Logarithmic Tree-Numbers for Acyclic Complexes

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Abstract

For a d-dimensional cell complex Γ with $\tilde{H}_i(\Gamma) = 0$ for $-1 \leqslant i < d$, an i-dimensional tree is a non-empty collection B of i-dimensional cells in Γ such that $\tilde{H}_i(B \cup \Gamma^{(i-1)}) = 0$ and $w(B) := |\tilde{H}_{i-1}(B \cup \Gamma^{(i-1)})|$ is finite, where $\Gamma^{(i)}$ is the i-skeleton of Γ. The i-th tree-number is defined $k_i := \sum_B w(B)^2$, where the sum is over all i-dimensional trees. In this paper, we will show that if Γ is acyclic and $k_i > 0$ for $-1 \leqslant i \leqslant d$, then k_i and the combinatorial Laplace operators Δ_i are related by $\sum_{i=-1}^{d} \omega_i \, x^{i+1} = (1+x)^2 \sum_{i=0}^{d-1} \kappa_i x^i$, where $\omega_i = \log\det\Delta_i$ and $\kappa_i = \log k_i$. We will discuss various consequences and applications of this equation.

1 Introduction

In this paper, we will extend Temperley's tree-number formula for finite graphs [13] to a class of cell complexes, called γ-complexes, and show applications to various acyclic complexes.

As the main object of study in this paper, we define a γ-*complex* to be a non-empty finite cell complex Γ whose integral cellular chain complex $\{C_i, \partial_i\}$ with $C_{-1} = \mathbb{Z}$ satisfies the following conditions:

(γ1) $\partial_i \neq 0$ for $0 \leqslant i \leqslant \dim\Gamma$, and

(γ2) the reduced integral homology $\tilde{H}_i(\Gamma) = 0$ for $i < \dim\Gamma$.

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This definition is intended to be a generalization of connected finite graphs. Other examples of γ-complexes are matroid complexes, standard simplexes, and cubical complexes [4] with the latter two being acyclic. Note that a γ-complex is a special case of APC (acyclic in positive codimension) complexes in the terminology of [4].

We define high-dimensional spanning trees for a γ-complex extending the ideas in [1]. Given a γ-complex Γ, let Γ_i be the set of all i-dimensional cells, and $\Gamma^{(i)}$ the i-skeleton of Γ. Given a subset $S \subset \Gamma_i$, define $\Gamma_S = S \cup \Gamma^{(i-1)}$ as a subcomplex of Γ. An *i-dimensional spanning tree* of Γ (or simply, an *i-tree*) is a non-empty subset $B \subset \Gamma_i$ such that $\tilde{H}_i(\Gamma_B) = 0$ and $w(B) := |\tilde{H}_{i-1}(\Gamma_B)|$ is finite. Define the *i-th tree-number* of Γ by

$$k_i(\Gamma) = k_i = \sum_B w(B)^2 \,,$$

where the sum is over all i-trees in Γ. We will see that $k_i > 0$ for all $-1 \leqslant i \leqslant \dim \Gamma$ where we define $k_{-1} = 1$. If Γ is a graph, then k_0 is the number of vertices and k_1 is the number of spanning trees in Γ.

An important method for computing the tree-numbers for Γ is given by the combinatorial Laplacians Δ_i ([1], [4], and [13]). For example, let $\Delta_0 = L + J$, where L is the Laplacian matrix of a finite graph G of order n, and J is the all 1's matrix. Temperley [13] showed that $\det \Delta_0 = n^2 k_1$ for G (refer to Theorem 4). This method is more efficient than the matrix-tree theorem for certain graphs. Indeed, for $\Gamma = K_n$ the complete graph on n vertices, we have $\Delta_0 = nI$ and $\det \Delta_0 = n^n$, from which the Cayley's Theorem $k_1(K_n) = n^{n-2}$ is immediate.

We will show that Temperley's formula can be extended to any γ-complex Γ (refer to Proposition 7). Also, if Γ is acyclic of dimension d, then each Δ_i is positive-definite, and the following polynomials are well-defined:

$$D(x) = \sum_{i=-1}^{d} (\log \det \Delta_i) x^{i+1} \text{ and } K(x) = \sum_{i=0}^{d-1} (\log k_i) x^i \,.$$

The main result of the paper is

$$D(x) = (1+x)^2 K(x) \,. \tag{1}$$

A refinement of this equation and its applicability to matroid complexes will be discussed through a simple example. (See Section 5.)

This paper is organized as follows. Section 2 is a review of useful facts from matrix theory and combinatorial Laplacians for γ-complexes. It also provides a proof of Temperley's tree-number formula. In Section 3, we will describe high-dimensional spanning trees for a γ-complex via the boundary operators of its chain complex. In Section 4, we will prove the main results of the paper which consist of a generalization of Temperley's tree-number formula and a logarithmic version (1) of this result for acyclic γ-complexes. In Section 5, we will discuss applications of (1) to standard simplexes [7], the cubical complexes [4], and an example of graphic matroid complex.

