

On the number of orientations of random graphs with no directed cycles of a given length*

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Abstract

Let \vec{H} be an orientation of a graph H . Alon and Yuster proposed the problem of determining or estimating $D(n, m, \vec{H})$, the maximum number of \vec{H} -free orientations a graph with n vertices and m edges may have. We consider the maximum number of \vec{H} -free orientations of typical graphs $G(n, m)$ with n vertices and m edges. Suppose $\vec{H} = C_\ell^\circ$ is the directed cycle of length $\ell \geq 3$. We show that if $m \gg n^{1+1/(\ell-1)}$, then this maximum is $2^{o(m)}$, while if $m \ll n^{1+1/(\ell-1)}$, then it is $2^{(1-o(1))m}$.

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1 Introduction

Given a simple graph H , an *orientation* \vec{H} of H is obtained by assigning an ‘orientation’ or ‘direction’ to each of its edges. Such an \vec{H} is called *oriented graph*. Given a simple graph G and an oriented graph \vec{H} , let $D(G, \vec{H})$ be the number of orientations of G with no copy of \vec{H} and let $D(n, \vec{H}) = \max\{D(G, \vec{H}) : |V(G)| = n\}$. The problem of estimating $D(n, \vec{H})$ was posed by Paul Erdős [5, p. 45] in 1974. We investigate a similar problem, where we consider ‘typical’ graphs $G(n, m)$ with n vertices and m edges. Our terminology and notation are standard (see, e.g., [3, 8]). In particular, we use the notation $a \ll b$ as shorthand for the statement $a/b \rightarrow 0$ as $n \rightarrow \infty$.

Alon and Yuster [2] proved that, if n is sufficiently large, then $D(n, T_k) = 2^{\text{ex}(n, K_k)}$ for any tournament T_k with k vertices, i.e., any orientation of the complete graph K_k . Here we work with *directed cycles* C_ℓ° of length ℓ as opposed to tournaments and we consider random graphs. Let us state our main theorem in terms of the binomial random graph $G(n, p)$ (one may derive corresponding results for the $G(n, m)$ model from the $G(n, p)$ results below by standard means; the reader is referred to [8, Section 1.4] for this ‘equivalence’ of the models $G(n, p)$ and $G(n, m)$).

Theorem 1.1. *Fix $\ell \geq 3$. For every $0 < \gamma < 1$, there exist c and C for which the following hold for $G = G(n, p)$ with probability tending to 1 as $n \rightarrow \infty$.*

- (i) *If $p \geq Cn^{-1+1/(\ell-1)}$, then $D(G, C_\ell^\circ) \leq 2^{\gamma \binom{n}{2} p}$.*
- (ii) *If $n^{-2} \ll p \leq cn^{-1+1/(\ell-1)}$, then $D(G, C_\ell^\circ) \geq 2^{(1-\gamma) \binom{n}{2} p}$.*

We refer to cases (i) and (ii) above, respectively, as the *supercritical* and the *subcritical* cases.

Since our results are mostly asymptotic, in the next sections, when convenient, we freely and tacitly suppose that n is larger than a suitably large constant.

2 Preliminaries

Let $0 < p \leq 1$ be given and let $\vec{G} = (V, E)$ be an oriented graph. Let A and B be non-empty disjoint subsets of V . We write $e_{\vec{G}}(A, B)$ for the number of edges of \vec{G} oriented from A to B . Define the p -density of (A, B) as $d_{\vec{G}, p}(A, B) = e_{\vec{G}}(A, B)/p|A||B|$. For $0 < \varepsilon \leq 1$, the pair (A, B) is called $(\varepsilon, \vec{G}, p)$ -regular if, for all $X \subset A$ and $Y \subset B$ such that $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have $|d_{\vec{G}, p}(X, Y) - d_{\vec{G}, p}(A, B)| \leq \varepsilon$. We consider analogous definitions for graphs. In particular, if $G = (A, B; E)$ is a bipartite graph and (A, B) is an (ε, G, p) -regular pair, then we say that G is (ε, G, p) -regular. If the bipartite graph G is (ε, G, d) -regular, where $d = e_G(A, B)/|A||B|$, then we say that G is (ε) -regular. More explicitly, in an (ε) -regular bipartite graph $G = (A, B; E)$, for all $X \subset A$ and $Y \subset B$ such that $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$, we have

$$\left| e_G(X, Y) - \frac{e_G(A, B)}{|A||B|} |X||Y| \right| \leq \varepsilon \frac{e_G(A, B)}{|A||B|} |X||Y|. \quad (1)$$

Given an oriented n -vertex graph \vec{G} and a constant $\varepsilon > 0$, a partition $\{V_0, V_1, \dots, V_k\}$ of the vertex set of \vec{G} is called (ε, k) -*equitable* if $|V_0| \leq \varepsilon n$ and $|V_1| = \dots = |V_k|$. Furthermore, we say that an (ε, k) -equitable partition is $(\varepsilon, \vec{G}, p)$ -*regular* if, for $1 \leq i, j \leq k$, at most εk^2 pairs (V_i, V_j) are not $(\varepsilon, \vec{G}, p)$ -regular. Given $0 < \eta$, $p = p(n) \leq 1$, an n -vertex graph G is called (η, p) -*upper uniform* if, for all subsets $X, Y \subset V(G)$ with $X \cap Y = \emptyset$ such that $|X| \geq \eta n$ and $|Y| \geq \eta n$, we have $e_G(X, Y) \leq (1 + \eta)p|X||Y|$. Now we give a version of Szemerédi's Regularity Lemma for sparse oriented graphs (see, e.g., [10, 12], where the non-oriented case is considered).

Theorem 2.1. *For all $\varepsilon > 0$ and positive integer m , there exist $\eta > 0$ and $M \geq m$, such that, for any $0 < p \leq 1$ and any orientation \vec{G} of a sufficiently large (η, p) -upper uniform graph G , there exists an $(\varepsilon, \vec{G}, p)$ -regular partition of the vertex set of \vec{G} into $k + 1$ classes, where $m \leq k \leq M$.*

Let \vec{G} be an oriented n -vertex graph and let $W_1, \dots, W_\ell \subset V(\vec{G})$ be a family of pairwise disjoint sets with $|W_1| = \dots = |W_\ell| \geq \sigma n$. We say that (W_1, \dots, W_ℓ) induces an $(\varepsilon, \delta, \sigma)$ -*blow-up* of C_ℓ° in \vec{G} if, for $1 \leq i \leq \ell$, the pairs (W_i, W_{i+1}) are $(\varepsilon, \vec{G}, p)$ -regular with $d_{\vec{G}, p}(W_i, W_{i+1}) \geq \delta$, where we set $W_{\ell+1} = W_1$. Naturally, here, we are thinking of the oriented subgraph $\bigcup_{1 \leq i \leq \ell} \vec{G}[W_i, W_{i+1}]$ of \vec{G} , where $\vec{G}[W_i, W_{i+1}]$ has vertex set $W_i \cup W_{i+1}$ and contains precisely the oriented edges of \vec{G} starting in W_i and ending in W_{i+1} .

