Total colorings of F_5-free planar graphs with maximum degree 8 *

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Mathematics Subject Classifications: 05C15, 05C10

Abstract

The total chromatic number of a graph G, denoted by $\chi''(G)$, is the minimum number of colors needed to color the vertices and edges of G such that no two adjacent or incident elements get the same color. It is known that if a planar graph G has maximum degree $\Delta \geq 9$, then $\chi''(G) = \Delta + 1$. The join $K_1 \vee P_n$ of K_1 and P_n is called a fan graph F_n. In this paper, we prove that if G is a F_5-free planar graph with maximum degree 8, then $\chi''(G) = 9$.

Key words: Planar graph; Total coloring; Cycle

1 Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow [2] for the terminology and notation not defined here. For a graph G, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$ (or simply V, E and Δ), respectively. For a face f of G, the *degree* $d(f)$ is the number of edges incident with it, where each cut-edge is counted twice. The join $K_1 \vee P_n$ of K_1 and P_n is called a *fan graph* F_n. We say that a graph G is *F_n-free* if G contains no F_n as a subgraph. A *k-cycle* is a cycle of length k. We say that two cycles are *adjacent* if they share at least one edge.

A *total k-coloring* of G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest integer k such that G has a total k-coloring. Clearly, $\chi''(G) \geq \Delta + 1$. Behzad [1] and Vizing [12] independently posed the following famous conjecture, which is known as the total coloring conjecture (**TCC**).

Conjecture A. *For any graph G, $\chi''(G) \leq \Delta + 2$.*

*This work is supported by research grants NSFC (11271006, 11201440).

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This conjecture was confirmed for general graphs with $\Delta \leq 5$. In recent years, the study of total colorings for the class of planar graphs has attracted considerable attention. For planar graphs the only open case is $\Delta = 6$ ([7, 9]), and for planar graphs with large maximum degree, there is a stronger result. It is shown that $\chi''(G) = \Delta + 1$ if G is a planar graph with $\Delta \geq 9$ ([8]). This stronger result does not hold for planar graphs of maximum degree at most 3. For $4 \leq \Delta \leq 8$, it is unknown that $\chi''(G) = \Delta + 1$ if G is a planar graph with maximum degree Δ. For $\Delta = 8$, the following three results have been recently proved.

Theorem A. ([6]) *Let G be a planar graph with $\Delta = 8$. If G contains no adjacent 3-cycles, then $\chi''(G) = \Delta + 1$.*

Theorem B. ([11]) *Let G be a planar graph with $\Delta \geq 8$. If G contains no adjacent 4-cycles, then $\chi''(G) = \Delta + 1$.*

Theorem C. ([10]) *Let G be a planar graph with $\Delta \geq 8$. If G contains no 5- or 6-cycles with chords, then $\chi''(G) = \Delta + 1$.*

Theorem D. ([5]) *Let G be a planar graph with $\Delta \geq 8$. If G contain no 5-cycles with two chords, then $\chi''(G) = \Delta + 1$.*

Here, we generalize these results and get the following result.

Theorem 1. *If G be a F_5-free planar graph with $\Delta \geq 8$, then $\chi''(G) = \Delta + 1$.*

Now, we introduce some more notations and definitions. Let G be a planar graph with a plane drawing, denote by F the face set of G. For a vertex v of G, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex*, *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at least k, respectively. Similarly, we can define a *k-face*, *k^--face* and a *k^+-face*. We use $(v_1, v_2, \cdots, v_k)$ to denote a cycle (or a face) whose boundary vertices are $v_1, v_2, \cdots, v_k$ in the clockwise order in G. Denote by $n_d(v)$ the number of d-vertices adjacent to v, by $f_d(v)$ the number of d-faces incident with v.

2 Proof of Theorem 1

According to [8], planar graphs with $\Delta \geq 9$ has a total $(\Delta + 1)$-coloring, so to prove Theorem 1, in the following we assume that $\Delta = 8$. Let $G = (V, E, F)$ be a minimal counterexample to Theorem 1, such that $|V| + |E|$ is minimum. Then every proper subgraph of G has a total 9-coloring. Let L be the color set $\{1, 2, \cdots, 9\}$ for simplicity. It is easy to prove that G is 2-connected and hence the boundary of each face f is exactly a cycle. We first show some known properties on G.
(a) G contains no edge uv with $min\{d(u), d(v)\} \leq 4$ and $d(u) + d(v) \leq 9$. (see([3]))

(b) G contains no even cycle $(v_1, v_2, \cdots, v_{2t})$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$. (see([3]))

It follows from (a) that, the two neighbors of a 2-vertex are all 8-vertices, and any two 4^--vertices are not adjacent. Note that in all figures of the paper, vertices marked $\bullet$ have no edges of G incident with them other than those shown.

Lemma 2. ([5], [6]) *G has no configurations depicted in Fig.* 1(1) – (6).

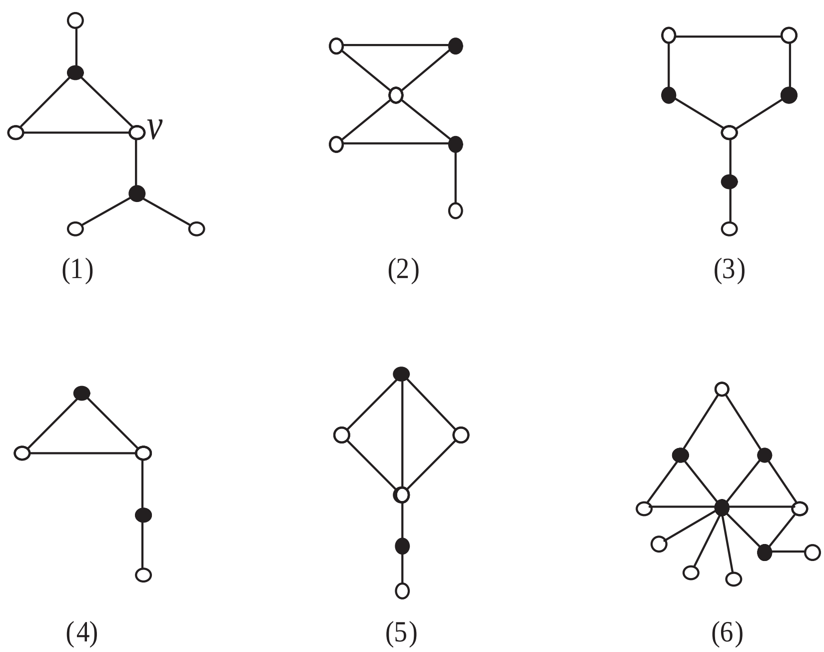

Fig. 1. Reducible Configuration: $d(v) = 7$ in (1)

Lemma 3. ([4]) *Suppose that v is a 8-vertex and $v_1, v_2, \cdots, v_k$ are consecutive neighbors of v with $d(v_1) = d(v_k) = 2$ and $d(v_i) \geq 3$ for $2 \leq i \leq k-1$, where $k \in \{3, 4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $1 \leq i \leq k-1$, then at least one vertex in $\{v_2, v_3, \cdots, v_{k-1}\}$ is a 4^+-vertex.*

Lemma 4. ([13]) *Suppose that v is a 8-vertex and $u, v_1, v_2, \cdots, v_k$ are consecutive neighbors of v with $d(u) = d(v_1) = 2$ and $d(v_i) \geq 3$ for $2 \leq i \leq k$, where $k \in \{3, 4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $1 \leq i \leq k-2$, and the face incident with v, v_{k-1}, v_k is a 3-face, then at least one vertex in $\{v_2, v_3, \cdots, v_{k-1}\}$ is a 4^+-vertex.*

Lemma 5. ([5]) *Suppose that v is a 8-vertex and $u, v_1, v_2, \cdots, v_k$ are consecutive neighbors of v with $d(u) = 2$ and $d(v_i) \geq 3$ for $1 \leq i \leq k$, where $k \in \{4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $2 \leq i \leq k-2$, and the face incident with v, v_j, v_{j+1} is a 3-face for all $j \in \{1, k-1\}$, then at least one vertex in $\{v_2, v_3, \cdots, v_{k-1}\}$ is a 4^+-vertex.*

Let φ be a (partial) total 9-coloring of G. For a vertex v of G, we denote by $C(v)$ the set of colors of edges incident with v. Call φ is *nice* if only some 4^--vertices are not colored. Note that every nice coloring can be greedily extended to a 9-total-coloring of G, since each 4^--vertex is adjacent to at most four vertices and incident with at most four edges. Therefore, in the rest of this paper, we shall always suppose that such vertices are colored at the very end.

