A Set and Collection Lemma

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Abstract

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent. Let $\alpha(G)$ stand for the cardinality of a largest independent set.

In this paper we prove that if Λ is a nonempty collection of maximum independent sets of a graph G, and S is an independent set, then

- there is a matching from $S - \bigcap\Lambda$ into $\bigcup\Lambda - S$, and
- $|S| + \alpha(G) \leqslant \left|\bigcap\Lambda \cap S\right| + \left|\bigcup\Lambda \cup S\right|$.

Based on these findings we provide alternative proofs for a number of well-known lemmata, such as the "*Maximum Stable Set Lemma*" due to Claude Berge and the "*Clique Collection Lemma*" due to András Hajnal.

Keywords: matching; independent set; stable set; core; corona; clique

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subseteq V$, then $G[X]$ is the subgraph of G spanned by X. By $G - W$ we mean the subgraph $G[V - W]$, if $W \subseteq V(G)$, and we use $G - w$, whenever $W = \{w\}$.

The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, while the *neighborhood* of $A \subseteq V$ is $N(A) = N_G(A) = \{v \in V : N(v) \cap A \neq \emptyset\}$. By $\overline{G}$ we denote the complement of G.

A set $S \subseteq V(G)$ is *independent* (*stable*) if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the set of all the independent sets of G. An independent set of maximum cardinality will be referred to as a *maximum independent set* of G, and the *independence number* of G is $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*.

If $\alpha(G) + \mu(G) = |V(G)|$, then G is called a König-Egerváry graph [5, 14].

Lemma 1 (*Maximum Stable Set Lemma*)**.** *[1], [2] An independent set X is maximum if and only if every independent set S disjoint from X can be matched into X.*

Let $\Omega(G)$ denote the family of all maximum independent sets of G and

$$\text{core}(G) = \bigcap\{S : S \in \Omega(G)\} \text{ [11], while}$$
$$\text{corona}(G) = \bigcup\{S : S \in \Omega(G)\} \text{ [3].}$$

A set $A \subseteq V(G)$ is a *clique* in G if A is independent in $\overline{G}$, and $\omega(G) = \alpha\left(\overline{G}\right)$.

Our main motivation has been the "*Clique Collection Lemma*" due to Hajnal [8]. Some recent applications may be found in [4, 9, 13].

Lemma 2 (*Clique Collection Lemma*)**.** *[8] If Γ is a collection of maximum cliques in G, then*

$$\left|\bigcap\Gamma\right| \geqslant 2 \cdot \omega(G) - \left|\bigcup\Gamma\right|.$$

In this paper we introduce the "*Matching Lemma*". It is both a generalization and strengthening of a number of observations including the "*Maximum Stable Set Lemma*" due to Berge, and the "*Clique Collection Lemma*"due to Hajnal.

2 Results

It is clear that the statement "*there exists a matching from a set A into a set B*" is stronger than just saying that $|A| \leqslant |B|$. The "*Matching Lemma*" offers a tool validating existence of matchings and their corresponding inequalities.

Lemma 3 (Matching Lemma)**.** *Let $S \in \text{Ind}(G), X \in \Lambda \subseteq \Omega(G)$, and $|\Lambda| \geqslant 1$. Then the following assertions are true:*

(i) *there exists a matching from $S - \bigcap\Lambda$ into $\bigcup\Lambda - S$;*

(ii) *there exists a matching from $S \cap X - \bigcap\Lambda$ into $\bigcup\Lambda - (X \cup S)$.*

Proof. *(i)* In order to prove that there is a matching from $S - \bigcap\Lambda$ into $\bigcup\Lambda - S$, we use Hall's Theorem, i.e., we show that for every $A \subseteq S - \bigcap\Lambda$ we must have

$$|A| \leqslant \left|N(A) \cap \left(\bigcup\Lambda\right)\right| = \left|N(A) \cap \left(\bigcup\Lambda - S\right)\right|.$$

Assume, by way of contradiction, that Hall's condition is not satisfied. Let us choose a minimal subset $\tilde{A} \subseteq S - \bigcap\Lambda$, for which $\left|\tilde{A}\right| > \left|N\left(\tilde{A}\right) \cap \left(\bigcup\Lambda\right)\right|$.

There exists some $W \in \Lambda$ such that $\tilde{A} \nsubseteq W$, because $\tilde{A} \subseteq S - \bigcap \Lambda$. Further, the inequality $\left|\tilde{A} \cap W\right| < \left|\tilde{A}\right|$ and the inclusion

$$N(\tilde{A} \cap W) \cap \left(\bigcup \Lambda\right) \subseteq N(\tilde{A}) \cap \left(\bigcup \Lambda\right) - W$$

imply

$$\left|\tilde{A} \cap W\right| \leqslant \left|N(\tilde{A} \cap W) \cap \left(\bigcup \Lambda\right)\right| \leqslant \left|N(\tilde{A}) \cap \left(\bigcup \Lambda\right) - W\right|,$$

because we have selected $\tilde{A}$ as a minimal subset satisfying $\left|\tilde{A}\right| > \left|N\left(\tilde{A}\right) \cap \left(\bigcup \Lambda\right)\right|$.