A *digraph* \vec{D} is a pair (V, E) where V is the vertex set of \vec{D} and E is a set of *arcs*, or *oriented edges*, that is, ordered pairs (u, v) , where $u, v \in V$ and $u \neq v$. Let $0 < \varepsilon, \delta, p = p(n) \leq 1$ be given and let \vec{G} be an n -vertex oriented graph. Let $\mathcal{P} = \{V_0, V_1, \dots, V_k\}$ be an (ε, k) -equitable partition of the vertex set of \vec{G} . The *coloured reduced digraph* $\vec{R} = \vec{R}(\vec{G}, \varepsilon, \delta, \mathcal{P})$ associated with \mathcal{P} has vertex set $[k] = \{1, \dots, k\}$ and, for every $\{i, j\} \in \binom{[k]}{2}$, the pair (i, j) is an arc in \vec{R} if and only if the pair (V_i, V_j) is $(\varepsilon, \vec{G}, p)$ -regular; moreover, an arc (i, j) in \vec{R} is coloured *grey* if $d_{\vec{G}, p}(V_i, V_j) < \delta$ and is coloured *blue* if $d_{\vec{G}, p}(V_i, V_j) \geq \delta$.

Given a graph H , a real number $\varepsilon > 0$, and positive integers n and m , we define $\mathcal{G}(H, n, m, \varepsilon)$ as the family of graphs G with vertex set $V = \bigcup_{x \in V(H)} V_x$, where the sets V_x have cardinality n each and are pairwise disjoint, and with edge set of the form $E = \bigcup_{\{x, y\} \in E(H)} E_{xy}$, where each E_{xy} is the edge set of an (ε) -regular graph with exactly m edges between V_x and V_y for all $\{x, y\} \in E(H)$. We denote by $\mathcal{F}(H, n, m, \varepsilon)$ the set of graphs in $\mathcal{G}(H, n, m, \varepsilon)$ that do *not* contain *transversal copies* of H , i.e., copies of H with exactly one vertex in each V_x . The following result is proved in [6] (for a weaker result, see [11]).

Theorem 2.2. *Let C_ℓ be a cycle with $\ell \geq 3$. For any $0 < \alpha < 1$, there exists $\varepsilon_0 > 0$ such that, for all $\mu > 0$, there exists $C \geq 1$ such that, for all $p \geq Cn^{-1+1/(\ell-1)}$ and $0 < \varepsilon < \varepsilon_0$, the random graph $G(n, p)$ contains a.a.s. no subgraphs in $\mathcal{F}(C_\ell, \tilde{n}, \lfloor \alpha \tilde{n}^2 p \rfloor, \varepsilon)$ for any $\tilde{n} \geq \mu n$.*

Now we give a definition and some results that are useful in the proof of Theorem 1.1(i).

Definition 2.3. Let $\sigma, \varepsilon > 0$ and $0 < p = p(n) \leq 1$ be given. We say that an n -vertex graph G satisfies $\text{EDGE}(\sigma, \varepsilon, p)$ if, for every $U \subset V(G)$ with $|U| \geq \sigma n$, we have $|e_G(U) - \binom{|U|}{2}p| < \varepsilon p \binom{|U|}{2}$.

Lemma 2.4. For every $0 < \delta < 1/2$, there exists $\varepsilon > 0$ such that, for every $\sigma > 0$, if \vec{G} is an orientation of an n -vertex graph G that satisfies $\text{EDGE}(\sigma, \varepsilon, p)$ for some $0 < p = p(n) \leq 1$, then, for every disjoint $X, Y \subset V(G)$ with $|X| = |Y| \geq \sigma n$, we have $e_G(X, Y) \geq (1 - 3\delta)p|X||Y|$. Furthermore, either $d_{\vec{G},p}(X, Y) \geq \delta$ or $d_{\vec{G},p}(Y, X) \geq \delta$.

Proof. Fix $0 < \delta < 1/2$ and put $\varepsilon = \min\{(1 - 2\delta)/3, \delta\}$. Fix $\sigma > 0$. Let \vec{G} be an orientation of an n -vertex graph G and consider $0 < p = p(n) \leq 1$. Suppose G satisfies $\text{EDGE}(\sigma, \varepsilon, p)$ and let X and Y be disjoint subsets of $V(G)$ with $|X| = |Y| \geq \sigma n$. Then

$$\begin{aligned} e_G(X, Y) &= e_G(X \cup Y) - e_G(X) - e_G(Y) \\ &> (1 - \varepsilon) \binom{|X| + |Y|}{2} p - (1 + \varepsilon) \binom{|X|}{2} p - (1 + \varepsilon) \binom{|Y|}{2} p \\ &= (1 - 3\varepsilon + 2\varepsilon/|X|)p|X||Y| > (1 - 3\varepsilon)p|X||Y|, \end{aligned} \tag{2}$$

where the first inequality follows from the definition of $\text{EDGE}(\sigma, \varepsilon, p)$. Thus, since $\varepsilon \leq \delta$, we deduce that $e_G(X, Y) \geq (1 - 3\delta)p|X||Y|$. Furthermore, using the fact $\varepsilon \leq (1 - 2\delta)/3$, we have $e_G(X, Y) \geq 2\delta p|X||Y|$. Therefore, we conclude that $d_{\vec{G},p}(X, Y) \geq \delta$ or $d_{\vec{G},p}(Y, X) \geq \delta$. \square

The following three simple facts will be useful in Section 3. For a proof of Fact 2.5, the reader is referred to [7]. Fact 2.6 follows from Chernoff bounds and Fact 2.7 follows from calculations similar to those in the proof of Lemma 2.4.

Fact 2.5. For all $0 < \varepsilon \leq 1/6$, there exists a constant $C > 0$ such that any (ε) -regular bipartite graph $B = (V_1, V_2; E)$ contains a spanning (2ε) -regular subgraph with exactly m edges, for all $C(|V_1| + |V_2|) \leq m \leq |E|$.

