Treewidth of the Kneser Graph and the Erdős-Ko-Rado Theorem

Daniel J. Harvey *
Department of Mathematics and Statistics
The University of Melbourne
Melbourne, Australia
d.harvey@pgrad.unimelb.edu.au

David R. Wood †
School of Mathematical Sciences
Monash University
Melbourne, Australia
david.wood@monash.edu

Submitted: Dec 17, 2013; Accepted: Feb 20, 2014; Published: Feb 28, 2014
Mathematics Subject Classifications: 05C75, 05D05

Abstract

Treewidth is an important and well-known graph parameter that measures the complexity of a graph. The *Kneser graph* Kneser(n, k) is the graph with vertex set $\binom{[n]}{k}$, such that two vertices are adjacent if they are disjoint. We determine, for large values of n with respect to k, the exact treewidth of the Kneser graph. In the process of doing so, we also prove a strengthening of the Erdős-Ko-Rado Theorem (for large n with respect to k) when a number of disjoint pairs of k-sets are allowed.

Keywords: graph theory; Kneser graph; treewidth; separators; Erdős-Ko-Rado

1 Introduction

A *tree decomposition* of a graph G is a pair $(T, (B_x \subset V(G) : x \in V(T)))$ where T is a tree and $(B_x \subseteq V(G) : x \in V(T))$ is a collection of sets, called *bags*, indexed by the nodes of T. The following properties must also hold:

- for each $v \in V(G)$, the nodes of T that index the bags containing v induce a non-empty connected subtree of T,
- for each $vw \in E(G)$, there exists some bag containing both v and w.

*Supported by an Australian Postgraduate Award.
†Supported by the Australian Research Council.

The *width* of a tree decomposition is the size of the largest bag, minus 1. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G.

Treewidth is an important concept in modern graph theory. Treewidth was initially defined by Halin [6] (with different nomenclature to the modern standard) and then later by Robertson and Seymour [16], who used it in their famous series of papers proving the Graph Minor Theorem [15]. The treewidth of a graph essentially describes how "tree-like" it is, where lower treewidth implies a more "tree-like" structure. (A forest has treewidth at most 1, for example.) Treewidth is also of key interest in the field of algorithm design—for example, treewidth is a key parameter in fixed-parameter tractability [1].

Let $[n] = \{1, \ldots, n\}$. For any set $S \subseteq [n]$, a subset of S of size k is called a *k-set*, or occasionally a k-set in S. Let $\binom{S}{k}$ denote the set of all k-sets in S. We say two sets *intersect* when they have non-empty intersection.

The *Kneser graph* $\mathrm{Kneser}(n, k)$ is the graph with vertex set $\binom{[n]}{k}$, such that two vertices are adjacent if they are disjoint.

Kneser graphs were first investigated by Kneser [9]. The chromatic number of the graph $\mathrm{Kneser}(n, k)$ was shown to be $n - 2k + 2$ by Lovász [11], as Kneser originally conjectured. This was an important proof due to the development of the topological methods involved. Many other proofs of this result have been found, for example consider [19], which gives a more combinatorial version. The Kneser graph is also of interest with regards to fractional chromatic number [17]. The famous Erdős-Ko-Rado Theorem [2] has a well-known relationship to the Kneser graph, as does the generalisation to cross-intersecting families by Pyber [14]. We discuss these in more detail in Section 2, and shall use both of these results to prove the following two theorems about the treewidth of the Kneser graph.

Theorem 1. *Let G be a Kneser graph with $n \geqslant 4k^2 - 4k + 3$ and $k \geqslant 3$. Then*

$$\mathrm{tw}(G) = \binom{n-1}{k} - 1.$$

This theorem is our main result, giving an exact answer for the treewidth of the Kneser graph when n is sufficiently large. In order to prove this, we show that $\binom{n-1}{k} - 1$ is both an upper bound and lower bound on the treewidth. We construct a tree decomposition directly in Section 3 to prove an upper bound. In Section 4 we prove the lower bound by using the relationship between treewidth and separators. In Section 6, we further conjecture that Theorem 1 extends to lower values of n.

We also prove the following more precise result when $k = 2$.

Theorem 2. *Let G be a Kneser graph with $k = 2$. Then*

$$\mathrm{tw}(G) = \begin{cases} 0 & \text{if } n \leqslant 3 \\ 1 & \text{if } n = 4 \\ 4 & \text{if } n = 5 \\ \binom{n-1}{2} - 1 & \text{if } n \geqslant 6. \end{cases}$$

The upper bounds for Theorem 2 are proved in Section 3, and the lower bounds in Section 5.