2 Preliminaries

2.1 Lemmas from Matrix Theory

We will review several important facts about symmetric matrices. For definitions and basic facts from matrix theory, one may refer to [6]. All matrices are assumed to have real entries. For a square matrix M, let P_M denote the multiset of all non-zero eigenvalues of M, and let $\pi_M = \prod_{\lambda \in P_M} \lambda$. The following two lemmas and their proofs appear in [1]. We will sketch the proofs here.

Lemma 1. *Let A and B be $n \times n$ symmetric matrices such that $AB = BA = 0$. Then, $P_{A+B} = P_A \cup P_B$ as a multiset. In particular, if $A + B$ is non-singular,*

$$\det(A + B) = \pi_A \pi_B \,. \tag{2}$$

Proof. Since A and B are symmetric and they commute, they are simultaneously diagonalizable. For each $i \in [1, n]$, let λ_i and μ_i be the eigenvalues of A and B, respectively, so that the collection $\{\lambda_i + \mu_i \,|\, i \in [1, n]\}$ is the multiset of all eigenvalues of $A + B$. Since $AB = 0$, we have either $\lambda_i = 0$ or $\mu_i = 0$ for each i. Therefore $\alpha = \lambda_i + \mu_i \in P_{A+B}$ if and only if $\alpha = \lambda_i \in P_A$ or $\alpha = \mu_i \in P_B$. □

Lemma 2. *Let M be a rectangular matrix of rank r ($r > 0$). Let $\mathcal{B}(M)$ be the collection of all non-singular $r \times r$ submatrices of M. If $A = MM^t$, or M^tM, then*

$$\pi_A = \sum_{B \in \mathcal{B}(M)} (\det B)^2 \,. \tag{3}$$

Proof. This result follows from Binet-Cauchy theorem and the fact that the product of all non-zero eigenvalues of a *diagonalizable* matrix of rank r equals the sum of all principal minors of order r. Equation (3) holds for both MM^t and M^tM because they have the same multiset of non-zero eigenvalues. Details will be omitted. □

2.2 Combinatorial Laplacians for γ-complexes

We will assume familiarity with basic definitions concerning finite cell complexes and reduced homology groups. One may refer to standard texts such as [10] for details.

Let X be a finite cell complex of dimension d. For $i \in [0, d]$, let X_i denote the set of all i-dimensional cells in X. The i-skeleton $X^{(i)}$ of X is $X_0 \cup X_1 \cup \cdots \cup X_i$. Since our main object of study is a γ-complex, we will consider only those X such that $X_i \neq \emptyset$ for all $i \in [0, d]$. This condition on X allows one to represent the boundary maps ∂_i of its chain complex as matrices. Also we define X_{-1} to be a set with one element.

For $i \in [-1, d]$, the i-th chain group of X is the free abelian group $C_i \cong \mathbb{Z}^{|X_i|}$ generated by X_i. Let $\{C_i, \partial_i\}$ be an augmented chain complex of X with the augmentation $\partial_0 : C_0 \to C_{-1} \cong \mathbb{Z}$ given by $\partial_0(v) = 1$ for every $v \in X_0$. The i-th reduced homology group of X is defined $\tilde{H}_i(X) = \operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i+1}$ where we define ∂_{d+1} and ∂_{-1} to be zero maps. Hence, we have $\tilde{H}_d(X) = \operatorname{Ker} \partial_d$ and $\tilde{H}_{-1}(X) = 0$. X is *acyclic* if $\tilde{H}_i(X) = 0$ for all $i \in [-1, d]$.

For $i \in [0,d]$, regard the boundary map $\partial_i : C_i \to C_{i-1}$ as a $|X_{i-1}|\times|X_i|$ integer matrix whose rows and columns are indexed by X_{i-1} and X_i, respectively. In particular, the augmentation ∂_0 is an all 1's row matrix of size $|X_0|$. The coboundary map $\partial_i^t : C_{i-1} \to C_i$ is the transpose of ∂_i.

For $i \in [-1,d]$, the i-th combinatorial Laplacian $\Delta_i : C_i \to C_i$ is defined by

$$\Delta_i = \partial_{i+1}\partial_{i+1}^t + \partial_i^t\partial_i \text{ if } i \in [0,d-1]\,,$$
$$\Delta_{-1} = \partial_0\partial_0^t\,,\ \text{and } \Delta_d = \partial_d^t\partial_d\,.$$

Note that $L_i := \partial_{i+1}\partial_{i+1}^t$ and $J_i := \partial_i^t\partial_i$ are symmetric, non-negative definite, and $L_iJ_i = J_iL_i = 0$ because $\partial_i\partial_{i+1} = 0$. Hence, each Δ_i is also symmetric and non-negative definite by Lemma 1.