On the other hand,

$$\left|\tilde{A} \cap W\right| + \left|\tilde{A} - W\right| = \left|\tilde{A}\right| > \left|N(\tilde{A}) \cap \left(\bigcup \Lambda\right)\right| = \left|N(\tilde{A}) \cap \left(\bigcup \Lambda\right) - W\right| + \left|N(\tilde{A}) \cap W\right|.$$

Consequently, since $\left|\tilde{A} \cap W\right| \leqslant \left|N(\tilde{A}) \cap \left(\bigcup \Lambda\right) - W\right|$, we can infer that $\left|\tilde{A} - W\right| > \left|N(\tilde{A}) \cap W\right|$. Therefore,

$$\tilde{A} \cup \left(W - N(\tilde{A})\right) = W \cup \left(\tilde{A} - W\right) - \left(N(\tilde{A}) \cap W\right)$$

is an independent set of size greater than $|W| = \alpha(G)$, which is a contradiction that proves the claim.

(ii) By part *(i)*, there exists a matching from $S - \bigcap \Lambda$ into $\bigcup \Lambda - S$. Since X is independent, there are no edges between

$$\left(S - \bigcap \Lambda\right) - (S - X) = (S \cap X) - \bigcap \Lambda \text{ and } X - S.$$

Therefore, there exists a matching

$$\text{from } (S \cap X) - \bigcap \Lambda \text{ into } \left(\bigcup \Lambda - S\right) - (X - S) = \bigcup \Lambda - (X \cup S),$$

as claimed. □

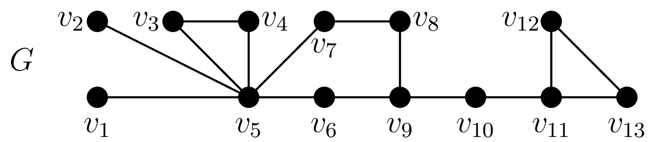

Figure 1: $\{v_1, v_2, v_3, v_6, v_8, v_{10}, v_{12}\}$, $\{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{13}\}$, $\{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{12}\}$ are maximum independent sets.

Example 4. Let us consider the graph G from Figure 1 and $S = \{v_1, v_4, v_7\} \in \mathrm{Ind}(G)$, $\Lambda = \{S_1, S_2\}$, where $S_1 = \{v_1, v_2, v_3, v_6, v_8, v_{10}, v_{12}\}$ and $S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{13}\}$. Then there is a matching from $S - \bigcap \Lambda = \{v_4, v_7\}$ into $\bigcup \Lambda - S = \{v_2, v_3, v_6, v_8, v_{10}, v_{12}, v_{13}\}$, namely, $M = \{v_3v_4, v_7v_8\}$.

Remark 5. The conclusions of the Matching Lemma may be false, if the family Λ is not included in $\Omega(G)$. Note that in Figure 1, if $S = \{v_1, v_2, v_4, v_7, v_9, v_{12}\} \in \mathrm{Ind}(G)$, $\Lambda = \{S_1, S_2\}$, where $S_1 = \{v_2, v_3, v_7\}$ and $S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{12}\}$, then there is no matching from $S - \bigcap \Lambda = \{v_1, v_4, v_9, v_{12}\}$ into $\bigcup \Lambda - S = \{v_3, v_6, v_{10}\}$.

The Matching Lemma allows us to give an alternative proof of the following result due to Berge.

Lemma 6 (*Maximum Stable Set Lemma*)**.** *[1, 2] An independent set X is maximum if and only if every independent set S disjoint from X can be matched into X.*

Proof. The "*only if*" part follows from the Matching Lemma *(i)*, by taking $\Lambda = \{X\}$.

For the "*if*" part we proceed as follows. According to the hypothesis, there is a matching from $S - X = S - S \cap X$ into X, in fact, into $X - S \cap X$, for each $S \in \mathrm{Ind}(G)$. Let $S \in \Omega(G)$. Hence, we obtain

$$\alpha(G) = |S| = |S - X| + |S \cap X| \leqslant |X - S \cap X| + |S \cap X| = |X| \leqslant \alpha(G),$$

which clearly implies $X \in \Omega(G)$. □

Applying the Matching Lemma *(i)* to $\Lambda = \Omega(G)$ we immediately obtain the following.