By the Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(2d(v)-6)+\sum_{f\in F}(d(f)-6) = -12 < 0.$$

We define ch to be the initial charge. Let $ch(v) = 2d(v) - 6$ for each $v \in V$ and $ch(f) = d(f) - 6$ for each $f \in F$. So $\sum_{x\in V\cup F} ch(x) = -12 < 0$. In the following, we will reassign a new charge denoted by $ch^{'}(x)$ to each $x \in V \cup F$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have $\sum_{x\in V\cup F} ch^{'}(x) = \sum_{x\in V\cup F} ch(x) = -12$. If we can show that $ch^{'}(x) \geq 0$ for each $x \in V \cup F$, then we get an obvious contradiction to $0 \leq \sum_{x\in V\cup F} ch^{'}(x) = \sum_{x\in V\cup F} ch(x) = -12$. which completes our proof.

For $f = (v_1, v_2, \cdots, v_k) \in F$, we use $(d(v_1), d(v_2), \cdots, d(v_k)) \to (c_1, c_2, \cdots, c_k)$ to denote that the vertex v_i sends f the amount of charge c_i for $i = 1, 2, \cdots, k$. Now we define the discharging rules as follows.

R1. Each 2-vertex receives 1 from each of its neighbors.

R2. For a 3-face (v_1, v_2, v_3), let
$(3^-, 7^+, 7^+) \to \big(0, \frac{3}{2}, \frac{3}{2}\big)$,
$(4, 6^+, 6^+) \to \big(\frac{1}{2}, \frac{5}{4}, \frac{5}{4}\big)$,
$(5^+, 5^+, 5^+) \to \big(1, 1, 1\big)$.

R3. For a 4-face (v_1, v_2, v_3, v_4), let
$(3^-, 7^+, 3^-, 7^+) \to \big(0, 1, 0, 1\big)$,
$(3^-, 7^+, 4^+, 7^+) \to \big(0, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}\big)$,
$(4^+, 4^+, 4^+, 4^+) \to \big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big)$.

R4. For a 5-face $(v_1, v_2, v_3, v_4, v_5)$, let
$(3^-, 7^+, 3^-, 7^+, 7^+) \to \big(0, \frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3}\big)$,
$(3^-, 7^+, 4^+, 4^+, 7^+) \to \big(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\big)$,
$(4^+, 4^+, 4^+, 4^+, 4^+) \to \big(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\big)$.

Next we show that $ch^{'}(x) \geq 0$ for each $x \in V \cup F$. Since our discharging rules are designed such that $ch^{'}(f) \geq 0$ for all $f \in F$ and $ch^{'}(v) \geq 0$ for all 2-vertices $v \in V$, it suffices to check that $ch'(v) \geq 0$ for all 3^+-vertices in G. Let $v \in V$. Suppose $d(v) = 3$. Then $ch'(v) = ch(v) = 0$. Suppose $d(v) = 4$. Then v sends at most $\frac{1}{2}$ to each of its incident faces and $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$. Suppose $d(v) = 5$. Then $f_3(v) \leq 3$, and v sends at most 1 to each of its incident 3-faces by R2, at most $\frac{1}{2}$ to each of its incident 4^+-faces by R3 and R4. So $ch'(v) \geq ch(v) - f_3(v) \times 1 - (5 - f_3(v)) \times \frac{1}{2} = \frac{3}{2} - \frac{1}{2}f_3(v) \geq 0$. Suppose $d(v) = 6$. Then $f_3(v) \leq 4$, and v sends at most $\frac{5}{4}$ to each of its incident 3-faces, at most $\frac{1}{2}$ to each of its incident 4^+-faces. So $ch'(v) \geq ch(v) - f_3(v) \times \frac{5}{4} - (6 - f_3(v)) \times \frac{1}{2} = 3 - \frac{3}{4}f_3(v) \geq 0$.

Call a 3-face is *bad* if it has a 3^--vertex, a 4-face is *bad* if it has two 3^--vertices, *good* otherwise.

Suppose $d(v) = 7$. Note that $f_3(v) \leq 5$. If $f_3(v) \leq 2$, then $ch'(v) \geq ch(v) - f_3(v) \times \frac{3}{2} - (7 - f_3(v)) \times 1 = 1 - \frac{1}{2}f_3(v) \geq 0$. Suppose $3 \leq f_3(v) \leq 5$, then v is incident with at most two bad 3-faces by Fig. 1(1). If $3 \leq f_3(v) \leq 4$, then $ch'(v) \geq ch(v) - \max\{2 \times \frac{3}{2} + (f_3(v) - 2) \times \frac{5}{4} + (7 - f_3(v)) \times \frac{1}{2}, \frac{3}{2} + (f_3(v) - 1) \times \frac{5}{4} + \frac{3}{4} + (7 - f_3(v) - 1) \times \frac{1}{2}, f_3(v) \times \frac{5}{4} + 2 \times 1 + (7 - f_3(v) - 2) \times \frac{3}{4}\} = \frac{9}{4} - \frac{1}{2}f_3(v) \geq \frac{1}{4} > 0$. If $f_3(v) = 5$, then $ch'(v) \geq ch(v) - \max\{2 \times \frac{3}{2} + 3 \times \frac{5}{4} + 2 \times \frac{1}{2}, \frac{3}{2} + 4 \times \frac{5}{4} + \frac{3}{4} + \frac{1}{2}\} = \frac{1}{4} > 0$.

Suppose $d(v) = 8$. Let $v_1, v_2, \cdots, v_8$ be neighbors of v and $f_1, f_2, \cdots, f_8$ be faces incident with v in an clockwise order, where f_i is incident with v_i, v_{i+1}, and $i \in \{1, 2, \cdots, 8\}$. Note that all the subscripts in the paper are taken modulo 8. First, we prove some lemmas.

Lemma 6. *Suppose that v is a 8-vertex and $v_1, v_2, \cdots, v_k, v_{k+1}, v_s, v_{s+1}$ are consecutive neighbors of v with $d(v_1) = 2$ and $d(v_i) = 3$ for $2 \leq i \leq k$, where $3 \leq k + 1 \leq s$ and $s \in \{3, 5, \cdots, 7\}$. If v is incident with 3-faces (v, v_k, v_{k+1}) and (v, v_s, v_{s+1}), and incident with 4-faces (v, v_j, x_j, v_{j+1}) for all $1 \leq j \leq k - 1$, then* $\min\{d(v_s), d(v_{s+1})\} \geq 4$.