An important property of Δ_i is that the dimension of the 0-eigenspace of Δ_i as an operator on a finite dimensional vector space over $\mathbb{Q}$ equals the dimension of the reduced rational homology $\tilde{H}_i(X;\mathbb{Q})$ [5, Proposition 2.1]. Therefore, if $\Delta_i \neq 0$,

$$\det\Delta_i > 0 \quad \text{if and only if} \quad \operatorname{rk}\tilde{H}_i(X) = 0\,. \tag{4}$$

Note that $\Delta_{-1} = L_{-1} : \mathbb{Z} \to \mathbb{Z}$ is a multiplication by $|X_0|$. Now the following lemma is immediate from the definition of γ-complex and (4).

Lemma 3. *If Γ is a γ-complex of dimension d, then $\det\Delta_i > 0$ for $i \in [-1,d)$. In addition, if Γ is acyclic, then $\det\Delta_d > 0$ also.* □

2.3 Temperley's tree-number formula

For a finite loopless graph G with n vertices and its Laplacian matrix $L(G)$, Temperley [13] showed the following analogue of the Matrix-Tree theorem [8] for the number of spanning trees $k(G)$ in G. Let J denote the all 1's matrix.

Theorem 4.

$$\det(L(G)+J) = n^2k(G)\,.$$

Proof. We will give a proof of this formula as a consequence of the multilinearity of determinant function and the Matrix-Tree theorem. We refer the readers to [2] for a proof via eigenvalues.

Let $L(G)+J = (C_1+D_1, C_2+D_2, \ldots, C_n+D_n)$, where C_i's and D_i's are the columns of $L(G)$ and J, respectively. Given a subset $S \subset [n]$, define $M_S = (X_1, X_2, \ldots, X_n)$, where $X_i = C_i$ if $i \notin S$ and $X_i = D_i$ if $i \in S$. By the multilinearity of determinant function (on columns), $\det(L(G)+J) = \sum_{S\subset[n]} \det M_S$, where the sum is over all subsets S of $[n]$.

Clearly, we have $\det M_\emptyset = \det L(G) = 0$ because $L(G)$ is singular. Also, if $|S| > 1$, then $\det M_S = 0$ because rank of J is 1. However, for every $i \in [n]$, we see that $\det M_{\{i\}} = nk(G)$ because every entry in D_i is 1 and every cofactor of $L(G)$ equals $k(G)$ by the Matrix-Tree theorem. Therefore, we have

$$\det(L(G)+J) = \sum_{0\leqslant i\leqslant n} \det M_{\{i\}} = n^2k(G)\,.$$

□

We make two observations about Theorem 4. First, unlike Matrix-Tree theorem, Temperley's formula does not require deletion of a row and a column from $L(G)$ to compute $k(G)$. Second, regarding G as a 1-dimensional γ-complex, one can check that $L(G) = \partial_1 \partial_1^t$ and $J = \partial_0^t \partial_0$. Hence, Theorem 4 says

$$\det \Delta_0 = n^2 k(G)\,. \tag{5}$$

As we shall see, similar observations can be made in computing high-dimensional tree-numbers for γ-complexes using combinatorial Laplacians. In particular, one can easily check that equation (5) is a consequence of Proposition 7.

3 High-dimensional trees for γ-complexes

We refer the readers to [1], [4], and [7] for details of high-dimensional trees and of the exact homology sequence used in the proof of Theorem 6. In this section, Γ will denote a γ-complex of dimension d. For a non-empty subset $S \subset \Gamma_i$, define $\Gamma_S := S \cup \Gamma^{(i-1)}$ as an i-dimensional subcomplex of Γ. For $i \in [-1, d]$, a non-empty subset $B \subset \Gamma_i$ is an *i-dimensional spanning tree* (or simply, *i-tree*) if

1. $\tilde{H}_i(\Gamma_B) = 0$,
2. $w(B) := |\tilde{H}_{i-1}(\Gamma_B)|$ is finite, and
3. $\tilde{H}_j(\Gamma_B) = 0$ for $j \leqslant i-2$.

Note that condition 3 is a consequence of the fact $\Gamma_B^{(i-1)} = \Gamma^{(i-1)}$. We will denote the set of all i-trees in Γ by $\mathcal{B}_i = \mathcal{B}_i(\Gamma)$ with $\mathcal{B}_{-1} = \{\emptyset\}$. It is clear that $\mathcal{B}_0$ is the set of all single 0-cells in Γ and $\mathcal{B}_1$ is the set of all graph theoretic spanning trees of $\Gamma^{(1)}$ as a finite graph.