Corollary 7. *[3] For every $S \in \Omega(G)$, there is a matching from $S - \mathrm{core}(G)$ into $\mathrm{corona}(G) - S$.*

The following inequality is a numerical interpretation of the Matching Lemma.

Lemma 8 (*Set and Collection Lemma*)**.** *If $S \in \mathrm{Ind}(G)$, $\Lambda \subseteq \Omega(G)$, and $|\Lambda| \geqslant 1$, then*

$$|S| + \alpha(G) \leqslant \left|\bigcap \Lambda \cap S\right| + \left|\bigcup \Lambda \cup S\right|.$$

Proof. Let $X \in \Lambda$. By the Matching Lemma *(ii)*, there is a matching from $S \cap X - \bigcap \Lambda$ into $\bigcup \Lambda - (X \cup S)$. Hence we infer that

$$\begin{aligned} |S \cap X| - \left|\bigcap \Lambda \cap S\right| &= |S \cap X| - \left|\bigcap \Lambda \cap S \cap X\right| \\ &= \left|S \cap X - \bigcap \Lambda\right| \leqslant \left|\bigcup \Lambda - (X \cup S)\right| \\ &= \left|\bigcup \Lambda \cup (X \cup S)\right| - |X \cup S| = \left|\bigcup \Lambda \cup S\right| - |X \cup S|. \end{aligned}$$

Therefore, we obtain

$$|S \cap X| - \left|\bigcap \Lambda \cap S\right| \leqslant \left|\bigcup \Lambda \cup S\right| - |X \cup S|,$$

which implies

$$|S| + \alpha(G) = |S| + |X| = |S \cap X| + |X \cup S| \leqslant \left|\bigcap \Lambda \cap S\right| + \left|\bigcup \Lambda \cup S\right|,$$

as claimed. □

The conclusions of the Set and Collection Lemma may be false, if the family Λ is not included in $\Omega(G)$. For instance, the graph G of Figure 1 has $\alpha(G) = 7$, and if $S = \{v_1, v_2, v_4, v_7, v_9, v_{12}\} \in \mathrm{Ind}(G)$, $\Lambda = \{S_1, S_2\}$, where $S_1 = \{v_2, v_3, v_7\}$ and $S_2 = \{v_1, v_2, v_4, v_6, v_7, v_{10}, v_{12}\}$, then

$$13 = |S| + \alpha(G) \nleqslant \left|\bigcap \Lambda \cap S\right| + \left|\bigcup \Lambda \cup S\right| = 11.$$

Corollary 9. *If* $\Lambda \subseteq \Omega(G), |\Lambda| \geqslant 1$, *then* $2 \cdot \alpha(G) \leqslant \left|\bigcap \Lambda\right| + \left|\bigcup \Lambda\right|$.

Proof. Let $S \in \Lambda$. Using the Set and Collection Lemma, we obtain

$$2 \cdot \alpha(G) = |S| + \alpha(G) \leqslant \left|\bigcap \Lambda \cap S\right| + \left|\bigcup \Lambda \cup S\right| = \left|\bigcap \Lambda\right| + \left|\bigcup \Lambda\right|,$$

as required. □

Since every maximum clique of G is a maximum independent set of $\overline{G}$, Corollary 9 is equivalent to the following result, due to Hajnal.

Lemma 10 (*Clique Collection Lemma*). *[8] If* Γ *is a collection of maximum cliques in* G, *then*

$$\left|\bigcap \Gamma\right| \geqslant 2 \cdot \omega(G) - \left|\bigcup \Gamma\right|.$$

If $\Lambda = \Omega(G)$, then Corollary 9 implies the following.

Corollary 11. *For every graph* G, *it is true that*

$$2 \cdot \alpha(G) \leqslant |\mathrm{core}(G)| + |\mathrm{corona}(G)|.$$

The graph G_1 from Figure 2 satisfies $2 \cdot \alpha(G_1) < |\mathrm{core}(G_1)| + |\mathrm{corona}(G_1)|$, because $\alpha(G_1) = 4$, $\mathrm{core}(G_1) = \{v_8, v_9\}$, and $\mathrm{corona}(G_1) = \{v_1, v_3, v_4, v_5, v_7, v_8, v_9\}$.

The *vertex covering number* of G, denoted by $\tau(G)$, is the number of vertices in a minimum vertex cover in G, that is, the size of any smallest vertex cover in G. Thus we have $\alpha(G) + \tau(G) = |V(G)|$. Since

$$|V(G)| - \left|\bigcup \{S : S \in \Omega(G)\}\right| = \left|\bigcap \{V(G) - S : S \in \Omega(G)\}\right|,$$

Corollary 11 immediately implies the following.