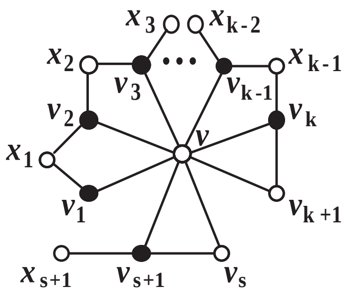

Fig. 2. Reducible Configuration in G

Proof. By Fig. 1(2), we have $\min\{d(v_s), d(v_{s+1})\} \geq 3$. Assume to be contradictory that $d(v_s) = 3$ or $d(v_{s+1}) = 3$. Without loss of generality, suppose that $d(v_{s+1}) = 3$, and $N(v_{s+1}) = \{v, v_s, x_{s+1}\}$ (see Fig. 2). Consider a nice coloring φ of $G' = G - vv_1$. If $\varphi(v_1x_1) \in C(v)$, then the forbidden colors for vv_1 number at most 8, so vv_1 can be properly colored. Then we can suppose $\varphi(v_1x_1) \notin C(v)$. Without loss of generality, suppose that $\varphi(v) = 9$, $\varphi(v_1x_1) = 1$, and $\varphi(vv_j) = j$ for $j \in \{2, \cdots, k, k+1, s, s+1\}$. It is easy to see that $1 \in C(v_j)$ for $j \in \{2, \cdots, k, s+1\}$, since otherwise, we can recolor vv_j with 1, color vv_1 with j, a contradiction. So $\varphi(v_2x_2) = \cdots = \varphi(v_{k-1}x_{k-1}) = \varphi(v_kv_{k+1}) = 1$ and $1 \in \{\varphi(v_sv_{s+1}), \varphi(v_{s+1}x_{s+1})\}$. Note that $\varphi(v_kx_{k-1}) = k+1$, since otherwise, we may get a contradiction by exchange the colors on vv_{k+1} and v_kv_{k+1}, color vv_1 with $k+1$. Thus $\varphi(v_{k-1}x_{k-2}) = k+1$, since otherwise, we exchange the colors on vv_{k+1} and v_kv_{k+1}, v_kx_{k-1} and $v_{k-1}x_{k-1}$, color vv_1 with $k+1$, also a contradiction. Similarly, $\varphi(v_{k-2}x_{k-3}) = \cdots = \varphi(v_2x_1) = k+1$.

If $k+1 = s$, then $\varphi(v_{s+1}x_{s+1}) = 1$. We exchange the colors on vv_k and vv_{s+1}, recolor v_kv_{k+1} with $\varphi(v_sv_{s+1})$, vv_{k+1} with 1, and v_sv_{s+1} with $k+1$, color vv_1 with $k+1$, a contradiction. So we can suppose $k+1 < s$. Then $k+1 \in \{\varphi(v_sv_{s+1}), \varphi(v_{s+1}x_{s+1})\}$, since otherwise, we can exchange the colors on vv_{k+1} and v_kv_{k+1}, v_kx_{k-1} and $v_{k-1}x_{k-1}$, $\cdots$, v_1x_1 and v_2x_1, recolor vv_{s+1} with $k+1$, color vv_1 with $s+1$, a contradiction. We first exchange the colors on vv_s and v_sv_{s+1}. If $\varphi(v_sv_{s+1}) = k+1$, we additionally exchange the colors on vv_{k+1} and v_kv_{k+1}, v_kx_{k-1} and $v_{k-1}x_{k-1}$, $\cdots$, v_1x_1 and v_2x_1. Then we color vv_1 with s, also a contradiction. □

Lemma 7. *Suppose that v is a 8-vertex and $v_1, v_2, v_3, v_k, v_{k+1}, v_s, v_{s+1}$ $(3 \le k < s \le 8)$ are consecutive neighbors of v with $d(v_2) = 3$. If the face incident with v, v_i, v_{i+1} is a 3-face for all $i \in \{1, 2, k, s\}$, then at most one vertex in $\{v_k, v_{k+1}, v_s, v_{s+1}\}$ is a 3−vertex.*

Proof. By Fig. 1(2), we have $\min\{d(v_k), d(v_{k+1}, d(v_s), d(v_{s+1})\} \ge 3$. Assume to be contradictory that there are two 3−vertices in $\{v_k, v_{k+1}, v_s, v_{s+1}\}$. Consider a nice coloring φ of $G' = G - vv_2$. Without loss of generality, suppose that $\varphi(v) = 2$ and $\varphi(vv_i) = i$ for $i \in \{1, 3, k, k+1, s, s+1\}$. First, we have $9 \in C(v_2)$, that is, $\varphi(v_1v_2) = 9$ or $\varphi(v_2v_3) = 9$, for otherwise, we can obtain a nice coloring of G by color vv_2 with 9, a contradiction. Second, for each 3−vertex v_j $(j \in \{k, k+1, s, s+1\})$, we note that if $j \notin C(v_2)$, then $9 \in C(v_j)$. Otherwise, we can recolor vv_j with 9, and color vv_2 with j to obtain a nice coloring of G, a contradiction.

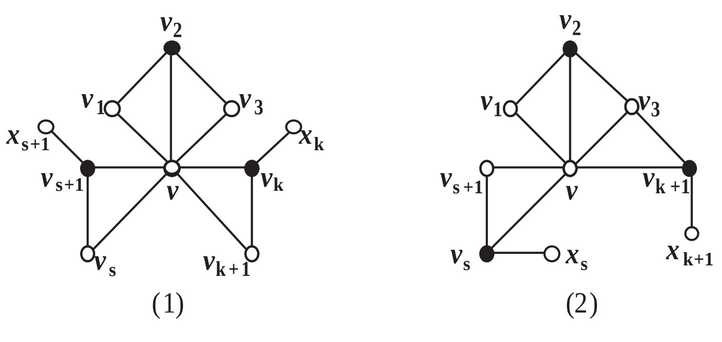

Fig. 3. Reducible Configuration in G

Case 1. $k > 3$ and $s < 8$.

Without loss of generality, suppose that $d(v_k) = d(v_{s+1}) = 3$, $N(v_k) = \{v, v_{k+1}, x_k\}$, $N(v_{s+1}) = \{v, v_s, x_{s+1}\}$ and $\varphi(v_2v_3) = 9$ (see Fig. 3(1)). We note that if $\varphi(v_1v_2) \notin \{3, k\}$, then $3 \in C(v_k)$, that is, $C(v_k) = \{k, 3, 9\}$. Otherwise, we exchange the colors on vv_3 and v_2v_3, recolor vv_k with 3, and color vv_2 with k, a contradiction. Similarly, if $\varphi(v_1v_2) \notin \{3, s+1\}$, then $C(v_{s+1}) = \{s+1, 3, 9\}$. Suppose $\varphi(v_1v_2) \notin \{3, k+1\}$. Since $\varphi(v_1v_2)$ is different from either k or $s+1$, we may assume that $\varphi(v_1v_2) \neq k$. Then $C(v_k) = \{k, 3, 9\}$. We exchange the colors on vv_{k+1} and v_kv_{k+1}, color vv_2 with $k+1$. If $\varphi(v_kv_{k+1}) = 3$, we additionally exchange the colors on vv_3 and v_2v_3. Thus we obtain a nice coloring of G, a contradiction. Suppose $\varphi(v_1v_2) = 3$. Then $9 \in C(v_k)$. If $1 \notin C(v_k)$, then we exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3, recolor vv_k with 1, and color vv_2 with k, a contradiction. So $C(v_k) = \{k, 1, 9\}$. We exchange the colors on vv_{k+1} and v_kv_{k+1}, color vv_2 with $k+1$. If $\varphi(v_kv_{k+1}) = 1$, we additionally exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3. Thus we obtain a nice coloring of G, a contradiction. Suppose $\varphi(v_1v_2) = k+1$. Then $C(v_k) = \{k, 3, 9\}$ and $C(v_{s+1}) = \{s+1, 3, 9\}$. We exchange the colors on vv_{k+1} and v_kv_{k+1}, recolor vv_{s+1} with $k+1$, and color vv_2 with $s+1$. If $\varphi(v_kv_{k+1}) = 3$, we additionally exchange the colors on vv_3 and v_2v_3. Thus we also obtain a nice coloring of G, a contradiction.