Finally, in the process of proving Theorem 1, we prove the following generalisation of the Erdős-Ko-Rado Theorem (Theorem 6 in Section 2), which says that if $n \geqslant 2k$ and H is a complete subgraph in the complement of Kneser(n,k) then $|H| \leqslant \binom{n-1}{k-1}$. We prove the same bound for balanced complete multipartite graphs.

Theorem 3. *Say $p \in [\frac{2}{3}, 1)$ and $n \geqslant \max(4k^2-4k+3, \frac{1}{1-p}(k^2-1)+2)$. If H is a complete multipartite subgraph of the complement of* Kneser(n,k) *such that no colour class contains more than $p|H|$ vertices, then $|H| \leqslant \binom{n-1}{k-1}$.*

Note that similar, but incomparable, generalisations of the Erdős-Ko-Rado Theorem have recently been explored in [5, 4, 18]. Theorem 3 is proven in Section 4, since it follows almost directly from our proof of the lower bound on the treewidth of a Kneser graph.

2 Basic Definitions and Preliminaries

From now on, we refer to the graph Kneser(n,k) as G, with n and k implicit.

Let $\Delta(H)$ be the maximum degree of a graph H and $\delta(H)$ be the minimum degree of a graph H. Also let $\alpha(H)$ be the size of the largest independent set of H, where an *independent set* is a set of pairwise non-adjacent vertices. If $k=1$, then G is the complete graph. If $n<2k$ then G contains no edges. If $n=2k$ then G is an induced matching. From now on, we shall assume that $n \geqslant 2k+1$ and $k \geqslant 2$, since the treewidth is trivial in the other cases.

In order to prove a lower bound on the treewidth of the Kneser graph, we use a known result about the relationship between treewidth and separators.

Definition Given a constant $p \in [\frac{2}{3}, 1)$, a *p-separator* (of order k) is a set $X \subset V(G)$ such that $|X| \leqslant k$ and no component of $G-X$ contains more than $p|G-X|$ vertices.

Theorem 4. *[16] For each $p \in [\frac{2}{3}, 1)$, every graph G has a p-separator of order* $\text{tw}(G)+1$.

It can easily be shown that we can partition the components of $G-X$ into two parts, such that the components in a part contain, in total, at most $p|G-X|$ vertices. This gives the following lemma.

Lemma 5. *Let X be a p-separator. Then $V(G-X)$ can be partitioned into two parts A and B, with no edge between A and B, such that*

- $(1-p)|G-X| \leqslant |A| \leqslant \frac{1}{2}|G-X|$,
- $\frac{1}{2}|G-X| \leqslant |B| \leqslant p|G-X|$.

We use a few important well known combinatorial results.

Theorem 6 (Erdős-Ko-Rado [2, 7])**.** *Let G be* Kneser(n, k) *for some $n \geqslant 2k$. Then $\alpha(G) = \binom{n-1}{k-1}$. If $n \geqslant 2k+1$ and $\mathcal{A}$ is an independent set such that $|\mathcal{A}| = \binom{n-1}{k-1}$, then $\mathcal{A} = \{v | i \in v\}$ for a fixed element $i \in [n]$.*

The original Erdős-Ko-Rado Theorem defines $\mathcal{A}$ as a set of k-sets in $[n]$, such that the k-sets of $\mathcal{A}$ pairwise intersect. Our formulation in terms of vertices in the Kneser graph is clearly equivalent. We will use Theorem 6 when determining an upper bound for $\mathrm{tw}(G)$.

The second major result is by Pyber [14]. Let $\mathcal{A}$ and $\mathcal{B}$ be sets of vertices of the Kneser graph G, such that for all $v \in \mathcal{A}$ and $w \in \mathcal{B}$ the pair vw is not an edge. Then we say the pair $(\mathcal{A}, \mathcal{B})$ are *cross-intersecting families*.