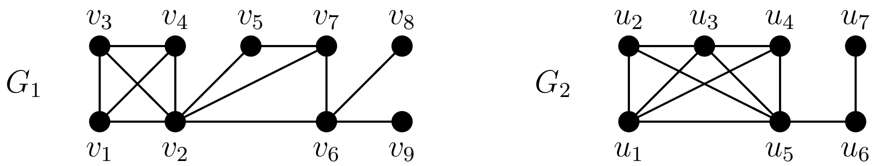

Figure 2: Both G_1 and G_2 satsify Corollary 11.

Corollary 12. *[7] If G is a graph, then*

$$\alpha(G) - |\mathrm{core}(G)| \leqslant \tau(G) - |\bigcap \{V(G) - S : S \in \Omega(G)\}|.$$

It is clear that $|\mathrm{core}(G)| + |\mathrm{corona}(G)| \leqslant \alpha(G) + |V(G)|$.

Proposition 13. *If G is a graph with a nonempty edge set, then*

$$|\mathrm{core}(G)| + |\mathrm{corona}(G)| \leqslant \alpha(G) + |V(G)| - 1.$$

Proof. Assume, to the contrary, that $|\mathrm{core}(G)| + |\mathrm{corona}(G)| \geqslant \alpha(G) + |V(G)|$.

If $S \in \Omega(G)$, then

$$|\mathrm{corona}(G) - S| = |\mathrm{corona}(G)| - \alpha(G) \geqslant |V(G)| - |\mathrm{core}(G)| = |V(G) - \mathrm{core}(G)|.$$

Since, clearly, $\mathrm{corona}(G) - S \subseteq V(G) - \mathrm{core}(G)$, we obtain $V(G) = \mathrm{corona}(G)$ and $\mathrm{core}(G) = S$. It follows that $N(\mathrm{core}(G)) = \emptyset$, since $\mathrm{corona}(G) \cap N(\mathrm{core}(G)) = \emptyset$.

On the other hand, since G has a nonempty edge set and S is a maximum independent set, we have $\emptyset \neq N(S) = N(\mathrm{core}(G))$.

This contradiction proves the claimed inequality. □

Remark 14. The complete bipartite graph $K_{1,n-1}$ satisfies $\alpha(K_{1,n-1}) = n - 1$, and hence

$$|\mathrm{core}(K_{1,n-1})| + |\mathrm{corona}(K_{1,n-1})| = 2(n-1) = \alpha(G) + |V(K_{1,n-1})| - 1.$$

In other words, the bound in Proposition 13 is tight.

It has been shown in [12] that

$$\alpha(G) + \left|\bigcap \{V - S : S \in \Omega(G)\}\right| = \mu(G) + |\mathrm{core}(G)|$$

is satisfied by every König-Egerváry graph G, and taking into account that

$$\left|\bigcap \{V - S : S \in \Omega(G)\}\right| = |V(G)| - \left|\bigcup \{S : S \in \Omega(G)\}\right|,$$

we infer that the König-Egerváry graphs enjoy the following.

Proposition 15. *If G is a König-Egerváry graph, then*

$$2 \cdot \alpha(G) = |\mathrm{core}(G)| + |\mathrm{corona}(G)|.$$

The converse of Proposition 15 is not true. For instance, see the graph G_2 from Figure 2, which has $\alpha(G_2) = 3$, $\mathrm{corona}(G_2) = \{u_2, u_4, u_6, u_7\}$, and $\mathrm{core}(G_2) = \{u_2, u_4\}$.

3 Conclusions

In this paper we have proved the "*Set and Collection Lemma*", which has been employed in order to obtain a number of alternative proofs and/or strengthenings of some known results.

By Proposition 15 we know that $2 \cdot \alpha(G) = |\text{core}(G)| + |\text{corona}(G)|$ holds for every König-Egerváry graph G. Therefore, it is true for each very well-covered graph G [10]. Recall that G is a *very well-covered* graph if it has no isolated vertices, $2\alpha(G) = |V(G)|$, and all its maximal independent sets are of the same cardinality [6]. It is worth noting that there are other graphs enjoying this equality, e.g., every graph G having a unique maximum independent set, because, in this case, $\alpha(G) = |\text{core}(G)| = |\text{corona}(G)|$.

Problem 16. Characterize graphs satisfying $2 \cdot \alpha(G) = |\text{core}(G)| + |\text{corona}(G)|$.

Let us consider a dual problem. It is clear that for every graph G there exists a collection of maximum independent sets Λ such that $2 \cdot \alpha(G) = \left|\bigcup \Lambda\right| + \left|\bigcap \Lambda\right|$. Just take $\Lambda = \{X\}$ for some maximum independent set X.

Problem 17. For a given graph G find the cardinality of a largest collection of maximum independent sets Λ such that $2 \cdot \alpha(G) = \left|\bigcup \Lambda\right| + \left|\bigcap \Lambda\right|$.

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