Define the *i-th tree-number* of Γ to be

$$k_i = k_i(\Gamma) = \sum_{B \in \mathcal{B}_i} w(B)^2\,.$$

We have $k_{-1} = 1$ by definition, and $k_0 = |\Gamma_0|$. If Γ is a connected graph, then k_1 is the number of spanning trees in Γ because $w(B) = 1$ for $B \in \mathcal{B}_1$. However, $w(B)$ may not equal 1 for $B \in \mathcal{B}_i$ when $i > 1$. (See [7].)

Next, we will describe i-trees via the boundary operator ∂_i of Γ, which will show that $k_i > 0$ for $i \geqslant 0$. Since $\partial_i \neq 0$ for $i \in [0, d]$, both Γ_{i-1} and Γ_i are non-empty. Given a non-empty subset $T \subset \Gamma_i$, define ∂_T to be the $|\Gamma_{i-1}| \times |T|$ submatrix of ∂_i consisting of the columns of ∂_i indexed by T. Recall that if Γ is a connected finite graph of order n with the incidence matrix ∂_1, then $T \subset \Gamma_1$ is a spanning tree of Γ iff $|T| = \operatorname{rk} \partial_T = \operatorname{rk} \partial_1 = n-1$. (Refer to [2] for details.) More generally, we have the following useful fact.

Proposition 5. *Let Γ be a γ-complex of dimension d. Let $r_i = \operatorname{rk} \partial_i$ for $i \in [0, d]$. Then $\mathcal{B}_i$ is non-empty, and it is given by*

$$\mathcal{B}_i = \{\, B \subset \Gamma_i \mid |B| = \operatorname{rk} \partial_B = r_i \,\}\,. \tag{6}$$

Moreover, we have $r_i = |\Gamma_{i-1}| - r_{i-1}$, where $r_{-1} = 0$.

Proof. Suppose $B \in \mathcal{B}_i$. Since $\operatorname{Ker}\partial_B = \tilde{H}_i(\Gamma_B) = 0$, we have $\operatorname{rk}\partial_B = |B|$. Since $\Gamma_B^{(i-1)} = \Gamma^{(i-1)}$ and $\tilde{H}_{i-1}(\Gamma_B)$ is finite, we must have $\operatorname{rk}\partial_B = n_{i-1}$ the rank of $\operatorname{Ker}\partial_{i-1}$. However, $\tilde{H}_{i-1}(\Gamma) = 0$ implies $r_i = n_{i-1}$, and we have $|B| = \operatorname{rk}\partial_B = r_i$. The inclusion of the right-hand side of (6) in $\mathcal{B}_i$ is proved similarly. The second statement follows from $n_{i-1} = |\Gamma_{i-1}| - r_{i-1}$. □

Remarks 1. In matroid theoretic terms, $\mathcal{B}_i$ is the set of all bases of a matroid whose ground set is Γ_i and the independent sets are the subsets $I \subset \Gamma_i$ such that $\operatorname{Ker}\partial_I = 0$ or $I = \emptyset$. (Refer to [11] for the definition of a matroid.)
2. If Γ is also acyclic, then there is exactly one d-tree, namely $B = \Gamma_d$. Since $\operatorname{Ker}\partial_d = \tilde{H}_d(\Gamma) = 0$, the only base of the matroid just mentioned is Γ_d. In this case, it also follows that $k_d = 1$ because $\tilde{H}_{d-1}(\Gamma_B) = \tilde{H}_{d-1}(\Gamma) = 0$.
3. If X is a cell complex satisfying $(\gamma 2)$ but $r_i = 0$ for some i, then X has no i-tree. Indeed, for any non-empty subset $S \subset \Gamma_i$, we would have $\tilde{H}_i(\Gamma_S) = \mathbb{Z}^{|S|} \neq 0$.

The following theorem will play an essential role in Section 4. Given non-empty subsets $S \subset \Gamma_{i-1}$ and $T \subset \Gamma_i$, let $\partial_{S,T}$ be the $|S| \times |T|$ submatrix of ∂_i whose rows and columns are indexed by S and T, respectively. Denote $\bar{S} = \Gamma_{i-1} \setminus S$.

Theorem 6. *Let Γ be a γ-complex of dimension d. Let $r_i = \operatorname{rk}\partial_i$ for $i \in [0,d]$. Then the set of all $r_i \times r_i$ non-singular submatrices of ∂_i is given by*

$$\mathcal{B}(\partial_i) := \{\, \partial_{\bar{A},B} \mid A \in \mathcal{B}_{i-1} \text{ and } B \in \mathcal{B}_i \,\}\,.$$

Moreover, we have $|\det\partial_{\bar{A},B}| = w(A)w(B)$ for $\partial_{\bar{A},B} \in \mathcal{B}(\partial_i)$.

