| = |r_i' \cap f_1'| = 2$ and $r_i \cap r_i \cap f_1' = \emptyset$. Let there be j distinct edges in

$$E^* = \bigcup_{i=1}^{4} \{r_i, r_i'\}.$$

If $j = 2$, then since $|f_2| \geqslant 3$ there are two twins in $f_2 \setminus \{x\}$, a contradiction. Hence, $j \geqslant 3$. If two distinct edges in E^* intersect in the same set of two vertices in f_1', then there are two twins in f_1, a contradiction. Hence every two distinct edges in E^* have a different intersection in f_1'. Since $j \geqslant 3$, there will therefore be at least three edges with both ends in f_1'. This implies that $d_{G_X}(w) \geqslant 2$ for some $w \in f_1'$, a contradiction. □

Lemma 8. *If H is a 3-regular hypergraph of rank at most 6 and X is a 8-clique in H, then there exists a vertex $x \in X$ with $|N(x)| \leqslant 14$.*

Proof. Let X be a 8-clique in H. By Lemma 4(a), $\theta_X(H) \geqslant 2$. Let e_1 and e_2 be two overlapping edges in $H[X]$ and let $x \in e_1 \cap e_2$. Then, $|N(x)| \leqslant 14$. □

We are now in a position to prove Theorem 3. Recall its statement.

Theorem 3. *If H is a twin-free 3-regular hypergraph of order n and rank at most 6, then $\alpha(H) \geqslant 2n/23$.*

Proof. Let G be the graph with vertex set $V(G) = V(H)$ and where two vertices are adjacent in G if and only if they are adjacent in H. Clearly, $\alpha(G) = \alpha(H)$. Since H is 3-regular of rank at most 6, we note that $\Delta(G) \leqslant 15$. Let $p = 23$. If X is a clique of size at most 7 in G, then for each vertex $x \in X$ we have $d_G(x) = |N_H(x)| \leqslant 15 < p - |X|$. If X is a clique of size 8 in G (and therefore in H), there exists an $x \in X$, such that $d_G(x) = |N_H(x)| < 15 = p - |X|$, by Lemma 8. If X is a clique of size 9 in G (and therefore in H) there exists an $x \in X$, such that $d_G(x) = |N_H(x)| < 14 = p - |X|$, by Lemma 7. If X is a clique of size 10 in G (and therefore in H) there exists an $x \in X$, such that $d_G(x) = |N_H(x)| < 13 = p - |X|$, by Lemma 6. Furthermore there is no clique in G (or H) of size greater than 10 by Lemma 5. Therefore condition (A) holds in Theorem 1, implying that $\alpha(H) = \alpha(G) \geqslant 2|V(G)|/p = 2|V(H)|/23 = 2n/23$, by Theorem 1. □

We conjecture that the following holds.

Conjecture 9. If H is a twin-free 3-regular hypergraph of order n and rank at most 6, then $\alpha(H) \geqslant n/10$.

We remark that if Conjecture 9 is true, then the bound is tight due to the following example. Let H_{10} be the 6-uniform hypergraph with five edges, e_1, e_2, e_3, e_4, e_5, and ten vertices defined by $V(H_{10}) = \{u_{i,j,k} \mid 1 \leqslant i < j < k \leqslant 5\}$, where the vertex $u_{i,j,k}$ belongs to edges e_i, e_j and e_k. Then, H_{10} is 3-regular and 6-uniform. Furthermore, H_{10} is twin-free as different vertices belong to different sets of edges. Further, for distinct vertices $u_{i,j,k}$ and $u_{i',j',k'}$ in H_{10}, we note that $\{i, j, k\} \cap \{i', j', k'\} \neq \emptyset$ as all indices cannot be distinct since they are between 1 and 5, implying that $u_{i,j,k}$ and $u_{i',j',k'}$ are adjacent. Hence, $\alpha(H_{10}) = 1 = n/10$, where $n = n(H_{10})$. Therefore, H_{10} would show that Conjecture 9 would be best possible.

4 Transversals in 6-uniform hypergraphs

Chvátal and McDiarmid [1] established the following upper bound on the transversal number of a uniform hypergraph in terms of its order and size.

Chvátal-McDiarmid Theorem. *For $k \geqslant 2$, if H is a k-uniform hypergraph of order n and size m, then*

$$\tau(H) \leqslant \frac{n + \left\lfloor \frac{k}{2} \right\rfloor m}{\left\lfloor \frac{3k}{2} \right\rfloor}.$$

Let $n_i(H)$ denote the number of vertices in H of degree i. As a consequence of the Chvátal-McDiarmid Theorem, we have the following two results.