Fact 2.6. For every $0 < \sigma, \varepsilon < 1$, if $p = p(n) \gg 1/n$, then, a.a.s., $G(n, p)$ satisfies $\text{EDGE}(\sigma, \varepsilon, p)$.

Fact 2.7. For every $\eta > 0$, there exist $\varepsilon > 0$ and $\sigma > 0$ such that, if $0 < p \leq 1$ and G is an n -vertex graph satisfying $\text{EDGE}(\sigma, \varepsilon, p)$, then G is (η, p) -upper uniform.

3 Proof of Theorem 1.1

Let us briefly sketch the main idea of the proof of part (i). Given $p \geq Cn^{-1+1/(\ell-1)}$ and a graph $G = G(n, p)$, we will consider the set of $(\varepsilon, \vec{G}, p)$ -regular partitions of its C_ℓ° -free orientations. Since these partitions are into a number of parts bounded by $M = M(\varepsilon)$, there are at most M^n of them, and the total number of coloured reduced digraphs is bounded by a constant. It follows that, if G has more than $2^{\gamma \binom{n}{2} p}$ orientations that are

C_ℓ° -free, then there is one partition of $V(G)$ and one coloured reduced digraph \vec{R} that account for more than $2^{\frac{\gamma}{2}} \binom{n}{2}^p$ of these orientations of G .

Now we consider the number of ways we can orient the edges of G so that we obtain the reduced digraph \vec{R} . We will see that if there are only few pairs of clusters with blue arcs in both directions, then there are at most $2^{\frac{\gamma}{2}} \binom{n}{2}^p$ possible orientations. But if there are many pairs of clusters with blue arcs in both directions—which there must be for the total number of C_ℓ° -free orientations of G to be at least $2^{\gamma \binom{n}{2}^p}$ —then we can find a triple of clusters in \vec{R} which contains a directed blue 3-cycle plus one extra blue arc. We will prove that, since $p \geq Cn^{-1+1/(\ell-1)}$, the graph G has a.a.s. the property that the directed edges in \vec{G} between the clusters of any such triple must contain a copy of C_ℓ° , and this contradiction will imply the result.

The only step at which we used the probability bound of part (i) was the final step of finding a copy of C_ℓ° given the triple with a directed blue 3-cycle plus a blue arc in \vec{R} . That step fails if $p \leq cn^{-1+1/(\ell-1)}$ for some small constant c , and it does so for the simple reason that for such values of p , most edges of G are not contained in any copy of C_ℓ . We can fix a C_ℓ° -free orientation of those edges of G that *are* contained in copies of C_ℓ s, and then any orientation of the remaining edges is automatically C_ℓ° -free. Since almost all edges do not belong to C_ℓ s, part (ii) follows.

3.1 Two auxiliary lemmas

We now give two lemmas that concern orientations \vec{G} of n -vertex graphs G for which the following two conditions hold, for certain parameters σ , ε_E , $p = p(n)$, \tilde{n} , δ and ε_F , to be specified later:

- (H1) G satisfies $\text{EDGE}(\sigma, \varepsilon_E, p)$,
- (H2) G contains no member of $\mathcal{F}(C_\ell, \tilde{n}, [\delta \tilde{n}^2 p], \varepsilon_F)$.

In the first lemma (Lemma 3.1), we suppose that \vec{G} contains an ℓ -tuple of sets (W_1, \dots, W_ℓ) inducing an appropriate blow-up of C_ℓ° , and deduce that \vec{G} itself contains a C_ℓ° . In the second lemma (Lemma 3.2), we suppose that the regularity lemma has been applied to \vec{G} and that the associated coloured reduced digraph \vec{R} contains a certain subdigraph D° made up of blue arcs; we then conclude that \vec{G} contains a C_ℓ° , finding first a suitable ℓ -tuple of sets (W_1, \dots, W_ℓ) and then applying Lemma 3.1.

Lemma 3.1. *For every $0 < \delta$, $\varepsilon_F < 1/3$ and $\ell \geq 3$, there exists ε_E such that, for every $0 < \sigma < 1$, the following holds.*

Suppose \vec{G} is an orientation of G , where G is an n -vertex graph that satisfies both (H1) and (H2) above for some $\sigma n \leq \tilde{n} \leq n$ and $p = p(n) \gg 1/n$. If (W_1, \dots, W_ℓ) with $|W_1| = \dots = |W_\ell| = \tilde{n}$ induces an $(\varepsilon_R, \delta, \sigma)$ -blow-up of C_ℓ° , where $\varepsilon_R = \varepsilon_F(1 - 3\delta)/2$, then \vec{G} contains a copy of C_ℓ° .

Proof. Fix $0 < \delta, \varepsilon_F < 1/3$ and $\ell \geq 3$. Let ε_E be given by an application of Lemma 2.4 with parameter δ . Fix $0 < \sigma < 1$ and define $C_{(F.2.5)}$ as the constant obtained by an application of Fact 2.5 with parameter $\varepsilon_F/2$. Now suppose that G, \vec{G} and (W_1, \dots, W_ℓ) are as in the statement of the lemma and $p = p(n) \gg 1/n$.

For convenience, put $W_{\ell+1} = W_1$. Let $\vec{G}_\ell = \bigcup_{1 \leq i \leq \ell} \vec{G}[W_i, W_{i+1}]$ be the subdigraph of \vec{G} associated with the $(\varepsilon_R, \delta, \sigma)$ -blow-up of C_ℓ° induced by the sets W_i . Also, let G_ℓ be the underlying graph of \vec{G}_ℓ , that is, the graph obtained from \vec{G}_ℓ by ignoring the orientation of its edges.

Fix $i \in [\ell]$. Since $p \gg 1/n$ we have $\lfloor \delta |W_i| |W_{i+1}| p \rfloor \geq C_{(F.2.5)}(|W_i| + |W_{i+1}|)$ and, by the definition of $(\varepsilon_R, \delta, \sigma)$ -blow-up of C_ℓ° , we have $\lfloor \delta |W_i| |W_{i+1}| p \rfloor \leq e_{G_\ell}(W_i, W_{i+1})$. Furthermore, since G satisfies $\text{EDGE}(\sigma, \varepsilon_E, p)$, by Lemma 2.4, we have $e_{G_\ell}(W_i, W_{i+1}) \geq (1 - 3\delta)p|W_i||W_{i+1}|$. Thus, since (W_i, W_{i+1}) is $(\varepsilon_R, G_\ell, p)$ -regular, we conclude that the induced bipartite subgraph $G_\ell[W_i \cup W_{i+1}]$ of G_ℓ is $(\varepsilon_F/2)$ -regular. Therefore, we can apply Fact 2.5 on the bipartite graphs $G_\ell[W_i \cup W_{i+1}]$ for $1 \leq i \leq \ell$, obtaining spanning (ε_F) -regular subgraphs $G_\ell^{\text{span}}[W_i \cup W_{i+1}]$ of $G_\ell[W_i \cup W_{i+1}]$ with exactly $\lfloor \delta |W_i| |W_{i+1}| p \rfloor$ edges.