Case 2. $k = 3$.

Then $d(v_{k+1}) = 3$. Without loss of generality, suppose that $d(v_s) = 3$ and $s \neq k+1$, $N(v_{k+1}) = \{v, v_3, x_{k+1}\}$, $N(v_s) = \{v, v_{s+1}, x_s\}$ (see Fig. 3(2)).

Case 2.1. $\varphi(v_2v_3) = 9$.

We note that if $\varphi(v_1v_2) \notin \{3, k+1\}$, then $3 \in C(v_{k+1})$, that is, $C(v_{k+1}) = \{k+1, 3, 9\}$. Otherwise, we exchange the colors on vv_3 and v_2v_3, recolor vv_{k+1} with 3, and color vv_2 with $k+1$, a contradiction. Similarly, if $\varphi(v_1v_2) \notin \{3, s\}$, then $C(v_s) = \{s, 3, 9\}$. Suppose $\varphi(v_1v_2) \notin \{3, k+1\}$. Then $C(v_{k+1}) = \{k+1, 3, 9\}$. So $\varphi(v_3v_{k+1}) = 3$ or 9, a contradiction. Suppose $\varphi(v_1v_2) = 3$. Then $9 \in C(v_{k+1})$. If $1 \notin C(v_{k+1})$, then we exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3, recolor vv_{k+1} with 1, and color vv_2 with $k+1$, a contradiction. So $C(v_{k+1}) = \{k+1, 1, 9\}$. Similarly, $C(v_s) = \{s, 1, 9\}$. If $v_1 = v_{s+1}$, then $\varphi(v_1v_s) = 9$ and $\varphi(v_3v_{k+1}) = 1$. We exchange the colors on v_1v_2 and v_1v_s, recolor v_2v_3 with 1, vv_3 with 9, v_3v_{k+1} with 3, and color vv_2 with 3, a contradiction. Otherwise, $v_1 \neq v_{s+1}$. We exchange the colors on vv_{s+1} and v_sv_{s+1}, and color vv_2 with $s+1$. If $\varphi(v_sv_{s+1}) = 1$, we additionally exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3. Thus we obtain a nice coloring of G, also a contradiction. Suppose $\varphi(v_1v_2) = k+1$. Then $C(v_s) = \{s, 3, 9\}$. We exchange the colors on vv_{s+1} and v_sv_{s+1}, color vv_2 with $s+1$. If $\varphi(v_1v_s) = 3$, we additionally exchange the colors on vv_3 and v_2v_3. Thus we also obtain a nice coloring of G, a contradiction.

Case 2.2. $\varphi(v_1v_2) = 9$.

By Case 2.1, we may assume that $v_1 \neq v_{s+1}$. We note that if $\varphi(v_2v_3) \notin \{1, k+1\}$, then $1 \in C(v_{k+1})$, that is, $C(v_{k+1}) = \{k+1, 1, 9\}$. Otherwise, we exchange the colors on vv_1 and v_1v_2, recolor vv_{k+1} with 1, and color vv_2 with $k+1$, a contradiction. Similarly, if $\varphi(v_2v_3) \notin \{1, s\}$, then $C(v_s) = \{s, 1, 9\}$. Suppose $\varphi(v_2v_3) \notin \{1, k+1\}$. Then $C(v_{k+1}) = \{k+1, 1, 9\}$. We exchange the colors on vv_3 and v_3v_{k+1}, and color vv_2 with 3. If $\varphi(v_3v_{k+1}) = 1$, we additionally exchange the colors on vv_1 and v_1v_2. Thus we obtain a nice coloring of G, a contradiction. Suppose $\varphi(v_2v_3) = 1$.Then $9 \in C(v_{k+1})$. If $3 \notin C(v_{k+1})$, then we exchange the colors on vv_1 and v_1v_2, vv_3 and v_2v_3, recolor vv_{k+1} with 3, and color vv_2 with $k+1$, a contradiction. So $C(v_{k+1}) = \{k+1, 3, 9\}$. Similarly, $C(v_s) = \{s, 3, 9\}$. We exchange the colors on vv_1 and v_1v_2, v_2v_3 and v_3v_{k+1}, recolor vv_s with 1, and color vv_2 with s, a contradiction. Suppose $\varphi(v_2v_3) = k+1$. Then $C(v_s) = \{s, 1, 9\}$. We exchange the colors on vv_{s+1} and v_sv_{s+1}, color vv_2 with $s+1$. If $\varphi(v_sv_{s+1}) = 1$, we additionally exchange the colors on vv_1 and v_1v_2. Thus we also obtain a nice coloring of G, a contradiction.

Case 3. $s+1 = 1$. Completely similar with the Case 2. □

Lemma 8. *Suppose that $d(v_i) = d(v_k) = 2$ and $d(v_j) \geq 3$ for all $j = i+1, \cdots, k-1$, where $k \geq i+2$. If $\min\{d(f_i), d(f_{i+1}), \cdots, d(f_{k-1})\} \geq 4$, then v sends at most $\frac{3}{2} + (k-i-2)$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$.*

Proof. By Lemma 3, $\max\{d(v_{i+1}), \cdots, d(v_{k-1})\} \geq 4$ or $\max\{d(f_i), \cdots, d(f_{k-1})\} \geq 5$. If $\max\{d(v_{i+1}), \cdots, d(v_{k-1})\} \geq 4$, then v sends at most $2 \times \frac{3}{4} + (k-i-2)$ (in total) to $f_i, \cdots, f_{k-1}$ by R3. If $\max\{d(f_i), \cdots, d(f_{k-1})\} \geq 5$, then or v sends at most $\frac{1}{3} + (k-i-1)$(in total) to $f_i, \cdots, f_{k-1}$ by R3 and R4. Since $2 \times \frac{3}{4} > 1 + \frac{1}{3}$, v sends at most $\frac{3}{2} + (k-i-2)$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. □

Lemma 9. *Suppose that $d(v_i) = d(v_{i+4}) = 2$ and $d(v_j) \geq 3$ for all $j = i+1, i+2, i+3$. If $\min\{d(f_i), d(f_{i+2}), d(f_{i+3})\} \geq 4$ and $d(f_{i+1}) = 3$, then v sends at most $\frac{15}{4}$ (in total) to f_i, f_{i+1}, f_{i+2} and f_{i+3}.*

Proof. If $d(v_{i+1}) = 3$, then $d(v_{i+2}) \geq 7$, and $d(f_i) \geq 5$ by Lemma 4, so v sends at most $\frac{1}{3} + \frac{3}{2} + \frac{3}{4} + 1 = \frac{43}{12}$ to f_i, f_{i+1}, f_{i+2} and f_{i+3}. If $d(v_{i+2}) = 3$, then $d(v_{i+1}) \geq 7$, and $d(v_{i+3}) \geq 4$ or there is at least one 5^+-face in $\{f_{i+2}, f_{i+3}\}$ by Lemma 4, so v sends at most $\frac{3}{4} + \frac{3}{2} + \max\{2 \times \frac{3}{4}, 1 + \frac{1}{3}\} = \frac{15}{4}$ to f_i, f_{i+1}, f_{i+2} and f_{i+3}. If $\min\{d(v_{i+1}), d(v_{i+2})\} \geq 4$, then v sends at most $\frac{3}{4} + \frac{5}{4} + \frac{3}{4} + 1 = \frac{15}{4}$ to f_i, f_{i+1}, f_{i+2} and f_{i+3}. Since $\frac{43}{12} < \frac{15}{4}$, v sends at most $\frac{15}{4}$ (in total) to f_i, f_{i+1}, f_{i+2} and f_{i+3}. □