Proof. Let $S \subset \Gamma_{i-1}$ with $|S| = r_{i-1}$ and let $T \subset \Gamma_i$ with $|T| = r_i$. Then $\partial_{\bar{S},T}$ is a square submatrix of ∂_i of order r_i by Prop. 5. First, we will show that $\partial_{\bar{S},T}$ is singular if $S \notin \mathcal{B}_{i-1}$ or $T \notin \mathcal{B}_i$. Regard $\partial_{\bar{S},T}$ as the top boundary operator for the relative complex (Γ_T, Γ_S). Note that $\tilde{H}_i(\Gamma_T) = \operatorname{Ker}\partial_T$, $\tilde{H}_i(\Gamma_T,\Gamma_S) = \operatorname{Ker}\partial_{\bar{S},T}$, and $\tilde{H}_{i-1}(\Gamma_S) = \operatorname{Ker}\partial_S$. Since $\tilde{H}_i(\Gamma_S) = 0$, we obtain the following exact sequence from the long exact homology sequence of the pair (Γ_T, Γ_S):

$$0 \to \operatorname{Ker}\partial_T \to \operatorname{Ker}\partial_{\bar{S},T} \to \operatorname{Ker}\partial_S \to \tilde{H}_{i-1}(\Gamma_T)\,.$$

If $T \notin \mathcal{B}_i$, then $\operatorname{Ker}\partial_T \neq 0$ by Remark 1 above. Hence, we have $\operatorname{Ker}\partial_{\bar{S},T} \neq 0$. Similarly, if $S \notin \mathcal{B}_{i-1}$, then $\operatorname{rk}(\operatorname{Ker}\partial_S) \neq 0$. If $T \notin \mathcal{B}_i$, we are done. If $T \in \mathcal{B}_i$, then $\operatorname{Ker}\partial_T = 0$ and $\tilde{H}_{i-1}(\Gamma_T)$ is finite. Therefore, it is clear that $\operatorname{Ker}\partial_{\bar{S},T} \neq 0$.

Now we proceed to prove the second statement, which will also complete the proof of the first statement. Consider the following portion of the long exact homology sequence of the pair (Γ_B, Γ_A) with $A \in \mathcal{B}_{i-1}$ and $B \in \mathcal{B}_i$:

$$\tilde{H}_{i-1}(\Gamma_A) \to \tilde{H}_{i-1}(\Gamma_B) \to \tilde{H}_{i-1}(\Gamma_B,\Gamma_A) \to \tilde{H}_{i-2}(\Gamma_A) \to \tilde{H}_{i-2}(\Gamma_B)\,.$$

Since $\tilde{H}_{i-1}(\Gamma_A) = \tilde{H}_{i-2}(\Gamma_B) = 0$, it follows that

$$|\tilde{H}_{i-1}(\Gamma_B,\Gamma_A)| = |\tilde{H}_{i-2}(\Gamma_A)| \cdot |\tilde{H}_{i-1}(\Gamma_B)| = w(A)w(B)\,.$$

Note that $C_j(\Gamma_B,\Gamma_A) = \mathbb{Z}^{r_i}$ if $j = i-1$, and 0 if $j < i-1$. Therefore, we have $|\tilde{H}_{i-1}(\Gamma_B,\Gamma_A)| = |\mathbb{Z}^{r_i}/\operatorname{Im}\partial_{\bar{A},B}| = |\det\partial_{\bar{A},B}|$. □

4 Main Results

The following proposition is a generalization of Temperley's tree-number formula (5) for γ-complexes.

Proposition 7. *Let Γ be a γ-complex of dimension d, and let Δ_i be its combinatorial Laplacians for $i \in [-1, d]$. Then*

(1) $\det \Delta_{-1} = k_0$*,*

(2) $\det \Delta_i = k_{i-1} k_i^2 k_{i+1}$ *for* $i \in [0, d-1]$*, and*

(3) $\det \Delta_d = k_{d-1}$ *if* Γ *is acyclic, and* 0 *otherwise.*

Proof. (1) In Section 2, we noted that $\Delta_{-1} = L_{-1} : \mathbb{Z} \to \mathbb{Z}$ is a multiplication by $|\Gamma_0|$. In Section 3, we also saw that $k_0 = |\Gamma_0|$. Hence $\det \Delta_{-1} = k_0$.
(2) Note that we have $\operatorname{rk} \partial_i \partial_i^t = \operatorname{rk} \partial_i > 0$ for $i \in [0, d]$. Therefore, $\partial_i \partial_i^t$ has non-zero eigenvalues. Let π_i denote the product of all non-zero eigenvalues of $\partial_i \partial_i^t$. By Lemma 2 and Theorem 6, we have

$$\pi_i = \sum_{\substack{A \in \mathcal{B}_{i-1} \\ B \in \mathcal{B}_i}} (\det \partial_{\bar{A}, B})^2 = \sum_{\substack{A \in \mathcal{B}_{i-1} \\ B \in \mathcal{B}_i}} w(A)^2 w(B)^2 = k_{i-1} k_i \,.$$