Corollary 10. *If H is a 6-uniform hypergraph with $\Delta(H) \leqslant 3$, then*

$$18\tau(H) \leqslant 3n_1(H) + 4n_2(H) + 5n_3(H).$$

Proof. Let H be a 6-uniform hypergraph of order n and size m satisfying $\Delta(H) \leqslant 3$. For notational simplicity, let $n_i = n_i(H)$ for $i \in \{1, 2, 3\}$. Applying the Chvátal-McDiarmid Theorem to the hypergraph H, we have that

$$\tau(H) \leqslant \frac{n + 3m}{9} = \frac{2n + 6m}{18} = \frac{2(n_1 + n_2 + n_3) + (n_1 + 2n_2 + 3n_3)}{18} = \frac{3n_1 + 4n_2 + 5n_3}{18},$$

or, equivalently, $18\tau(H) \leqslant 3n_1(H) + 4n_2(H) + 5n_3$. $\square$

Corollary 11. *If H is a 3-regular 6-uniform hypergraph of order n, then $\tau(H) \leqslant 5n/18$.*

Our aim in this section is to lower the best known upper bound on the transversal number of a 3-regular 6-uniform hypergraph of order n from $5n/18 \approx 0.27777777\,n$ (see Corollary 11) to $37n/138 \approx 0.268115942\,n$. In order to state our result, let

$$c_1 = \frac{1}{6}, \quad c_2 = \frac{2}{9}, \quad \text{and} \quad c_3 = \frac{37}{138}.$$

We first prove the following result on 6-uniform hypergraphs. We remark that if we allow edges of size less than 6, then the result of Theorem 12 is not true anymore. For example, 3-regular 3-uniform hypergraphs on n vertices may have transversal number $n/2$ (see, [5]).

Theorem 12. *If H is a 6-uniform hypergraph with $\Delta(H) \leqslant 3$, then*

$$\tau(H) \leqslant c_1 n_1(H) + c_2 n_2(H) + c_3 n_3(H).$$

Proof. We proceed by induction on the order of a 6-uniform hypergraph H satisfying $\Delta(H) \leqslant 3$. For a hypergraph H' with $\Delta(H') \leqslant 3$, let

$$\theta(H') = c_1 n_1(H') + c_2 n_2(H') + c_3 n_3(H').$$

Hence we wish to show that $\tau(H) \leqslant \theta(H)$. If $m(H) = 0$, then $\tau(H) = 0$ and the result is immediate. Hence we may assume that $m(H) \geqslant 1$, implying that $|V(H)| \geqslant 6$. If $|V(H)| = 6$, then $\tau(H) = 1 \leqslant \theta(H)$. This establishes the base cases when $|V(H)| \leqslant 6$. Let H be a 6-uniform hypergraph such that $\Delta(H) \leqslant 3$ and assume the theorem holds for all 6-uniform hypergraphs H' satisfying $\Delta(H') \leqslant 3$ and $n(H') < n(H)$.

If $\Delta(H) \leqslant 2$, then $n_3(H) = 0$ and the theorem holds by Corollary 10 of the Chvátal-McDiarmid Theorem. Hence we may assume that $\Delta(H) = 3$. We consider two cases, depending on whether H has twins of degree 3 and or not.

Suppose first that H contains two twins, x_1 and x_2, of degree 3. Let $X = \{x_1, x_2\}$ and let $H' = H - \{x_1, x_2\}$. Thus, H' is obtained from H by removing the vertices X from H removing the three hyperedges that intersect X, and removing all resulting isolated vertices, if any. Let T' be a minimum transversal in H'. Then, $T = T' \cup \{x_1\}$ is a transversal in H, and so $\tau(H) \leqslant |T| = |T'| + 1 = \tau(H') + 1$. We note that by removing the three edges that contain X, the degrees of x_1 and x_2 drop from 3 to zero. Further, if some vertex $v \notin X$ belongs to i of the deleted edges its degree drops to $d_H(v) - i$ in H', implying that the sum of the degrees of vertices not in X decrease by 12 in H' due to the 6-uniformity of H. If the degree of a vertex drops from 1 to 0 in H', then it decreases $\theta(H)$ by c_1. If its degree drops from 2 to 1 in H', then it decreases $\theta(H)$ by $c_2 - c_1$, while if its drops from 3 to 2 in H', then it decreases $\theta(H)$ by $c_3 - c_2$. Since $c_1 \geqslant c_2 - c_1 \geqslant c_3 - c_2$, we therefore have that whenever the degree of a vertex drops by 1 in H', then it decreases $\theta(H)$ by at least $c_3 - c_2$. Therefore,

$$\theta(H') \leqslant \theta(H) - 2c_3 - 12(c_3 - c_2) = \theta(H) + 12c_2 - 14c_3 < \theta(H) - 1,$$

implying that

$$\tau(H) \leqslant |T| = |T'| + 1 \leqslant \theta(H') + 1 < \theta(H).$$

Hence if H contains two twins, x_1 and x_2, of degree 3, then $\tau(H) < \theta(H)$. We may therefore assume H has no twins of degree 3, for otherwise the desired result holds.