Lemma 10. *Suppose that $d(v_i) = d(v_k) = 2$ and $d(v_j) \geq 3$ for all $j = i+1, \cdots, k-1$, where $k \geq i+3$. If $\min\{d(f_i), d(f_{k-1})\} \geq 4$ and $d(f_{i+1}) = \cdots = d(f_{k-2}) = 3$, then v sends at most $\frac{11}{4} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$.*

Proof. We note that if $k \geq i+4$, then $\min\{d(v_{i+2}), \cdots, d(v_{k-2})\} \geq 4$ by Fig. 1(5). If $d(f_i) = d(f_{k-1}) = 4$, then $\min\{d(v_{i+1}), d(v_{k-1})\} \geq 4$ by Lemma 4, so v sends at most $2 \times \frac{3}{4} + (k-i-2) \times \frac{5}{4} = \frac{11}{4} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. If one of f_i and f_{k-1} is 4−face, then v sends at most $\frac{3}{4}+\frac{1}{3}+\frac{3}{2}+(k-i-3)\times\frac{5}{4} = \frac{31}{12}+(k-i-3)\times\frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. If $\min\{d(f_i), d(f_{k-1})\} \geq 5$, then v sends at most $2\times\frac{1}{3}+2\times\frac{3}{2}+(k-i-4)\times\frac{5}{4} = \frac{29}{12} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. Since $\max\{\frac{11}{4}, \frac{31}{12}, \frac{29}{12}\} = \frac{11}{4}$, v sends at most $\frac{11}{4} + (k-i-3) \times \frac{5}{4}$ (in total) to $f_i, f_{i+1}, \cdots, f_{k-1}$. □

Now, we come back to check the new charge of 8-vertex v and consider nine cases in the following.

Case 1. $n_2(v) = 8$. Note that $f_{6^+}(v) = 8$ by Fig. 1(3) and (4). Then, no charge is discharged from v to its incident faces. So $ch'(v) = ch(v) - 8 \times 1 = 10 - 8 = 2 > 0$ by R1.

Case 2. $n_2(v) = 7$. Then $f_{6^+}(v) \geq 6$ and $f_3(v) = 0$ by Fig. 1(4). So $ch'(v) \geq ch(v) - 7 \times 1 - 2 \times 1 = 10 - 9 = 1 > 0$.

Case 3. $n_2(v) = 6$. Then there are four possibilities in which 2-vertices are located. They are shown as configurations in Fig. 4. For Fig. 4(1), $f_{6^+}(v) \geq 5$ and $f_3(v) \leq 1$. So $ch'(v) \geq ch(v) - 6 \times 1 - \frac{3}{2} - 2 \times 1 = \frac{1}{2} > 0$. For Fig. 4(2)–(4), $f_{6^+}(v) \geq 4$ and $f_3(v) = 0$. So $ch'(v) \geq ch(v) - 6 \times 1 - 4 \times 1 = 0$.

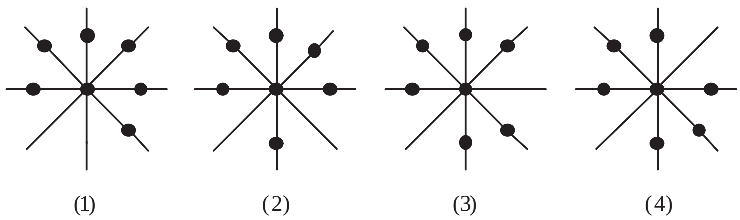

Fig. 4. $n_2(v) = 6$

Case 4. $n_2(v) = 5$. Then there are five possibilities in which 2-vertices are located. They are shown as configurations in Fig. 5.

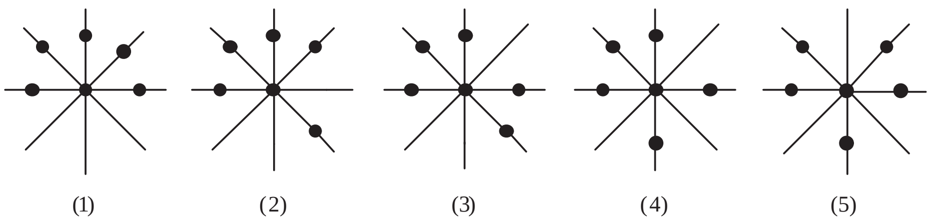

Fig. 5. $n_2(v) = 5$

For Fig. 5(1), $f_{6^+}(v) \geq 4$ and $f_3(v) \leq 2$. So $ch^{'}(v) \geq ch(v) - 5\times 1 - 2\times \frac{3}{2} - 2\times 1 = 0$. For Fig. 5(2) and (3), $f_{6^+}(v) \geq 3$ and $f_3(v) \leq 1$. So $ch^{'}(v) \geq ch(v) - 5\times 1 - \frac{3}{2} - \max\{\frac{11}{4}, \frac{3}{2}+1\} = \frac{3}{4} > 0$ by Lemma 8 and Lemma 10. For Fig. 5(4) and (5), $f_{6^+}(v) \geq 2$ and $f_3(v) = 0$. So $ch^{'}(v) \geq ch(v) - 5\times 1 - 3\times \frac{3}{2} = \frac{1}{2} > 0$.

Case 5. $n_2(v) = 4$. Then there are eight possibilities in which 2-vertices are located. They are shown as configurations in Fig. 6.