Now recall that $\partial_i^t \partial_i$ and $\partial_i \partial_i^t$ have the same multiset of non-zero eigenvalues. Therefore, for $i \in [0, d-1]$, Lemma 1 and Lemma 3 imply

$$\det \Delta_i = \det(\partial_i^t \partial_i + \partial_{i+1} \partial_{i+1}^t) = \pi_i \pi_{i+1} = k_{i-1} k_i^2 k_{i+1} \,.$$

(3) If Γ is acyclic, then $k_d = 1$ because Γ_d is the only d-tree in Γ. Therefore,

$$\det \Delta_d = \det(\partial_d^t \partial_d) = \pi_d = k_{d-1} k_d = k_{d-1} \,.$$

If Γ is not acyclic, then $\operatorname{rk} \tilde{H}_d(\Gamma) > 0$ and $\det \Delta_d = 0$ by (4). □

As we discussed relations between Theorem 4 and the Matrix-Tree theorem in Section 2.3, we can make similar observations about Proposition 7 and the *Cellular Matrix-Tree Theorem* [4, Theorem 2.8] as follows. Proposition 7 may be derived from the Cellular Matrix-Tree Theorem for general APC complexes X, which states that $\pi_i = k_{i-1} k_i / |\tilde{H}_{i-2}(X)|^2$. Since $\tilde{H}_{i-2}(X) = 0$ if X is a γ-complex, this formula reduces to $\pi_i = k_{i-1} k_i$ as in the above proof. Also, if X is an acyclic γ-complex, then the following theorem, a logarithmic version of Proposition 7 for acyclic γ-complexes, shows that high-dimensional tree numbers of X can be obtained without using reduced Laplacians. Refer to Section 5 for examples.

Theorem 8. *Let Γ be an acyclic γ-complex of dimension d. Let $D(x) = \sum_{i=-1}^{d} \omega_i x^{i+1}$ and $K(x) = \sum_{i=0}^{d-1} \kappa_i x^i$, where $\omega_i = \log \det \Delta_i$ and $\kappa_i = \log k_i$. Then we have*

$$D(x) = (1 + x)^2 K(x) \,. \tag{7}$$

Proof. Since Γ is a γ-complex, we have $\mathcal{B}_i \neq \emptyset$ and $k_i \geqslant 1$ for $i \in [0,d]$. Hence $K(x)$ is well defined. By Proposition 7, we see that $\det \Delta_i \geqslant 1$ for $i \in [-1,d]$, and $D(x)$ is well defined. The rest of the proof is checking the following details. Proposition 7 (1) implies $\omega_{-1} = \kappa_0$. Proposition 7 (2) implies $\omega_i = \kappa_{i-1} + 2\kappa_i + \kappa_{i+1}$ for $i \in [0, d-1]$. In particular, $k_{-1} = 1$ implies $\omega_0 = 2\kappa_0 + \kappa_1$, which also follows from (5). Also, $k_d = 1$ because Γ is acyclic, and we have $\omega_{d-1} = \kappa_{d-2} + 2\kappa_{d-1}$. Finally, Proposition 7 (3) implies $\omega_d = \kappa_{d-1}$. The result follows. □

The requirement that Γ be a γ-complex is important in Theorem 8. For example, one can construct an acyclic cell complex consisting of one 0-cell, one 2-cell, and one 3-cell which is not a γ-complex because $\partial_1 = \partial_2 = 0$ in its cellular chain complex. In this case, $\Delta_1 = 0$ and the above theorem cannot be applied.

As a corollary to Theorem 8, we obtain the following interesting property of the combinatorial Laplacians for acyclic γ-complexes. Refer to [12] for further discussions.