Let $J \subset G_\ell$ be the graph with vertex set $\bigcup_{i=1}^\ell W_i$ and edge set $\bigcup_{i=1}^\ell E(G_\ell^{\text{span}}[W_i \cup W_{i+1}])$. The graph J is clearly a graph of the family $\mathcal{G}(C_\ell, |W_1|, \lfloor \delta |W_1|^2 p \rfloor, \varepsilon_F)$. Recalling that G contains no subgraphs from $\mathcal{F}(C_\ell, |W_1|, \lfloor \delta |W_1|^2 p \rfloor, \varepsilon_F)$, we deduce that $J \subset G_\ell$ contains a C_ℓ . This C_ℓ corresponds to a C_ℓ° in $\vec{G}_\ell \subset \vec{G}$, and the proof is complete. \square

Let D_3° be obtained from the cycle C_3° of length 3 by the addition of an extra arc, forming a directed cycle of length 2 with some arc of C_3° . As in the previous lemma, we shall consider in our next lemma an orientation \vec{G} of an n -vertex graph G that satisfies both (H1) and (H2). However, in Lemma 3.2 below, our hypothesis will be that \vec{G} has been regularized with some suitably small parameter ε_R and that the associated coloured reduced digraph $\vec{R} = \vec{R}(\vec{G}, \varepsilon_R, \delta, \mathcal{P})$ contains a blue D_3° . In the lemma below, we are interested in *odd* cycles C_ℓ° . (The case in which ℓ is even is similar and simpler. For even ℓ , it is enough that \vec{R} should contain a blue directed 2-cycle C_2° ; we omit the details.)

Lemma 3.2. *Fix $z \geq 1$ and set $\ell = 2z + 1$. For every positive $\delta < 1/3$ and $\varepsilon_F < 1/z$ and $M \geq 1$, there exist ε_E and σ such that the following holds. Suppose G is an n -vertex graph that satisfies (H1) and (H2) for some \tilde{n} and $p = p(n) \gg 1/n$. Furthermore, suppose that \vec{G} is an orientation of G and that $\mathcal{P} = \{V_0, V_1, \dots, V_k\}$ is an $(\varepsilon_R, \vec{G}, p)$ -regular (ε_R, k) -equitable partition of the vertex set of \vec{G} , where $\varepsilon_R = \min\{\varepsilon_F(1 - 3(\delta/2))/2, \delta/2\}$ and $k \leq M$. If $\tilde{n} = \lfloor |V_1|/z \rfloor$ and the coloured reduced digraph $\vec{R} = \vec{R}(\vec{G}, \varepsilon_R, \delta, \mathcal{P})$ associated with \mathcal{P} contains a blue copy of D_3° , then \vec{G} contains a copy of $C_\ell^\circ = C_{2z+1}^\circ$.*

Proof. Fix $z \geq 1$, $0 < \delta < 1/3$, $0 < \varepsilon_F < 1/z$ and $M \geq 1$. Set $\ell = 2z + 1$. Let $\varepsilon_E = \varepsilon_{E(L.3.1)}$ be given by an application of Lemma 3.1 with parameters $\delta_{(L.3.1)} = \delta/2$, $\varepsilon_{F(L.3.1)} = z\varepsilon_F$ and ℓ . In order to apply Lemma 3.1, put $\varepsilon_{R(L.3.1)} = \varepsilon_{F(L.3.1)}(1 - 3\delta_{(L.3.1)})/2$. Let $\varepsilon_R = \min\{\varepsilon_{R(L.3.1)}/z, \delta/2\}$ and set $\sigma = (1 - \varepsilon_R)/zM$ (we shall apply Lemma 3.1 with this σ). We have now defined all the required constants.

We now suppose that G , \vec{G} , $\mathcal{P} = \{V_0, \dots, V_k\}$, and $\vec{R} = \vec{R}(\vec{G}, \varepsilon_R, \delta, \mathcal{P})$ are as in the statement of the lemma and $p = p(n) \gg 1/n$. We prove that \vec{G} must contain a C_ℓ° .

Adjusting the notation suitably, assume that (v_1, v_2) , (v_2, v_1) , (v_2, v_3) and (v_3, v_1) are the arcs of a blue D_3° in \vec{R} . Thus, we know that the corresponding pairs (V_i, V_j) are $(\varepsilon_R, \vec{G}, p)$ -regular with density at least δ in \vec{G} .

For $z = 1$, the result follows directly from Lemma 3.1. Suppose $z \geq 2$. For $i = 1, 2$ and 3 , consider pairwise disjoint subsets V_i^1, \dots, V_i^z of V_i with $|V_i^1| = \dots = |V_i^z| = \lfloor |V_i|/z \rfloor$. To apply Lemma 3.1, we shall prove that the sequence $(V_1^1, V_2^1, \dots, V_1^z, V_2^z, V_3^1)$ induces an $(\varepsilon_{R(L.3.1)}, \delta_{(L.3.1)}, \sigma)$ -blow-up of C_ℓ° in \vec{G} . If $u, v \in [z]$ and $X \subset V_1^u$ and $Y \subset V_2^v$ have cardinalities $|X| \geq \varepsilon_{R(L.3.1)}|V_1^u| \geq \varepsilon_R|V_1|$ and $|Y| \geq \varepsilon_{R(L.3.1)}|V_2^v| \geq \varepsilon_R|V_2|$, then, since (V_1, V_2) is $(\varepsilon_R, \vec{G}, p)$ -regular, we have that $|d_{\vec{G},p}(X, Y) - d_{\vec{G},p}(V_1^u, V_2^v)|$ is at most

$$|d_{\vec{G},p}(X, Y) - d_{\vec{G},p}(V_1, V_2)| + |d_{\vec{G},p}(V_1, V_2) - d_{\vec{G},p}(V_1^u, V_2^v)| \leq 2\varepsilon_R \leq z\varepsilon_R \leq \varepsilon_{R(L.3.1)}. \quad (3)$$

Applying the same argument we have $|d_{\vec{G},p}(Y, X) - d_{\vec{G},p}(V_2^v, V_1^u)| \leq \varepsilon_{R(L.3.1)}$. Therefore, for all $u, v \in [z]$, the pairs (V_1^u, V_2^v) and (V_2^u, V_1^v) are $(\varepsilon_{R(L.3.1)}, \vec{G}, p)$ -regular. Similarly, for all $u, v \in [z]$, the pairs (V_3^1, V_1^u) and (V_2^v, V_3^1) are $(\varepsilon_{R(L.3.1)}, \vec{G}, p)$ -regular.