Recall that by our earlier assumption, $\Delta(H) = 3$. Let R contain all vertices in H of degree 3. Then, $H[R]$ is a 3-regular hypergraph of rank at most six and with no twins. By Theorem 3 there exists a strongly independent set, I, in $H[R]$ of size at least $2|R|/23$. Let $H' = H - I$ and let $E^* = \{e_1^*, e_2^*, \ldots, e_{3|I|}^*\}$ be the set of $3|I|$ edges containing vertices from I. As observed earlier, when we delete an edge e from a 6-uniform hypergraph H with maximum degree at most 3 and if $v \in e$, then $\theta(H)$ drops by $c_3 - c_2$ if $d_H(v) = 3$, $\theta(H)$ drops by $c_2 - c_1$ if $d_H(v) = 2$, and $\theta(H)$ drops by c_1 if $d_H(v) = 1$. Further, $c_1 \geqslant c_2 - c_1 \geqslant c_3 - c_2$. Thinking of H' as being obtained from H by removing the edges e_1^*, e_2^*, e_3^*, $\ldots$, $e_{3|I|}^*$ in that order, we note that exactly $|R|$ times we drop $\theta(H)$ by $c_3 - c_2$, once for each vertex in R (noting that each vertex in R is contained in at least one edge in E^*). Further, at least $|I|$ times we drop $\theta(H)$ by c_1 since all edges are removed from the vertices in the independent set I. The total sum of the degrees of vertices decrease by $6|E^*|$ in H' due to the 6-uniformity of H. We therefore obtain the following.

$$\begin{array}{rcl}
\theta(H') + |I| & \leqslant & |I| + \theta(H) - (c_3 - c_2)|R| - c_1|I| - (c_2 - c_1)(6|E^*| - |R| - |I|) \\
& = & |I| + \theta(H) - (c_3 - c_2)|R| - c_1|I| - (c_2 - c_1)(18|I| - |R| - |I|) \\
& = & \theta(H) - (c_3 - c_2 - c_2 + c_1)|R| - (c_1 + 17c_2 - 17c_1 - 1)|I| \\
& = & \theta(H) - (c_3 - 2c_2 + c_1)|R| - (17c_2 - 16c_1 - 1)|I| \\
& \leqslant & \theta(H) - (c_3 - 2c_2 + c_1)|R| - 2(17c_2 - 16c_1 - 1)|R|/23 \\
& = & \theta(H) - (23c_3 - 46c_2 + 23c_1 + 34c_2 - 32c_1 - 2)|R|/23 \\
& = & \theta(H) - (23c_3 - 12c_2 - 9c_1 - 2)|R|/23 \\
& = & \theta(H).
\end{array}$$

Applying the inductive hypothesis to H', we have that $\tau(H') \leqslant \theta(H')$. Every transversal in H' can be extended to a transversal in H by adding to it the set I, implying that

$$\tau(H) \leqslant \tau(H') + |I| \leqslant \theta(H') + |I| \leqslant \theta(H),$$

which completes the proof. $\square$

As a consequence of Theorem 12, we have the following result for 6-uniform hypergraphs.

Corollary 13. *If H is a 3-regular 6-uniform hypergraph of order n, then $\tau(H) \leqslant 37n/138 \approx 0.268115942\, n$.*

Proof. If H is a 3-regular 6-uniform hypergraph of order n, then by Theorem 12 we have that $\tau(H) \leqslant c_1n_1(H) + c_2n_2(H) + c_3n_3(H) = 0 + 0 + 37n/138$. $\square$

We remark that Corollary 13 gives support for the following long-standing conjecture due to Tuza and Vestergaard [8], in that it lowers the best known upper bound on the transversal number of a 3-regular 6-uniform hypergraph of order n from $5n/18$ to $37n/138$.

Tuza-Vestergaard Conjecture. *If H is a 3-regular 6-uniform hypergraph of order n, then $\tau(H) \leqslant n/4 = 0.25n$.*

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