Corollary 9. *Let Γ be an acyclic γ-complex of dimension d. Then*

$$\sum_{q=0}^{d} (-1)^{q+1} q \log\det \Delta_q = 0\,.$$

Proof. Letting $x = -1$ in (7), we obtain $\sum_{q=-1}^{d} (-1)^{q+1} \log\det \Delta_q = 0$. The result follows by differentiating (7), letting $x = -1$, and applying this formula. □

Remarks. Theorem 8 can be refined as follows. Let $\det \Delta_i = \prod_p p^{\epsilon_{p,i}}$ be the prime decomposition of the positive integer $\det \Delta_i$. Let $\mathcal{P}$ be the set of all distinct primes that appear in these prime decompositions. For each $p \in \mathcal{P}$, define

$$D_p(x) = \log p \sum_{i=-1}^{d} \epsilon_{p,i}\, x^{i+1}\,.$$

Then, $D(x) = \sum_{p\in\mathcal{P}} D_p(x)$. Also, we claim that each $D_p(x)$ is divisible by $(1+x)^2$. Indeed, suppose $D_p(x) \equiv \log p\,(a_p x + b_p) \bmod (1+x)^2$ for some integers a_p and b_p. Since $D(x) \equiv 0 \bmod (1+x)^2$, we must have $\sum_{p\in\mathcal{P}} \log p\,(a_p x + b_p) = 0$. From this equation, one can show that $a_p = 0$ and $b_p = 0$ for each $p \in \mathcal{P}$. See Section 5.3 for an example.

5 Examples

5.1 Standard simplexes

Let Σ be the standard simplex on n vertices (hence $\dim \Sigma = n-1$). Σ is acyclic and $|\Sigma_i| = \binom{n}{i+1}$ for $i \in [-1, n-1]$. If $[\sigma]$ denotes an oriented simplex for $\sigma \in \Sigma_i$, one can check that $\Delta_i[\sigma] = n[\sigma]$, which follows directly from the definition of the boundary operators

∂_i and ∂_{i+1} (and their transpose). Therefore, we have $\Delta_i = nI$, where I is the identity matrix of order $\binom{n}{i+1}$, and $\det \Delta_i = n^{\binom{n}{i+1}}$. Letting $\omega_i = \log_n \det \Delta_i = \binom{n}{i+1}$, we see that

$$D(x) = \sum_{i=-1}^{n-1} \omega_i x^{i+1} = \sum_{i=-1}^{n-1} \binom{n}{i+1} x^{i+1} = (1+x)^n \,.$$

By Theorem 8, we obtain

$$K(x) = \sum_{i=0}^{n-2} \kappa_i x^i = (1+x)^{n-2} \,,$$

where $\kappa_i = \log_n k_i = \binom{n-2}{i}$. Hence, we have $k_i = n^{\binom{n-2}{i}}$ for $i \in [0, n-2]$. This result was originally obtained by Kalai [7].

5.2 Cubical complexes

The n-cube Q_n ($n \geqslant 1$) is an n-dimensional cell complex that is the n-fold product $I \times \cdots \times I$, where I is the unit interval regarded as a cell complex with two 0-cells and one 1-cell. Hence Q_n is a cell complex of dimension n, and is the convex hull of the 2^n points in $\mathbb{R}^n$ whose coordinates are all 0 or 1. One can see that Q_n is acyclic by induction on n together with the fact that Q_{n-1} is a deformation retract of Q_n for $n \geqslant 2$.

In [4], Duval, Klivans, and Martin showed that the tree-numbers for Q_n are

$$k_i = \prod_{j=2}^{n} (2j)^{\binom{j-2}{i-1}\binom{n}{j}} \qquad (i \in [1, n-1]) \tag{8}$$

based on the spectra (the multisets of eigenvalues) of $\partial_i \partial_i^t$, which are, in turn, obtained from those of Δ_i's. In what follows, we will derive (8) directly from the spectra $Spec(\Delta_i)$ of Δ_i via Theorem 8. We will start with the following generating function for the eigenvalues of Δ_i's for Q_n ([4, Theorem 3.4]):

$$\sum_{i=0}^{\dim Q_n} \sum_{\lambda \in Spec(D_i)} t^i r^\lambda = (1 + r^2 + tr^2)^n = \sum_{k=0}^{n} t^k \binom{n}{k} r^{2k} (1+r^2)^{n-k} \,, \tag{9}$$

where $D_i = \Delta_i$ for $i \geqslant 1$ and $D_0 = \partial_1 \partial_1^t$. From (9), one can deduce that $\det \Delta_i = \prod_{j=1}^{n} (2j)^{\binom{n}{j}\binom{j}{i}}$ for $i \in [1, n]$, and that $\pi_{D_0} = k_0 k_1 = \prod_{j=1}^{n} (2j)^{\binom{n}{j}}$. By Theorem 4, we also obtain $\det \Delta_0 = 2^n \prod_{j=1}^{n} (2j)^{\binom{n}{j}}$. Now, let $\omega_i = \log_2 \det \Delta_i$, and let $\alpha_j = \binom{n}{j} \log_2 (2j)$. Then,

$$\omega_{-1} = n, \quad \omega_0 = n + \sum_{j=1}^{n} \alpha_j \,, \quad \text{and } \; \omega_i = \sum_{j=1}^{n} \binom{j}{i} \alpha_j \text{ for } i \in [1, n] \,,$$