As $d_{\vec{G},p}(V_i, V_j) \geq \delta$ and (V_i, V_j) is $(\varepsilon_R, \vec{G}, p)$ -regular for every pair (i, j) in the set $\{(1, 2), (2, 1), (2, 3), (3, 1)\}$, we have, for all $u, v \in [z]$, that

$$d_{\vec{G},p}(V_1^u, V_2^v), d_{\vec{G},p}(V_2^v, V_1^u), d_{\vec{G},p}(V_2^v, V_3^1), d_{\vec{G},p}(V_3^1, V_1^u) \geq \delta - \varepsilon_R \geq \delta/2 = \delta_{(L.3.1)}, \quad (4)$$

where the last inequality follows from $\varepsilon_R \leq \delta/2$. Therefore, $V_1^1, V_2^1, \dots, V_1^z, V_2^z, V_3^1$ does induce an $(\varepsilon_{R(L.3.1)}, \delta_{(L.3.1)}, \sigma)$ -blow-up of $C_{2z+1}^\circ = C_\ell^\circ$ in \vec{G} . Recalling that (H1) and (H2) hold for G , we see that G satisfies $\text{EDGE}(\sigma, \varepsilon_{E(L.3.1)}, p)$ and G contains no member of

$$\mathcal{F}(C_\ell, \lfloor |V_1|/z \rfloor, \lfloor \delta_{(L.3.1)} \lfloor |V_1|/z \rfloor^2 p \rfloor, \varepsilon_{F(L.3.1)}).$$

Thus, all conditions of Lemma 3.1 are satisfied and we conclude that \vec{G} contains a copy of C_ℓ° . \square

3.2 Main lemma for the supercritical case

The next result (Lemma 3.3) is the main lemma in the proof of Theorem 1.1(i). As in Lemmas 3.1 and 3.2 in Section 3.1, in Lemma 3.3 below we consider an n -vertex graph G that satisfies (H1) and (H2) given at the beginning of Section 3.1, but suppose that (H2) holds for a range of values of \tilde{n} , namely, for all $\tilde{n} \geq cn$ for a certain constant $c > 0$. The conclusion in Lemma 3.3 is that, for any given $\gamma > 0$, with the constants in (H1) and (H2) chosen suitably, such a graph G has at most $2^{\gamma \binom{n}{2} p}$ orientations avoiding C_ℓ° .

Lemma 3.3. *Fix $\ell \geq 3$. For every $0 < \gamma < 1$, there exists $\delta > 0$ such that, for every $0 < \varepsilon_F < 1/\ell$, there exist ε_E and $\sigma > 0$ and an integer M such that, if $1/n \ll p = p(n) \leq 1$, then the following holds. Let G be an n -vertex graph satisfying (H1) and (H2) for all $\tilde{n} \geq \lfloor n/\ell M \rfloor$. Then we have*

$$D(G, C_\ell^\circ) \leq 2^{\gamma \binom{n}{2} p}. \quad (5)$$

Proof. We assume that ℓ is odd, since the even case is similar and easier. Suppose $\ell = 2z + 1$. Fix $0 < \gamma < 1$. Let $\delta > 0$ satisfy $4\delta + 3H(\delta) < \gamma/32$ and $\delta < 1/3$, where H is the binary entropy function, that is, $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ for all $0 < x < 1$ and $H(0) = H(1) = 0$. Fix $\varepsilon_{F(L.3.2)} = \varepsilon_F < 1/\ell$ and put $\varepsilon_R = \min\{\varepsilon_F(1 - 3(\delta/2))/2, \delta/2, (\gamma/64)^2\}$.

Let η_1 and M be the constants obtained by an application of Theorem 2.1 with parameters ε_R and $m = \lceil 1/\varepsilon_R \rceil$. Put $\eta = \min\{\eta_1, 1/2M, \varepsilon_R/2\}$ and let $\varepsilon_{E(L.3.2)}$ and $\sigma_{E(L.3.2)}$ be obtained by an application of Lemma 3.2 with parameters z, δ, ε_F and M .

Now let $\varepsilon_{E(F.2.7)}$ and $\sigma_{E(F.2.7)}$ be obtained by an application of Fact 2.7 with parameter η . Set $\sigma = \min\{\sigma_{E(L.3.2)}, \sigma_{E(F.2.7)}, \eta\}$ and let $\varepsilon_{E(L.2.4)}$ be obtained by an application of Lemma 2.4 with parameter δ . Set $\varepsilon_E = \min\{\varepsilon_{E(L.2.4)}, \varepsilon_{E(F.2.7)}, \varepsilon_{E(L.3.2)}\}$. Suppose $1/n \ll p = p(n) \leq 1$ and n is large enough.

Now let G be an n -vertex graph that satisfies (H1) and (H2) for all $\tilde{n} \geq \lfloor n/\ell M \rfloor$. We shall show that $D(G, C_\ell^\circ) \leq 2^{\gamma \binom{n}{2} p}$. For convenience, let us state our hypotheses on G explicitly here: we suppose that G satisfies $\text{EDGE}(\sigma, \varepsilon_E, p)$ and suppose that G contains no graph in $\mathcal{F}(C_\ell, \tilde{n}, \lfloor (\delta/2)\tilde{n}^2 p \rfloor, z\varepsilon_F)$ as a subgraph for any $\tilde{n} \geq \lfloor n/\ell M \rfloor$.

Denote by \mathcal{R} the set of digraphs $\vec{R} = \vec{R}(\vec{G}, \varepsilon_R, \delta, \mathcal{P})$ whose arcs are coloured grey and blue such that there exists a C_ℓ° -free orientation \vec{G} of G where \vec{R} is the coloured reduced digraph of an $(\varepsilon_R, \vec{G}, p)$ -regular partition \mathcal{P} of \vec{G} with $k+1$ classes with $m \leq k \leq M$. Given $\vec{R} \in \mathcal{R}$, we denote by $D_{\vec{R}}(G, C_\ell^\circ)$ the number of C_ℓ° -free orientations \vec{G} of G satisfying the extra condition that \vec{R} should be a coloured reduced digraph of an $(\varepsilon_R, \vec{G}, p)$ -regular partition of \vec{G} .