Theorem 7 (Erdős-Ko-Rado for Cross-Intersecting Families [14, 13])**.** *Let $n \geqslant 2k$ and let $(\mathcal{A}, \mathcal{B})$ be cross-intersecting families in the Kneser graph G. Then $|\mathcal{A}||\mathcal{B}| \leqslant \binom{n-1}{k-1}^2$. If $n \geqslant 2k+1$ and $(\mathcal{A}, \mathcal{B})$ are cross-intersecting families such that $|\mathcal{A}||\mathcal{B}| = \binom{n-1}{k-1}^2$, then $\mathcal{A} = \mathcal{B} = \{v | i \in v\}$ for a fixed element $i \in [n]$.*

As with Theorem 6, the original formulation by Pyber of Theorem 7 is more general. We have given the result in an equivalent form that is sufficient for our requirements.

Let X be a $\frac{2}{3}$-separator and A,B the parts of the vertex partition of $G - X$ as in Lemma 5. Now for all $v \in A$ and $w \in B$, v and w are in different components and as such are non-adjacent. So (A, B) are cross-intersecting families. We know $|A| = c|G - X|$ where $\frac{1}{3} \leqslant c \leqslant \frac{1}{2}$. By Theorem 7, it follows that $c(1-c)|G - X|^2 \leqslant \binom{n-1}{k-1}^2$. It follows that $|G - X| \leqslant \frac{3}{\sqrt{2}}\binom{n-1}{k-1}$. (We leave the precise calculation to the reader.) This gives a lower bound on $|X|$, and as such a lower bound on the treewidth (by Theorem 4). Hence $\mathrm{tw}(G) \geqslant \binom{n}{k} - \frac{3}{\sqrt{2}}\binom{n-1}{k-1} - 1$.

However, note that the parts A and B of $V(G - X)$ are vertex disjoint, but that the definition of a pair of cross-intersecting families does not require this. In fact, Theorem 7 shows that in the case where $|\mathcal{A}||\mathcal{B}|$ is maximised, $\mathcal{A} = \mathcal{B}$. We show we can do better than the above naïve lower bound on $\mathrm{tw}(G)$ when $\mathcal{A}$ and $\mathcal{B}$ are disjoint.

Before considering our final preliminary, we provide the following definitions. Consider all of the a-sets in $[b]$. Define the *colexicographic* or *colex* ordering on the a-sets as follows: if x and y are distinct a-sets, then $x < y$ when $\max(x - y) < \max(y - x)$. This is a strict total order. A set X of a-sets in $[b]$ is *first* if X consists of the first $|X|$ a-sets in the colex ordering of all the a-sets in $[b]$.

Now consider the colex ordering of a-sets in $[b]$. All of the a-sets in $[i]$ (where $i < b$) come before any a-set containing an element greater than or equal to $i + 1$. To see this, note if x is an a-set in $[i]$ and y is an a-set with $j \in y$ such that $j \geqslant i + 1$, then $\max(x - y) \leqslant \max(x) \leqslant i$, and $\max(y - x) \geqslant j \geqslant i + 1$ since $j \in y - x$. We will use this when determining the make-up of first sets in Section 4.

Let X be a set of a-sets in $[b]$. For $c \leqslant a$, the *c-shadow of X* is the set $\{x : |x| = c$, and $\exists y \in X$ such that $x \subseteq y\}$. That is, the c-shadow contains all c-sets that are contained within a-sets of X. If x is an a-set in $[b]$, let the *complement* of x be the $(b - a)$-set $y = [b] - x$. If X is a set of a-sets on $[b]$, then the *complement* of X is $\overline{X} := \{y : y$ is the complement of some $x \in X\}$. Note $|X| = |\overline{X}|$.

Lemma 8 (A first set minimises the shadow [10, 8] (see [3] for a short proof))**.** *Let X be a set of a-sets on $[b]$, $c \leqslant a$ and S be the c-shadow of X. Suppose $|X|$ is fixed but X is not. Then $|S|$ is minimised when X is first.*

This idea is also used by Pyber [14] and Matsumoto and Tokushige [13]. Intuitively, the shadow S should be minimised whenever the a-sets of X "overlap" as much as possible, so that each c-set in S is a subset of as many a-sets as possible.

3 Upper Bound for Treewidth

This section proves the upper bounds on $\mathrm{tw}(G)$ in Theorems 1 and 2.

In both Theorem 1 and 2, the upper bound is almost always $\binom{n-1}{k} - 1$. The only exceptions are the trivial cases (when $n \leqslant 2k$), and the case when $k = 2$ and $n = 5$, which is the Petersen graph. The Petersen graph is well-known to have treewidth 4 ([12], for example). What follows is a general upper bound on the treewidth of any graph, which is sufficient to prove the remaining cases.