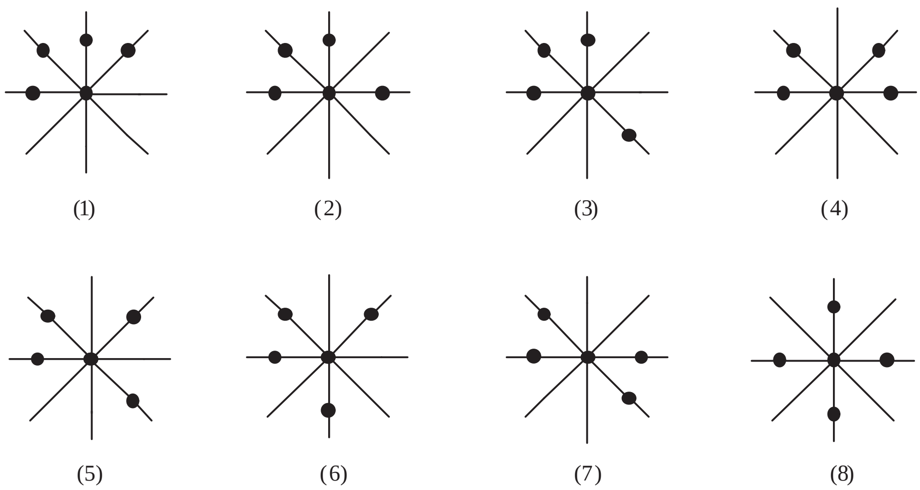

Fig. 6. $n_2(v) = 4$

For Fig. 6(1), $f_{6^+}(v) \geq 3$ and $f_3(v) \leq 3$. If $f_3(v) = 3$, then $ch^{'}(v) \geq ch(v) - 4\times 1 - (\frac{11}{4} + 2\times \frac{5}{4}) = \frac{3}{4} > 0$. Otherwise, $ch^{'}(v) \geq ch(v) - 4\times 1 - f_3(v)\times \frac{3}{2} - (5 - f_3(v))\times 1 = 1 - \frac{1}{2}f_3(v) \geq 0$. For Fig. 6(2) and (4), $f_{6^+}(v) \geq 2$ and $f_3(v) \leq 2$. If $f_3(v) = 2$, then $ch^{'}(v) \geq ch(v) - 4\times 1 - \frac{3}{2} - (\frac{11}{4} + \frac{5}{4}) = \frac{1}{2} > 0$ by Lemma 8 and Lemma 10. Otherwise, $ch^{'}(v) \geq ch(v) - 4\times 1 - \frac{3}{2} - f_3(v)\times \frac{3}{2} - (4 - f_3(v))\times 1 = \frac{1}{2} - \frac{1}{2}f_3(v) \geq 0$. For Fig. 6(3) and (7), $f_{6^+}(v) \geq 2$ and $f_3(v) \leq 2$. So $ch^{'}(v) \geq ch(v) - 4\times 1 - f_3(v)\times \frac{11}{4} - (2 - f_3(v))\times (\frac{3}{2}+1) = 1 - \frac{1}{4}f_3(v) > 0$ by Lemma 8 and Lemma 10. For Fig. 6(5) and (6), $f_{6^+}(v) \geq 1$ and $f_3(v) \leq 1$. So $ch^{'}(v) \geq ch(v) - 4\times 1 - 2\times \frac{3}{2} - f_3(v)\times \frac{11}{4} - (1 - f_3(v))\times (\frac{3}{2}+1) = \frac{1}{2} - \frac{1}{4}f_3(v) > 0$. For Fig. 6(8), $f_3(v) = 0$. So $ch^{'}(v) \geq ch(v) - 4\times 1 - 4\times \frac{3}{2} = 0$.

Case 6. $n_2(v) = 3$. Then there are five possibilities in which 2-vertices are located. They are shown as configurations in Fig. 7.

For Fig. 7(1), note that $\min\{d(f_1), d(f_2)\} \geq 6$, $\min\{d(f_3), d(f_8)\} \geq 4$, and $f_3(v) \leq 3$. If $f_3(v) \leq 2$, then $ch^{'}(v) \geq ch(v) - 3\times 1 - f_3(v)\times \frac{3}{2} - (6 - f_3(v))\times 1 = 1 - \frac{1}{2}f_3(v) \geq 0$. Suppose $f_3(v) = 3$, Then $\min\{d(f_4), d(f_7)\} = 3$. Without loss of generality, suppose that $d(f_4) = 3$, then v sends at most $\frac{3}{4}$ to f_3 by Lemma 4. If $d(f_7) = 3$, then $ch^{'}(v) \geq ch(v) - 3\times 1 - 1 - 2\times \frac{3}{4} - 3\times \frac{3}{2} = 0$. Otherwise, $d(f_4) = d(f_5) = d(f_6) = 3$, then f_5 is good by Fig. 1(5). So $ch^{'}(v) \geq ch(v) - 3\times 1 - 2\times 1 - \frac{3}{4} - \frac{5}{4} - 2\times \frac{3}{2} = 0$.

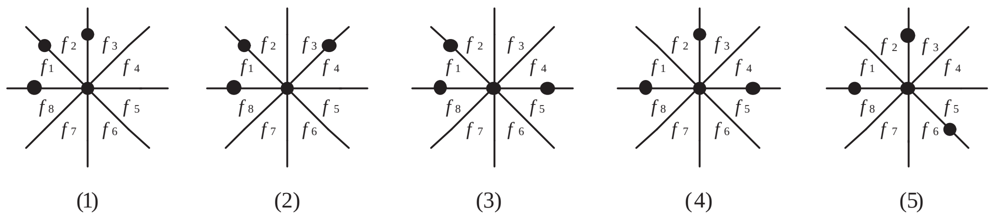

Fig. 7. $n_2(v) = 3$

For Fig. 7(2), $d(f_1) \geq 6$, $\min\{d(f_2), d(f_3), d(f_4), d(f_8)\} \geq 4$ and $f_3(v) \leq 3$. If $f_3(v) \leq 1$, then $ch'(v) \geq ch(v) - 3 \times 1 - \frac{3}{2} - f_3(v) \times \frac{3}{2} - (5 - f_3(v)) \times 1 = \frac{1}{2} - \frac{1}{2} f_3(v) \geq 0$ by Lemma 8. If $f_3(v) = 3$, then $d(f_5) = d(f_6) = d(f_7) = 3$, so $ch'(v) \geq ch(v) - 3 \times 1 - \frac{3}{2} - (\frac{11}{4} + 2 \times \frac{5}{4}) = \frac{1}{4} > 0$ by Lemma 8 and Lemma 10. Suppose $f_3(v) = 2$. If $\max\{d(f_4), d(f_8)\} \geq 5$, then $ch'(v) \geq ch(v) - 3 \times 1 - \frac{3}{2} - 2 \times \frac{3}{2} - \frac{1}{3} - 2 \times 1 = \frac{1}{6} > 0$. Otherwise, without loss of generality, suppose that $d(f_5) = 3$. If $d(f_6) = 3$, then f_4 and f_5 are good by Fig. 1(5) and Lemma 4. So $ch'(v) \geq ch(v) - 3 \times 1 - \frac{3}{2} - \frac{3}{4} - \frac{5}{4} - \frac{3}{2} - 2 \times 1 = 0$. If $d(f_7) = 3$, then f_4 and f_8 are good. So $ch'(v) \geq ch(v) - 3 \times 1 - \frac{3}{2} - 2 \times \frac{3}{4} - 2 \times \frac{3}{2} - 1 = 0$.

For Fig. 7(3), $d(f_1) \geq 6$, $\min\{d(f_2), d(f_4), d(f_5), d(f_8)\} \geq 4$, and $f_3(v) \leq 3$. If $f_3(v) = 3$, then $ch'(v) \geq ch(v) - 3 \times 1 - (\frac{11}{4} + \frac{5}{4}) - \frac{11}{4} = \frac{1}{4} > 0$ by Lemma 10. Otherwise, $f_3(v) \leq 2$. If $d(f_3) = 3$, then $ch'(v) \geq ch(v) - 3 \times 1 - \frac{11}{4} - \max\{\frac{15}{4}, 4 \times 1\} = \frac{1}{4} > 0$. If $d(f_3) \geq 4$, then $ch'(v) \geq ch(v) - 3 \times 1 - (\frac{3}{2} + 1) - \max\{\frac{11}{4} + \frac{5}{4}, \frac{15}{4}, 4 \times 1\} = \frac{1}{2} > 0$.

For Fig. 7(4), $f_3(v) \leq 2$. So $ch'(v) \geq ch(v) - 3 \times 1 - 2 \times \frac{3}{2} - \max\{\frac{11}{4} + \frac{5}{4}, \frac{15}{4}, 4 \times 1\} = 0$. For Fig. 7(5), $f_3(v) \leq 2$. So $ch'(v) \geq ch(v) - 3 \times 1 - \frac{3}{2} - f_3(v) \times \frac{11}{4} - (2 - f_3(v)) \times (\frac{3}{2} + 1) = \frac{1}{2} - \frac{1}{4} f_3(v) \geq 0$.

Case 7. $n_2(v) = 2$. Then there are four possibilities in which 2-vertices are located. They are shown as configurations in Fig. 8.