By Theorem 2.1, for every orientation \vec{G} of G , there exists an $(\varepsilon_R, \vec{G}, p)$ -regular partition of G with $k+1$ classes, where $m \leq k \leq M$. Note that, since there are 3^2 possible kinds of connections between two vertices in a coloured reduced digraph, there exist at most $9^{\binom{i}{2}}$ different coloured reduced graphs with i vertices. Thus,

$$D(G, C_\ell^\circ) \leq \sum_{m \leq i \leq M} \sum_{\vec{R}: |V(\vec{R})|=i} D_{\vec{R}}(G, C_\ell^\circ) \leq M 9^{\binom{M}{2}} \max_{\vec{R}} D_{\vec{R}}(G, C_\ell^\circ). \quad (6)$$

We now make the following claim.

Claim 3.4. *For any coloured reduced digraph \vec{R} , we have*

$$D_{\vec{R}}(G, C_\ell^\circ) \leq 2^{\frac{\gamma}{2} \binom{n}{2} p}. \quad (7)$$

Given that M is a constant and $p \gg 1/n$, inequalities (6) and (7) imply (5) for all large enough n , and the proof of Lemma 3.3 is complete. It now remains to prove Claim 3.4. \square

Proof of Claim 3.4. Fix a coloured reduced digraph \vec{R} . We first claim that \vec{R} does not contain too many blue directed 2-cycles C_2° . More precisely, our claim is as follows. Let R_2 be the graph on $[k] = V(\vec{R})$ where $\{i, j\}$ is an edge of R_2 if and only if both arcs (i, j) and (j, i) belong to \vec{R} and are blue. We claim that

$$e(R_2) = |E(R_2)| \leq \varepsilon_R^{1/2} k^2. \quad (8)$$

Suppose for a contradiction that (8) fails. Then there is a vertex a in R_2 with degree $d_{R_2}(a)$ larger than $2\varepsilon_R^{1/2}k$ in R_2 . Therefore $\binom{d_{R_2}(a)}{2} > \varepsilon_R k^2$, and it follows from the fact that \vec{R} is the coloured reduced digraph of an $(\varepsilon_R, \vec{G}, p)$ -regular partition $\mathcal{P} = \{V_0, \dots, V_k\}$ that there is a pair (b, c) with both b and c adjacent to a in R_2 with both (V_b, V_c) and (V_c, V_b) $(\varepsilon_R, \vec{G}, p)$ -regular pairs. By Lemma 2.4, we know that either $d_{\vec{G}}(V_b, V_c) \geq \delta$ or $d_{\vec{G}}(V_c, V_b) \geq \delta$, and hence at least one of (b, c) and (c, b) is blue in \vec{R} . It thus follows that \vec{R} contains a blue copy of D_3° with vertex set $\{a, b, c\}$. It now suffices to apply Lemma 3.2 to conclude that \vec{G} contains a copy C_ℓ° , contradicting that \vec{G} is a C_ℓ° -free orientation of G . This completes the proof of (8). We now move on to the proof of (7).

The digraph \vec{R} can be associated with at most $(M+1)^n$ partitions of $V(G)$ into at most $M+1$ parts. Let $\mathcal{P} = \{V_0, V_1, \dots, V_k\}$ be one of those partitions, where $m \leq k \leq M$. To estimate $D_{\vec{R}}(G, C_\ell^\circ)$, we shall estimate the number

$$D(\mathcal{P}, \vec{R}) \quad (9)$$

of C_ℓ° -free orientations \vec{G} of G that admit \mathcal{P} as an $(\varepsilon_R, \vec{G}, p)$ -regular partition and, furthermore, \vec{R} is the coloured reduced digraph $\vec{R}(\vec{G}, \varepsilon_R, \delta, \mathcal{P})$. We shall then have

$$D_{\vec{R}}(G, C_\ell^\circ) = \sum_{\mathcal{P}} D(\mathcal{P}, \vec{R}) \leq (M+1)^n D(\mathcal{P}, \vec{R}). \quad (10)$$

By property EDGE(σ, ε_E, p), there exist at most

$$(1 + \varepsilon_E)p \binom{n/k}{2} k \leq 2\varepsilon_R n^2 p \quad (11)$$

edges of G with both endpoints in the same V_i for some $1 \leq i \leq k$.

Since $\sigma \leq \eta \leq \varepsilon_R/2$ and $|V_0| \leq \varepsilon_R n$, there is $U_0 \subset V(G)$ that contains V_0 and with $|U_0| = \lfloor \varepsilon_R n \rfloor \geq \eta n \geq \sigma n$. Since G satisfies EDGE(σ, ε_E, p), by Fact 2.7, we know that G is (η, p) -upper uniform. Thus, using (η, p) -upper uniformity, we know that the number of edges with only one endpoint in U_0 is at most $(1 + \eta)\varepsilon_R n^2 p$. For the edges with both endpoints in U_0 , we use the fact G satisfies EDGE(σ, ε_E, p). Recalling that $|U_0| \geq \sigma n$, we may conclude that the number of edges inside U_0 is at most $(1 + \varepsilon_E)\varepsilon_R^2 n^2 p/2$. Therefore, the number of edges with at least one endpoint in $V_0 \subset U_0$ is at most

$$3\varepsilon_R n^2 p. \quad (12)$$

Let \vec{G} be a C_ℓ° -free orientation of G . Note that the orientation of the at most $5\varepsilon_R n^2 p$ edges counted in (11) and (12) does not affect whether or not \vec{G} should be counted in $D(\mathcal{P}, \vec{R})$ (recall the definition of $D(\mathcal{P}, \vec{R})$, given near (9)). Thus, the edges in (11) and (12) contribute

$$2^{5\varepsilon_R n^2 p} \quad (13)$$

to our count. Now, the structure of \vec{R} does impose restrictions on how the remaining edges of G may be oriented in any \vec{G} that is counted in $D(\mathcal{P}, \vec{R})$. We proceed to analyse

those restrictions. By the (η, p) -upper uniformity of G and the fact that $|V_1| = \dots = |V_k| \geq n/2M \geq \eta n$, we know that

$$e_G(V_i, V_j) \leq (1 + \eta) \left(\frac{n}{k}\right)^2 p \leq 2 \left(\frac{n}{k}\right)^2 p, \quad (14)$$

for all $1 \leq i < j \leq k$. Let us now count the number of ways those edges between V_i and V_j may be oriented in the \vec{G} counted in $D(\mathcal{P}, \vec{R})$. To do so, let us observe that, clearly, any given pair $\{i, j\}$ with $1 \leq i < j \leq k$ satisfies one of the following:

- (i) At least one of (i, j) and (j, i) is not an arc of \vec{R} .
- (ii) Both (i, j) and (j, i) are arcs of \vec{R} and one of them is grey and the other one is blue.
- (iii) Both (i, j) and (j, i) are arcs of \vec{R} and both of them are blue.
- (iv) Both (i, j) and (j, i) are arcs of \vec{R} and both of them are grey.