Lemma 9. *If H is any graph, then* $\mathrm{tw}(H) \leqslant \max\{\Delta(H), |V(H)| - \alpha(H) - 1\}$.

Proof. Let $\alpha := \alpha(H)$. We shall construct a tree decomposition with underlying tree T, where T is a star with $\alpha(H)$ leaves. Let R be the bag indexed by the central node of T, and label the other bags $B_1, \dots, B_\alpha$. Let $X := \{x_1, \dots x_\alpha\}$ be a maximum independent set in H. Let $R := V(H) - X$ and $B_i := N(x_i) \cup \{x_i\}$ for all $i \in \{1, \dots, \alpha\}$. We now show this is a tree decomposition:

Any vertex not in X is contained in R. Given the structure of the star, any induced subgraph containing the central node is connected. Alternatively, if a vertex is in X, then it appears only in bags indexed by leaves. However, since X is an independent set, $x_i \in X$ appears only in B_i, not in any other bag B_j. A single node is obviously connected. If vw is an edge of H, then at most one of v and w is in X. Say $v = x_i \in X$. Then v, w both appear in the bag B_i. Otherwise neither vertex is in X, and both vertices appear in R.

So this is a tree decomposition. The size of R is $|V(H)| - \alpha(H)$. The size of B_i is the degree of x_i, plus one, which is at most $\Delta(H) + 1$. From here our lemma is proven. □

We now consider this result for the Kneser graph itself.

Lemma 10. *If G is a Kneser graph with $k \geqslant 2$ and $n \geqslant 2k + 1$, then* $\mathrm{tw}(G) \leqslant \binom{n}{k-1} - 1$.

Proof. By Lemma 9 and Theorem 6, and since $n \geqslant 2k + 1$,

$$\mathrm{tw}(G) \leqslant \max\{\Delta(G), |V(G)| - \alpha(G) - 1\} = \max\left\{\binom{n-k}{k}, \binom{n}{k} - \binom{n-1}{k-1} - 1\right\}.$$

Since $k \geqslant 2$, $\mathrm{tw}(G) \leqslant \binom{n-1}{k} - 1$, as required. □

4 Separators in the Kneser Graph

To complete the proof of Theorem 1, it is sufficient to prove a lower bound on the treewidth. The following lemma, together with Theorem 4, provides this. It is the heart of the proof of Theorem 3.

Lemma 11. *Let X be a p-separator of the Kneser graph G. If $n \geqslant \max(4k^2 - 4k + 3, \frac{1}{1-p}(k^2-1)+2)$, then $|X| \geqslant \binom{n-1}{k}$.*

Proof. Assume, for the sake of a contradiction, that $|X| < \binom{n-1}{k}$. Then $|G - X| > \binom{n-1}{k-1}$. By Lemma 5, $G - X$ has two parts A and B such that $(1-p)|G-X| \leqslant |A| \leqslant \frac{1}{2}|G-X|$ and $\frac{1}{2}|G-X| \leqslant |B| \leqslant p|G-X|$ and no edge has an endpoint in both A and B.

For a given element $i \in [n]$, let $A_i := \{v \in A : i \in v\}$. Also define $A_{-i} := \{v \in A : i \notin v\}$. So A_i and A_{-i} partition the set A, for any choice of i. Define analogous sets for B.

Claim 1. There exists some i such that $|B_i| \geqslant \frac{1}{k}|B|$.

Proof. Since $|A| \geqslant (1-p)|G-X| > 0$, there is a vertex $v \in A$. Without loss of generality, $v = \{1, \dots, k\}$. Each $w \in B$ is not adjacent to v, and so w and v intersect. Thus each w must contain at least one of $1, \dots, k$. Hence at least one of these elements appears in at least $\frac{1}{k}|B|$ of the vertices of B, as required. □

Without loss of generality, $|B_n| \geqslant \frac{1}{k}|B|$.

Claim 2. $|B_n| > \binom{n-3}{k-2} + \binom{n-2}{k-2}$.