and we have

$$
\begin{aligned}
D(x) &= \sum_{i=-1}^{n} \omega_i x^{i+1} \\
&= n + \Big(n + \sum_{j=1}^{n} \alpha_j\Big)x + \sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\binom{j}{i}\alpha_j\Big)x^{i+1} \\
&= n(1+x) + \sum_{i=0}^{n}\Big(\sum_{j=1}^{n}\binom{j}{i}\alpha_j\Big)x^{i+1} \\
&= n(1+x) + x\sum_{j=1}^{n}\alpha_j(1+x)^j \quad \text{(by interchanging the sums)} \\
&= n(1+x)^2 + x\sum_{j=2}^{n}\alpha_j(1+x)^j \quad \text{(because } \alpha_1 = n\text{)}\,.
\end{aligned}
$$

By Theorem 8, we obtain

$$
K(x) = \sum_{i=0}^{n-1} \kappa_i x^i = n + x\sum_{j=2}^{n}\alpha_j(1+x)^{j-2}\,,
$$

where $\kappa_i = \log_2 k_i$. By identifying the coefficients of x^i for $i \in [1, n-1]$, we obtain $\kappa_i = \sum_{j=2}^{n}\binom{j-2}{i-1}\alpha_j$, and $k_i = \prod_{j=2}^{n}(2j)^{\binom{j-2}{i-1}\binom{n}{j}}$ for $i \in [1, n-1]$.

5.3 A non-acyclic example

Let X be a 2-dimensional simplicial complex on the vertex set $E = \{a, b, c, d, e\}$ given by $X_{-1} = \{\emptyset\}$, $X_0 = \binom{E}{1}$, $X_1 = \binom{E}{2}$, and $X_2 = \binom{E}{3} \setminus \{\{a, b, e\}, \{c, d, e\}\}$. One can check that X is the independent set complex of a cycle matroid of the graph $K_4 \setminus \{\text{an edge}\}$. (Refer to [3] for general matroid complexes.) In particular, X has the homotopy type of a bouquet of two-dimensional spheres. Hence it is a γ-complex of dimension 2.

For convenience, assume that simplices in each X_i ($i = 0, 1, 2$) are ordered lexicographically, and that the alphabetical ordering of vertices in each simplex gives the positive orientation for the corresponding oriented simplex. For example, C_2 for X is isomorphic to $\mathbb{Z}^8$ generated by the oriented simplices $[abc]$, $[abd]$, $[acd]$, $[ace]$, $[ade]$, $[bcd]$, $[bce]$, and $[bde]$. X is not acyclic because $\tilde{H}_2(X) = \operatorname{Ker}\partial_2 \cong \mathbb{Z}^2$ whose generators can be chosen to be $z_1 = [abc]-[abd]+[ace]-[ade]-[bce]+[bde]$ and $z_2 = [acd]-[ace]+[ade]-[bcd]+[bce]-[bde]$. The simplices in z_1 (or z_2) form a hollow triangular bipyramid. As column vectors, we may write

$$
z_1 = [1, -1, 0, 1, -1, 0, -1, 1]^t \text{ and } z_2 = [0, 0, 1, -1, 1, -1, 1, -1]^t\,.
$$

Now, let $C_3 = \mathbb{Z}^2$, and define $\partial_3 : C_3 \to C_2$ to be a 8×2 matrix whose columns are z_1 and z_2. One easily checks that the chain complex of X together with C_3 and ∂_3 is acyclic.

With this new "augmented" acyclic complex, one can show

$$\det \Delta_{-1} = 5,\ \det \Delta_0 = 5^5,\ \det \Delta_1 = 2^2 5^8,\ \det \Delta_2 = 2^4 5^5,\ \text{ and } \det \Delta_3 = 2^2 5\,.$$

By Proposition 7, $\det \Delta_2$ and $\det \Delta_3$ are independent of the choices of z_1 and z_2 because k_2 depends only on ∂_2 and $k_3 = 1$. The primes that appear in these prime decompositions are $\mathcal{P} = \{2, 5\}$, and we see that

$$D_2(x) = \log 2\,(2x^2 + 4x^3 + 2x^4) = \log 2\,(1+x)^2 2x^2 \text{ and}$$

$$D_5(x) = \log 5\,(1 + 5x + 8x^2 + 5x^3 + x^4) = \log 5\,(1+x)^2(1 + 3x + x^2)\,.$$

By the remarks at the end of Section 4,

$$D(x) = D_2(x) + D_5(x) = (1+x)^2(\log 5 + \log 5^3\, x + \log 2^2 5\, x^2)\,,$$

from which we get $k_0 = 5$, $k_1 = 5^3$, and $k_2 = 2^2 5$ by Theorem 8.

Question. Can one characterize tree numbers of a matroid complex via known matroid invariants?

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