Note that, because of Lemma 2.4, possibility (iv) is excluded. We now estimate in how many ways the edges in (14) may be oriented, according to the ‘type’ of the pair $\{i, j\}$.

Case 1. *The pair $\{i, j\}$ is of type (i).* The number of orientations of the edges in such pairs $\{i, j\}$ is at most $2^{\sum_{\{i,j\}} e_G(V_i, V_j)}$, where the sum ranges over all pairs $\{i, j\}$ of this type. Note that the number of such $\{i, j\}$ is at most $\varepsilon_R k^2$. Hence, recalling (14), we see that

$$2^{\sum_{(i,j)} e_G(V_i, V_j)} \leq 2^{2(n/k)^2 p \varepsilon_R k^2} = 2^{2\varepsilon_R n^2 p}. \quad (15)$$

Case 2. *The pair $\{i, j\}$ is of type (ii).* Recalling (14), we see that the number of possible orientations of the edges in such a pair (V_i, V_j) is at most

$$\begin{aligned} 2^{\sum_{i=0}^{\lfloor \delta(n/k)^2 p \rfloor} \binom{(n/k)^2 p(1+\eta)}{i}} &\leq 2\delta \left(\frac{n}{k}\right)^2 p \binom{(n/k)^2 p(1+\eta)}{\delta(n/k)^2 p(1+\eta)} \\ &\leq 2\delta \left(\frac{n}{k}\right)^2 p 2^{H(\delta)(n/k)^2 p(1+\eta)} \leq 2\delta \left(\frac{n}{k}\right)^2 p 2^{2H(\delta)(n/k)^2 p}. \end{aligned} \quad (16)$$

The first inequality follows from the fact that $\delta(1+\eta) < 1/2$, and the second follows from the estimate $\binom{x}{\beta x} < 2^{H(\beta)x}$, valid for all $0 < \beta < 1$ (see [9], Corollary 22.9). We now observe that the total contribution of the edges induced by pairs $\{V_i, V_j\}$ with $\{i, j\}$ of type (ii) is at most

$$\left(2\delta \left(\frac{n}{k}\right)^2 p\right)^{k^2} 2^{2H(\delta)(n/k)^2 p k^2} \leq n^{2k^2} 2^{2H(\delta)n^2 p} \leq 2^{3H(\delta)n^2 p}, \quad (17)$$

where we used that $p \gg 1/n$ to absorb n^{2k^2} into the exponential term.

Case 3. *The pair $\{i, j\}$ is of type (iii).* Recall that in (8) we proved that the number of pairs $\{i, j\}$ of type (iii) is at most $\varepsilon_R^{1/2} k^2$. Calculations similar to the calculations in

Case 1 show that the total contribution of the edges within pairs $\{i, j\}$ of type (iii) is at most

$$2^{2\varepsilon_R^{1/2}n^2p}. \quad (18)$$

We now note that, by (13), (15), (17) and (18) and the choices of δ and ε_R , we have

$$\log_2 D(\mathcal{P}, \vec{R}) \leq 5\varepsilon_R n^2 p + 2\varepsilon_R n^2 p + 3H(\delta)n^2 p + 2\varepsilon_R^{1/2}n^2 p \leq \frac{\gamma}{4} \binom{n}{2} p. \quad (19)$$

From (10), (19) and the fact that $p \gg 1/n$, we deduce that

$$D_{\vec{R}}(G, C_\ell^\circ) \leq 2^{n \log_2(M+1) + \frac{\gamma}{4} \binom{n}{2} p} \leq 2^{\frac{\gamma}{2} \binom{n}{2} p}, \quad (20)$$

concluding the proof of Claim 3.4. \square

3.3 Proof of Theorem 1.1

We consider the supercritical and subcritical cases, respectively, in Lemmas 3.5 and 3.6. Theorem 1.1 follows from Lemmas 3.5 and 3.6.

Lemma 3.5. *Fix $\ell \geq 3$. For every $\gamma > 0$ there exists $C > 0$ such that if $p \geq Cn^{-1+1/(\ell-1)}$ and $G = G(n, p)$, then $D(G, C_\ell^\circ) \leq 2^{\gamma \binom{n}{2} p}$ with probability $1 - o(1)$.*

Proof. Let $\ell \geq 3$ and $\gamma > 0$ be given. In what follows, we consider the case in which ℓ is odd (the case in which ℓ is even is similar).

Let $\delta_{(L.3.3)}$ be given by an application of Lemma 3.3 with parameters ℓ and γ . Let $\varepsilon_{R(T.2.2)}$ be obtained by an application of Theorem 2.2 with parameters ℓ and $\delta_{(L.3.3)}/2$. Following the quantification in Theorem 2.2 applied with $\mu = 1/2\ell M$, we obtain a constant C .

Suppose $p \geq Cn^{-1+1/(\ell-1)}$ and fix $\varepsilon_{F(L.3.3)} < \min\{\varepsilon_{R(T.2.2)}/\ell, 1/\ell\}$. Now we continue the application of Lemma 3.3 with parameter $\varepsilon_{F(L.3.3)}$, obtaining constants $\varepsilon_{E(L.3.3)}$, $\sigma_{(L.3.3)}$ and M .

Let $G = G(n, p)$. By Fact 2.6 applied with parameters $\sigma_{(L.3.3)}$ and $\varepsilon_{E(L.3.3)}$, we know G satisfies $\text{EDGE}(\sigma_{(L.3.3)}, \varepsilon_{E(L.3.3)}, p)$ with probability $1 - o(1)$. By Theorem 2.2, the graph G contains no member of $\mathcal{F}(C_\ell, \tilde{n}, \lfloor (\delta_{(L.3.3)}/2)\tilde{n}^2 p \rfloor, z\varepsilon_{F(L.3.3)})$ with probability $1 - o(1)$, for any $\tilde{n} \geq \lfloor n/\ell M \rfloor \geq \mu n$. Therefore, by Lemma 3.3, we conclude that $D(G, C_\ell^\circ) \leq 2^{\gamma \binom{n}{2} p}$ holds with probability $1 - o(1)$. \square

Lemma 3.6. *Fix $\ell \geq 3$. For every $\gamma > 0$ there exists $c > 0$ such that if $\binom{n}{2}^{-1} \ll p \leq cn^{-1+1/(\ell-1)}$ and $G = G(n, p)$, then $D(G, C_\ell^\circ) \geq 2^{(1-\gamma)\binom{n}{2} p}$ with probability $1 - o(1)$.*

Proof. Let $\ell \geq 3$ and $\gamma > 0$ be given. Put $c = (\gamma/16\ell)^{1/(\ell-1)}$ and denote by X_{C_ℓ} the number of copies of C_ℓ in G . We divide the proof in two cases, depending on the order of magnitude of p .