Proof. $|B| \geqslant \frac{1}{2}|G-X| \geqslant \frac{1}{2}\binom{n-1}{k-1}$. Then by Claim 1 and our subsequent assumption, $|B_n| \geqslant \frac{1}{k}|B| \geqslant \frac{1}{2k}|G-X| \geqslant \frac{1}{2k}\binom{n-1}{k-1}$. Assume for the sake of a contradiction that $|B_n| \leqslant \binom{n-3}{k-2} + \binom{n-2}{k-2}$. So

$$\frac{1}{2k}\binom{n-1}{k-1} \leqslant \binom{n-3}{k-2} + \binom{n-2}{k-2}.$$

Thus

$$(n-1)! \leqslant 2k(k-1)((n-k)(n-3)! + (n-2)!).$$

Hence

$$n^2 - 3n + 2 = (n-1)(n-2) \leqslant 2k(k-1)(2n-k-2) = 4k^2n - 4kn - 2k^3 - 2k^2 + 4k.$$

So $n^2 + (4k - 4k^2 - 3)n + 2k^3 + 2k^2 - 4k + 2 \leqslant 0$. Since $n \geqslant 4k^2 - 4k + 3$, it follows $2k^3 + 2k^2 - 4k + 2 \leqslant 0$. Given that $k \geqslant 1$, this provides our desired contradiction. □

Consider the set $\overline{A_{-n}}$, that is, the complements of the vertices in A that do not contain n. So every set in $\overline{A_{-n}}$ contains n. Let $\overline{A_{-n}}^* := \{\overline{v} - n : \overline{v} \in \overline{A_{-n}}\}$. That is, remove n from each set in $\overline{A_{-n}}$. There is clearly a one-to-one correspondence between $(n-k)$-sets in $\overline{A_{-n}}$ and $(n-k-1)$-sets in $\overline{A_{-n}}^*$.

Similarly, define $B_n^* := \{v - n : v \in B_n\}$. That is, remove from each vertex of B_n the element n, which they all contain. The resultant sets are $(k-1)$-sets in $[n-1]$.

Claim 3. If $v^* \in B_n^*$ and $\overline{w}^* \in \overline{A_{-n}}^*$, then $v^* \not\subseteq \overline{w}^*$.

Proof. Assume, for the sake of a contradiction, that $v^* \subseteq \overline{w}^*$. Then it follows that $v \subset \overline{w}$, by re-adding n to both sets. Thus v and w are adjacent. However, $v \in B_n \subset B$ and $w \in A_n \subset A$, which is a contradiction. □

Let S be the $(k-1)$-shadow of $\overline{A_{-n}}^*$. Hence if $v \in B_n^*$, then $v \notin S$, by Claim 3. So, it follows that

$$B_n^* \subseteq \binom{[n-1]}{k-1} - S.$$

Hence we have an upper bound for $|B_n^*|$ when we take $|S|$ to be minimised. By Lemma 8, $|S|$ is minimised when $\overline{A_{-n}}^*$ is first.

Claim 4. $|A_{-n}| \leqslant \binom{n-3}{k-2}$.

Proof. $|A_{-n}| = |\overline{A_{-n}}| = |\overline{A_{-n}}^*|$, so it is sufficient to show that $|\overline{A_{-n}}^*| \leqslant \binom{n-3}{k-2}$. Assume for the sake of contradiction that $|\overline{A_{-n}}^*| \geqslant \binom{n-3}{k-2} = \binom{n-3}{n-k-1}$.

Firstly, we show that $|S| \geqslant \binom{n-3}{k-1}$. It is sufficient to prove this lower bound when $|S|$ is minimised. Hence we can assume that $\overline{A_{-n}}^*$ is first, and contains the first $\binom{n-3}{n-k-1}$ $(n-k-1)$-sets in the colexicographic ordering. That is, it contains all $(n-k-1)$-sets on $[n-3]$. This is because there are $\binom{n-3}{n-k-1}$ such sets, and they come before all other sets in the ordering. In that case, S contains all $(k-1)$-sets in $[n-3]$. Since all of the $(k-1)$-sets in $[n-3]$ are in S, it follows that $|S| \geqslant \binom{n-3}{k-1}$, as required.

Then it follows that $|B_n^*| \leqslant \binom{n-1}{k-1} - \binom{n-3}{k-1} = \binom{n-3}{k-2} + \binom{n-2}{k-2}$. However, $|B_n^*| = |B_n| > \binom{n-3}{k-2} + \binom{n-2}{k-2}$ by Claim 2. This provides our desired contradiction. □

Claim 5. $|A_n| \geqslant \frac{k}{k+1}|A|$.