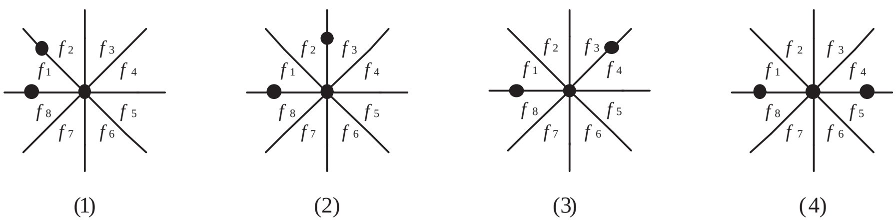

Fig. 8. $n_2(v) = 2$

For Fig. 8(1), note that $d(f_1) \geq 5$ and $f_3(v) \leq 4$. Suppose $f_3(v) = 4$. Then without loss of generality, let $d(f_3) = d(f_4) = d(f_7) = d(f_i) = 3$ ($i \in \{5, 6\}$). Then $d(v_4) \geq 4$ by Fig. 1(5), and v sends at most $\max\{\frac{1}{3} + \frac{3}{2}, \frac{3}{4} + \frac{5}{4}\} = 2$ (in total) to f_2 and f_3. If $d(f_8) \geq 5$, then $ch'(v) \geq ch(v) - 2 \times 1 - \frac{1}{3} - 2 - 3 \times \frac{3}{2} - \frac{3}{4} - \frac{1}{3} = \frac{1}{12} > 0$ by Lemma 5. Otherwise, $d(f_8) = 4$, then $d(v_8) \geq 4$ by Lemma 4, it follows that f_4 (if $i = 5$) or f_7 (if $i = 6$) is good, and v sends at most $\max\{\frac{3}{2}+\frac{3}{2}+\frac{1}{3}, \frac{3}{2}+\frac{5}{4}+\frac{3}{4}\}=\frac{7}{2}$ (in total) to f_5, f_6 and f_7 (or f_4). So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-2-\frac{5}{4}-\frac{7}{2}-\frac{3}{4}=\frac{1}{6}>0$.

Suppose $f_3(v)=3$. If $f_{5^+}(v)\geq 3$, then $ch^{'}(v)\geq ch(v)-2\times 1-f_{5^+}(v)\times\frac{1}{3}-3\times\frac{3}{2}-(5-f_{5^+}(v))\times 1=\frac{2}{3}f_{5^+}(v)-\frac{3}{2}>0$. If $f_{5^+}(v)=2$, then except f_1, there is one 5^+-face incident with v, and there is at least one good 4-face which incident with v. So $ch^{'}(v)\geq ch(v)-2\times 1-2\times\frac{1}{3}-3\times\frac{3}{2}-\frac{3}{4}-2\times 1=\frac{1}{12}>0$. If $f_{5^+}(v)=1$, then $d(f_i)\leq 4$ for $2\leq i\leq 8$. By symmetry, we need to consider the following cases in which 3-faces are located.

First, suppose $d(f_3)=d(f_4)=d(f_5)=3$. Then $\min\{d(v_3),d(v_4),d(v_5)\}\geq 4$ and $\max\{d(v_6),d(v_7),d(v_8)\}\geq 4$ by Fig. 1(5) and Lemma 4. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-2\times\frac{5}{4}-\frac{3}{2}-2\times\frac{3}{4}-2\times 1=\frac{1}{6}>0$. Second, suppose $d(f_4)=d(f_5)=d(f_6)=3$. Then $\min\{d(v_5),d(v_6)\}\geq 4$ by Fig. 1(5), $\max\{d(v_3),d(v_4)\}\geq 4$ and $\max\{d(v_7),d(v_8)\}\geq 4$ by Lemma 4. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-\max\{\frac{5}{4}+2\times\frac{3}{2}+3\times\frac{3}{4}+1, 2\times\frac{5}{4}+\frac{3}{2}+2\times\frac{3}{4}+2\times 1\}=\frac{1}{6}>0$. Third, suppose $d(f_3)=d(f_4)=d(f_6)=3$. Then $d(v_4)\geq 4$ by Fig. 1(5), $d(v_3)\geq 4$ and $\max\{d(v_7),d(v_8)\}\geq 4$ by Lemma 4, $\max\{d(v_5),d(v_6)\}\geq 4$ by Lemma 5. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-2\times\frac{3}{2}-\frac{5}{4}-3\times\frac{3}{4}-1=\frac{1}{6}>0$. Fourth, suppose $d(f_3)=d(f_4)=d(f_7)=3$. Then $\min\{d(v_3),d(v_4),d(v_8)\}\geq 4$ by Fig. 1(5) and Lemma 4, $\max\{d(v_5),d(v_6),d(v_7)\}\geq 4$ by Lemma 5. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-2\times\frac{3}{2}-\frac{5}{4}-3\times\frac{3}{4}-1=\frac{1}{6}>0$. Fifth, suppose $d(f_4)=d(f_5)=d(f_7)=3$. Then $d(v_5)\geq 4$ by Fig. 1(5), $d(v_8)\geq 4$ and $\max\{d(v_3),d(v_4)\}\geq 4$ by Lemma 4, $\max\{d(v_6),d(v_7)\}\geq 4$ by Lemma 5. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-2\times\frac{3}{2}-\frac{5}{4}-3\times\frac{3}{4}-1=\frac{1}{6}>0$. Sixth, suppose $d(f_3)=d(f_5)=d(f_7)=3$. Then f_2, f_4, f_6 and f_8 are good by Lemma 4 and Lemma 5, so $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-3\times\frac{3}{2}-4\times\frac{3}{4}=\frac{1}{6}>0$.

Suppose $f_3(v)=2$. Then without loss of generality, let $d(f_i)=d(f_j)=3$ $(3\leq i<j\leq 7)$. If $f_{5^+}(v)\geq 2$, then $ch^{'}(v)\geq ch(v)-2\times 1-f_{5^+}(v)\times\frac{1}{3}-2\times\frac{3}{2}-(6-f_{5^+}(v))\times 1=\frac{2}{3}f_{5^+}(v)-1\geq 0$. Otherwise, $d(f_t)\leq 4$ for all $2\leq t\leq 8$. If there is at least one good 3-face in $\{f_i,f_j\}$, then each face adjacent to good 3-face is good. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-\frac{5}{4}-\frac{3}{4}-\frac{3}{2}-4\times 1=\frac{1}{6}>0$. Now we suppose both f_i and f_j are bad. If $j=i+1$, then $i\notin\{3,6\}$ by Fig. 1(5) and Lemma 4. Without loss of generality, suppose $i=4$. Then at least two faces in $\{f_2,f_3,f_4\}$ are good by Fig. 1(5) and Lemma 4. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-\frac{3}{2}-3\times 1-\max\{\frac{3}{2}+2\times\frac{3}{4}, \frac{5}{4}+\frac{3}{4}+1\}=\frac{1}{6}>0$. Otherwise, there are two 7^+-vertices in $\{v_i,v_{i+1},v_j,v_{j+1}\}$. So $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-2\times\frac{3}{2}-2\times\frac{3}{4}-3\times 1=\frac{1}{6}>0$.

Suppose $f_3(v)\leq 1$. Then $ch^{'}(v)\geq ch(v)-2\times 1-\frac{1}{3}-f_3(v)\times\frac{3}{2}-(7-f_3(v))\times 1=\frac{2}{3}-\frac{1}{2}f_3(v)\geq 0$.