Case 1. $\binom{n}{2}^{-1} \ll p \ll n^{-1+1/(\ell-1)}$. It is easy to see that $\mathbb{E}(X_{C_\ell}) \leq (np)^\ell$. Using Markov's inequality we have

$$\mathbb{P}\left(X_{C_\ell} \geq \frac{\gamma}{2\ell} \binom{n}{2} p\right) \leq \frac{2\ell(np)^\ell}{\gamma \binom{n}{2} p}. \quad (21)$$

Since $p \ll n^{-1+1/(\ell-1)}$ we have $\mathbb{P}(X_{C_\ell} \geq \gamma \binom{n}{2} p/2\ell) = o(1)$. Furthermore, since $p \gg \binom{n}{2}^{-1}$, by Chernoff bounds, $e(G(n, p)) \geq (1 - \gamma/2) \binom{n}{2} p$ holds with probability $1 - o(1)$.

Assume that $X_{C_\ell} < (\gamma/2\ell) \binom{n}{2} p$. Consider the following procedure: fix the orientation of the edges that belong to cycles of length ℓ in G according to some total order of the vertices of G and orient the remaining edges in any way. An orientation generated by this procedure contains no copy of C_ℓ° . Therefore, with probability $1 - o(1)$, the number of such orientations is at least $2^{e(G(n, p)) - (\gamma/2) \binom{n}{2} p} \geq 2^{(1-\gamma) \binom{n}{2} p}$.

Case 2. $n^{-1} \ll p \leq cn^{-1+1/(\ell-1)}$. Since $p \gg n^{-1}$, we know that $X_{C_\ell} < 2(np)^\ell$ holds with probability $1 - o(1)$ (see Theorem 4.4.4 in [1]). Furthermore, note that

$$\mathbb{P}(X_{C_\ell} \geq 2(np)^\ell) \geq \mathbb{P}\left(X_{C_\ell} \geq \gamma \binom{n}{2} p/2\ell\right), \quad (22)$$

and hence $\mathbb{P}(X_{C_\ell} \geq \gamma \binom{n}{2} p/2\ell) = o(1)$. It now suffices to proceed as in Case 1. \square

4 Concluding remarks

For simplicity, we restricted our attention to counting C_ℓ° -free orientations. Using versions of Theorem 2.2 that work for general graphs H (see, e.g., [4, 13]), one may prove certain results on the number of \vec{H} -free orientations of $G(n, p)$ for orientations \vec{H} of any given graph H .

We have obtained satisfactory results for the random variable $D(G(n, p), C_\ell^\circ)$ for p close to the ‘threshold’. For p substantially below the threshold (the subcritical case), the value of $D(G(n, p), C_\ell^\circ)$ is a.a.s. $2^{e(G(n, p)) - \Theta(p^\ell n^\ell)}$. A simple analysis of our proof gives the lower bound part of this statement, while the upper bound part follows from the fact that $\Omega(p^\ell n^\ell)$ edge-disjoint copies of C_ℓ exist in $G(n, p)$ in this range.

For p substantially above the threshold (the supercritical case) we expect that our results can be substantially improved. One may check that $D(K_n, C_\ell^\circ)$ is approximately $C^n n!$, where $C \geq 1$ depends only on ℓ (and is equal to one if $\ell = 3$). But this ceases to be true for $D(G(n, p), C_\ell^\circ)$ when $p \ll 1$. It would be interesting to determine more accurately the asymptotic value of $D(G(n, p), C_\ell^\circ)$ as p varies from the threshold to 1.

References

- [1] N. Alon and J. H. Spencer, *The Probabilistic Method* (2nd Edition), Wiley-Interscience, New York, 2000.
- [2] N. Alon and R. Yuster, *The number of orientations having no fixed tournament*, *Combinatorica* **26** (2006), no. 1, 1–16.

- [3] B. Bollobás, *Random Graphs* (2nd Edition), Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001.
- [4] D. Conlon, T. Gowers, W. Samotij, and M. Schacht, *On the KLR conjecture in random graphs*, Submitted.
- [5] P. Erdős, *Some new applications of probability methods to combinatorial analysis and graph theory*, Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1974) (Winnipeg, Man.), Utilitas Math., 1974, pp. 39–51. Congressus Numerantium, No. X.
- [6] S. Gerke, Y. Kohayakawa, V. Rödl, and A. Steger, *Small subsets inherit sparse ε -regularity*, J. Combin. Theory Ser. B **97** (2007), no. 1, 34–56.
- [7] S. Gerke and A. Steger, *The sparse regularity lemma and its applications*, Surveys in Combinatorics 2005, London Math. Soc. Lecture Note Ser., vol. 327, Cambridge Univ. Press, Cambridge, 2005, pp. 227–258.
- [8] S. Janson, T. Łuczak, and A. Ruciński, *Random Graphs*, Wiley-Interscience, New York, 2000.
- [9] S. Jukna, *Extremal combinatorics* (2nd Edition), Texts in Theoretical Computer Science. An EATCS Series, Springer, Heidelberg, 2011.
- [10] Y. Kohayakawa, *Szemerédi’s regularity lemma for sparse graphs*, Foundations of computational mathematics (Rio de Janeiro, 1997), Springer, Berlin, 1997, pp. 216–230.
- [11] Y. Kohayakawa and B. Kreuter, *Threshold functions for asymmetric Ramsey properties involving cycles*, Random Structures Algorithms **11** (1997), no. 3, 245–276.
- [12] Y. Kohayakawa and V. Rödl, *Szemerédi’s regularity lemma and quasi-randomness*, Recent Advances in Algorithms and Combinatorics, CMS Books Math./Ouvrages Math. SMC, vol. 11, Springer, New York, 2003, pp. 289–351.
- [13] D. Saxton and A. Thomason, *Hypergraph containers*, 2013, submitted.