Proof. First we show that $|A_n| \geqslant k|A_{-n}|$. Suppose otherwise, for the sake of a contradiction. By Claim 4, $|A| = |A_n| + |A_{-n}| < (k+1)|A_{-n}| \leqslant (k+1)\binom{n-3}{k-2}$. But $|A| \geqslant (1-p)|G - X|$. Hence $(1-p)\binom{n-1}{k-1} < (k+1)\binom{n-3}{k-2}$. Thus $(n-1)(n-2) < \frac{1}{1-p}(k+1)(k-1)(n-k) \leqslant \frac{1}{1-p}(k+1)(k-1)(n-2)$. Thus $n < \frac{1}{1-p}(k^2-1)+1$, which contradicts our lower bound on n.

Then $|A_n| \geqslant k|A_{-n}| = k(|A| - |A_n|)$. So $(k+1)|A_n| \geqslant k|A|$ as required. □

Claim 6. $B_n = B$.

Proof. Suppose, for the sake of a contradiction, that there exists some vertex $v \in B$ such that $n \notin v$. So each $w \in A_n$ contains n (by definition) and some element of v (which is not n), since vw is not an edge. Any vertex of A_n can be constructed as follows—take element n, choose one of the k elements of v, and choose the remaining $k-2$ elements from the remaining $n-2$ elements of $[n]$. Thus

$$|A_n| \leqslant 1 \cdot k\binom{n-2}{k-2}.$$

Note this is actually a weak upper bound, since we have counted some of the vertices of A_n more than once. Recall $|A| \geqslant (1-p)|G-X| \geqslant (1-p)\binom{n-1}{k-1}$. So by Claim 5,

$$\frac{(1-p)k}{(k+1)}\binom{n-1}{k-1} \leqslant \frac{k}{k+1}|A| \leqslant k\binom{n-2}{k-2}.$$

Thus $\frac{n-1}{k-1} \leqslant \frac{1}{1-p}(k+1)$ and $n \leqslant \frac{1}{1-p}(k^2-1)+1$, which contradicts our lower bound on n. □

Claim 7. $A_n = A$.

Proof. This follows by essentially the same argument as Claim 6. Assume our claim does not hold and there exists $v \in A$ such that $n \notin v$. By Claim 6, $|B_n| = |B| \geqslant \frac{1}{2}\binom{n-1}{k-1}$. There is an upper bound on $|B_n|$ equal to the upper bound on $|A_n|$ in the previous proof. Then

$$\frac{1}{2}\binom{n-1}{k-1} \leqslant |B| = |B_n| \leqslant k\binom{n-2}{k-2},$$

and so $n \leqslant 2k(k-1)+1$. This contradicts our lower bound on n. □

Claims 6 and 7 show that every vertex in $G-X = A \cup B$ contains n. Thus $|G-X| \leqslant \binom{n-1}{k-1}$ and $|X| \geqslant \binom{n-1}{k}$, our desired contradiction. □

By Lemma 11, if X is a $\frac{2}{3}$-separator of the Kneser graph G and $n \geqslant 4k^2-4k+3$, then $|X| \geqslant \binom{n-1}{k}$. Hence by Theorem 4, $\text{tw}(G) \geqslant \binom{n-1}{k} - 1$. This proves Theorem 1.

Also, Lemma 11 allows us to prove Theorem 3.

Proof of Theorem 3. Let $C_1, \dots, C_r$ be the colour classes of H. Recall $G = \text{Kneser}(n,k)$. Let $X := V(\overline{G}) - V(H)$, so that $X, C_1, \dots, C_r$ is a partition of the vertex set of $\overline{G}$ (and also G). In G there are no edges between any pair C_i, C_j, and $|C_i| \leqslant p|H| = p|G-X|$ for each i. So X is a p-separator of G, and $|X| \geqslant \binom{n-1}{k}$ by Lemma 11. Hence $|H| \leqslant \binom{n-1}{k-1}$. □

5 Lower Bound for Treewidth in Theorem 2

To complete our proof of Theorem 2, we need to obtain a lower bound on the treewidth when $k=2$. If $n \leqslant 4$, then Theorem 2 is trivial. When $n=5$, then G is the Petersen graph, which contains a K_5-minor forcing $\text{tw}(G) \geqslant 4$. Hence we may assume that $n \geqslant 6$.

Assume, for the sake of a contradiction that $\text{tw}(G) < \binom{n-1}{2} - 1$. Let $(T, (B_x : x \in V(T)))$ be a minimum width tree decomposition for G, and normalise the tree decomposition such that if $xy \in E(T)$, then $B_x \nsubseteq B_y$ and $B_y \nsubseteq B_x$. By Theorem 4, there exists a $\frac{2}{3}$-separator X such that $|X| < \binom{n-1}{2}$. In fact, by the original proof in [16], we can go further and assert that X is a subset of a bag of $(B_x : x \in V(T))$.

Now $|G-X| = \binom{n}{2} - |X| > \binom{n-1}{1} = n-1$. By Lemma 5, $V(G-X)$ has two parts A and B such that $\frac{1}{3}|G-X| \leqslant |A|, |B| \leqslant \frac{2}{3}|G-X|$ and there is no edge with an endpoint in A and B. (Note that this bound on $|A|$ and $|B|$ is slightly weaker than in Lemma 5, but has the benefit of being the same on both parts.) Since $n \geqslant 6$, it follows that $|A|, |B| \geqslant 2$. By Theorem 6, $V(G-X)$ is too large to be an independent set, and so it contains an edge, with both endpoints in A or both endpoints in B.

Without loss of generality this edge is $\{1,2\}\{3,4\} \in A$. Then $B \subseteq \{\{1,3\},\{1,4\}, \{2,3\},\{2,4\}\}$. If B contains an edge, then $V(G-X) \subseteq \{\{1,2\},\{1,3\},\{1,4\},\{2,3\}, \{2,4\},\{3,4\}\}$ and has maximum order 6. Otherwise, without loss of generality, $B = \{\{1,3\},\{1,4\}\}$ and $A = \{\{3,4\},\{1,i\} | i \notin \{1,3,4\}\}$, so $|G-X| = n$. (Note A must be exactly that set, or $|G-X|$ is too small.)

If $n \geqslant 7$, then $|G-X| \geqslant 7$ and the first case cannot occur. However in the second case, $|B| = 2 < \frac{1}{3} \cdot 7 \leqslant \frac{1}{3}n$. So neither case can occur, and we have forced a contradiction on either $|G-X|$ or $|B|$. This completes the proof when $n \geqslant 7$. Hence, let $n = 6$, and note $|G-X| = 6$ in either case.

Now we use the fact that X is a subset of some bag B_x. Now for all $x \in V(T)$, $|B_x| \leqslant \binom{5}{2} - 1 = 9$. Since $|G-X| = 6$, it follows $|X| = 9$. Hence X is exactly a bag of maximum order. For either choice of $G-X$, note that A is a connected component. So there is some subtree of $T-x$ that contains all vertices of A. Let y be the node of this subtree adjacent to x. Also note, for either choice of $G-X$, that each vertex of X has a neighbour in A. So every vertex of B_x is also in bag B_y, which contradicts our normalisation.

Thus, if $n \geqslant 6$, then $\text{tw}(G) \geqslant \binom{n-1}{2} - 1$. This completes the proof of Theorem 2.

6 Open Questions

We conjecture that Theorem 1 should also hold for smaller values of n.

Conjecture 12. *Let G be a Kneser graph with $n \geqslant 3k$ and $k \geqslant 2$. Then* $\text{tw}(G) = \binom{n-1}{k} - 1$.

This conjecture follows directly from Theorem 2 when $k = 2$. The Petersen graph also shows that $n \geqslant 3k$ is a tight bound when $k = 2$.

In general, we can determine a slightly better tree decomposition when $n < 3k - 1$. Let $X = \{v \in V(G) : 1 \in v\}$, and let W be an independent set in $V(G) - X$ such that no two vertices of W have a common neighbour in X. We define a tree decomposition for G with underlying tree T as follows. Let r denote the root node of T, and let r have one child node for each vertex in W and each vertex in X adjacent to no vertex in W. Label each of these child nodes by their associated vertex of G. Let each node labeled by a vertex $w \in W$ have one child node for each vertex of $N(w) \cap X$. Label each of those child nodes by their associated vertex of G, and note that since every vertex of X has at most one neighbour in W, no vertex of G labels more than one node of T.

Define the bag indexed by r to be $V(G) - W - X$. Note this bag contains less than $\binom{n-1}{k}$ vertices when $W \neq \emptyset$. If a node is labeled by a vertex $v \in X$, let the corresponding bag be $N(v) \cup \{v\}$. These bags contain $\binom{n-k}{k} + 1$ vertices. If a node is labeled by a vertex $w \in W$, let the corresponding bag be $\{w\} \cup \{u : uw \in E(G), 1 \notin u\} \cup \{u : ux \in E(G)$

where $xw \in E(G)$ and $1 \in x\}$. These bags contain less than $\binom{n-1}{k}$ vertices whenever $|W| \geqslant 2$, since they contain no vertex in X, and each contains only one vertex from W. This is a valid tree decomposition, but we omit the proof. When $|W| \geqslant 2$, the width of this tree decomposition is less than the width given by Lemma 9.

However, when $|W| \leqslant 1$, this tree decomposition has the same width as given by Lemma 9. We can construct W such that $|W| \geqslant 2$ iff $n < 3k - 1$. For example, let $W = \{\{2, \ldots, (k+1)\}, \{(k+1), \ldots, 2k\}\}$. If $n \leqslant 3k - 2$, then any vertex of X must be non-adjacent to at least one vertex of W. Alternatively, if $n \geqslant 3k - 1$ and $|W| \geqslant 2$, then there exists two vertices $x, y \in W$ such that $|x \cup y| \leqslant 2k - 1$. Then X contains a vertex adjacent to both x and y. Hence, for general n, we cannot improve the lower bound on n in Theorem 1 to $3k - 2$ or below. This does leave a question about what may occur for $n = 3k - 1$. It is possible that Theorem 1 holds for $n \geqslant 3k - 1$, with the Petersen graph as a single exception.

Acknowledgements

Thanks to Alex Scott for helpful conversations, and for pointing out references [4, 5, 18].

References

[1] Hans L. Bodlaender. A tourist guide through treewidth. *Acta Cybernet.*, 11(1-2):1–21, 1993.

[2] Paul Erdős, Chao Ko, and Richard Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[3] Péter Frankl. A new short proof for the Kruskal-Katona theorem. *Discrete Math.*, 48(2-3):327–329, 1984.

[4] Dániel Gerbner, Nathan Lemons, Cory Palmer, Dömötör Pálvölgyi, Balázs Patkós, and Vajk Szécsi. Almost cross-intersecting and almost cross-sperner pairs of families of sets. *Graphs and Combinatorics*, 29:489–498, 2013.

[5] Dániel Gerbner, Nathan Lemons, Cory Palmer, Balázs Patkós, and Vajk Szécsi. Almost intersecting families of sets. *SIAM. Discrete Mathematics*, 26(4):1657–1669, 2012.

[6] Rudolf Halin. S-functions for graphs. *J. Geometry*, 8(1-2):171–186, 1976.

[7] Gyula O. H. Katona. A simple proof of the Erdős-Chao Ko-Rado theorem. *J. Combin. Theory Ser. B*, 13:183–184, 1972.

[8] Gyula O.H. Katona. A theorem of finite sets. In *Theory of graphs (Proc. Colloq., Tihany, 1966)*, pages 187–207. Academic Press, New York, 1968.

[9] Martin Kneser. Aufgabe 360. *Jahresber. Deutsch. Math.-Verein.*, 58:27, 1955.

[10] Joseph B. Kruskal. The number of simplices in a complex. In *Mathematical optimization techniques*, pages 251–278. Univ. of California Press, Berkeley, Calif., 1963.

[11] László Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[12] Lambertus Marchal. *Treewidth*. PhD thesis, Maastricht University, 2012.

[13] Makoto Matsumoto and Norihide Tokushige. The exact bound in the Erdős-Ko-Rado theorem for cross-intersecting families. *J. Combin. Theory Ser. A*, 52(1):90–97, 1989.

[14] László Pyber. A new generalization of the Erdős-Ko-Rado theorem. *J. Combin. Theory Ser. A*, 43(1):85–90, 1986.

[15] Neil Robertson and Paul D. Seymour. Graph minors I–XXIII. *J. Combin. Theory Ser. B*, 1983–2012.

[16] Neil Robertson and Paul D. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms*, 7(3):309–322, 1986.

[17] Edward R. Scheinerman and Daniel H. Ullman. *Fractional graph theory*. Wiley, 1997.

[18] Alex Scott and Elizabeth Wilmer. Hypergraphs of bounded disjointness. 2013. arXiv:1306.4236.

[19] Günter M. Ziegler. Generalized Kneser coloring theorems with combinatorial proofs. *Inventiones Math*, 147:671–691, 2001.