For Fig. 8(2), note that $f_3(v)\leq 3$, and v sends at most $\frac{3}{2}$ (in total) to f_1 and f_2 by Lemma 8. Suppose $f_3(v)=3$, without loss of generality, let $d(f_4)=d(f_5)=d(f_i)=3$ $(i\in\{6,7\})$. Then v sends at most $\max\{\frac{3}{2}+\frac{1}{2}, \frac{5}{4}+\frac{3}{4}\}=2$ (in total) to f_3 and f_4, and v sends at most $\max\{\frac{3}{2}+\frac{3}{2}+1+\frac{1}{3}, \frac{5}{4}+\frac{3}{2}+2\times\frac{3}{4}, \frac{5}{4}+\frac{5}{4}+\frac{3}{4}+1\}=\frac{13}{3}$ (in total) to f_5, f_6, f_7 and f_8 by Fig. 1(5), Lemma 4 and Lemma 5. So $ch'(v)\geq ch(v)-2\times 1-\frac{3}{2}-2-\frac{13}{3}=\frac{1}{6}>0$.

Suppose $f_3(v)=2$. Then without loss of generality, let $d(f_i)=d(f_j)=3$ $(4\leq i<j\leq 7)$. If there is at least one 5^+-face in $\{f_t|3\leq t\leq 8\}$, then $ch'(v)\geq ch(v)-2\times 1-\frac{3}{2}-2\times\frac{3}{2}-\frac{1}{3}-3\times 1=\frac{1}{6}>0$. Otherwise, $d(f_t)\leq 4$ for all $3\leq t\leq 8$. If there is at least one good 3-face in $\{f_i,f_j\}$, then each 4-face adjacent to good 3-face is good. So $ch'(v)\geq ch(v)-2\times 1-\frac{3}{2}-\frac{3}{2}-\frac{5}{4}-\frac{3}{4}-3\times 1=0$. Now we suppose both f_i and f_j are bad. If $j=i+1$, then $i=5$, f_3,f_4,f_7, and f_8 are good by Fig. 1(5) and Lemma 4. So $ch'(v)\geq ch(v)-2\times 1-\frac{3}{2}-4\times\frac{3}{4}-2\times\frac{3}{2}=\frac{1}{2}>0$. Otherwise, there are two 7^+-vertices in $\{v_i,v_{i+1},v_j,v_{j+1}\}$. So $ch'(v)\geq ch(v)-2\times 1-\frac{3}{2}-2\times\frac{3}{2}-\frac{1}{2}-3\times 1=0$.

Suppose $f_3(v)\leq 1$. Then $ch'(v)\geq ch(v)-2\times 1-\frac{3}{2}-f_3(v)\times\frac{3}{2}-(6-f_3(v))\times 1=\frac{1}{2}-\frac{1}{2}f_3(v)\geq 0$.

For Fig. 8(3), note that $f_3(v)\leq 4$. If $f_3(v)=4$, then $d(f_2)=d(f_5)=d(f_6)=d(f_7)=3$, so $ch'(v)\geq ch(v)-2\times 1-\frac{11}{4}-(\frac{11}{4}+2\times\frac{5}{4})=0$ by Lemma 10.

Suppose $f_3(v)=3$. If $d(f_2)\geq 4$, then $d(f_5)=d(f_6)=d(f_7)=3$, so $ch'(v)\geq ch(v)-2\times 1-(1+\frac{3}{2})-(\frac{11}{4}+2\times\frac{5}{4})=\frac{1}{4}>0$. If $d(f_2)=3$, then v sends at most $\frac{11}{4}$ (in total) to f_1,f_2 and f_3 by Lemma 10. Without loss of generality, let $d(f_5)=3$. If $d(f_6)=3$, then v sends at most 2 (in total) to f_4 and f_5, v sends at most $\frac{3}{4}$ to f_7. So $ch'(v)\geq ch(v)-2\times 1-\frac{11}{4}-2-\frac{3}{2}-\frac{3}{4}-1=0$. If $d(f_7)=3$, then v sends at most $\frac{3}{4}$ to f_4,f_6 and f_8, respectively. So $ch'(v)\geq ch(v)-2\times 1-\frac{11}{4}-2\times\frac{3}{2}-3\times\frac{3}{4}=0$.

Suppose $f_3(v)=2$. Then without loss of generality, let $d(f_i)=d(f_j)=3$ $(i<j)$. If $i=2$, then v sends at most $\frac{11}{4}$ (in total) to f_1,f_2 and f_3, v sends at most $\frac{3}{4}$ to f_{j-1} or f_{j+1}. So $ch'(v)\geq ch(v)-2\times 1-\frac{11}{4}-\frac{3}{2}-\frac{3}{4}-3\times 1=0$. Otherwise, v sends at most $\frac{5}{2}$ (in total) to f_1,f_2 and f_3 by Lemma 8, without loss of generality, let $i=5$. If $j=6$, then v sends at most 2 (in total) to f_4 and f_5. So $ch'(v)\geq ch(v)-2\times 1-\frac{5}{2}-2-\frac{3}{2}-2\times 1=0$. If $j=7$, then v sends at most $\frac{3}{4}$ to f_4 and f_8, respectively. So $ch'(v)\geq ch(v)-2\times 1-\frac{5}{2}-2\times\frac{3}{4}-2\times\frac{3}{2}-1=0$.

Suppose $f_3(v)\leq 1$. If $d(f_2)=3$, then $ch'(v)\geq ch(v)-2\times 1-\frac{11}{4}-5\times 1=\frac{1}{4}>0$. Otherwise, $ch'(v)\geq ch(v)-2\times 1-\frac{5}{2}-\frac{3}{2}-4\times 1=0$.

For Fig. 8(4), note that $f_3(v)\leq 4$. Suppose $f_3(v)=4$. Then $d(f_2)=d(f_3)=d(f_6)=d(f_7)=3$ and $ch'(v)\geq ch(v)-2\times 1-2\times(\frac{11}{4}+\frac{5}{4})=0$ by Lemma 10. Suppose $f_3(v)=3$. Then $ch'(v)\geq ch(v)-2\times 1-(\frac{11}{4}+\frac{5}{4})-\frac{15}{4}=\frac{1}{4}>0$ by Lemma 9. Suppose $f_3(v)=2$. If two 3-faces incident with v are adjacent, then $ch'(v)\geq ch(v)-2\times 1-(\frac{11}{4}+\frac{5}{4})-4\times 1=0$. Otherwise, $ch'(v)\geq ch(v)-2\times 1-2\times\frac{15}{4}=\frac{1}{2}>0$. Suppose $f_3(v)\leq 1$. Then $ch'(v)\geq ch(v)-2\times 1-f_3(v)\times\frac{15}{4}-(2-f_3(v))\times(4\times 1)=\frac{1}{4}f_3(v)\geq 0$.

Case 8. $n_2(v)=1$. Without loss of generality, let v_1 be the unique 2-vertex adjacent to v. First, we consider the case that v_1 is not incident with any 3-face. Note that $f_3(v)\leq 5$.

Suppose $f_3(v)=5$. Then $d(f_2)=d(f_3)=d(f_i)=d(f_6)=d(f_7)=3$ ($i\in\{4,5\}$), and at least two faces in $\{f_3,f_i,f_6\}$ are good by Fig. 1(5) and Lemma 5. If $\min\{f_1,f_8\}\geq 5$, then $ch^{'}(v)\geq ch(v)-1-2\times\frac{1}{3}-3\times\frac{3}{2}-2\times\frac{5}{4}-1=\frac{1}{3}\geq 0$. Otherwise, $\min\{f_1,f_8\}\leq 4$, without loss of generality, let $d(f_1)=4$. If $d(v_2)=3$, then f_3,f_i,f_6 and f_7 are good by Lemma 6, so $ch^{'}(v)\geq ch(v)-1-1-\frac{3}{2}-4\times\frac{5}{4}-2\times\frac{3}{4}=0$. If $d(v_2)\geq 4$, we may assume that $d(f_8)\geq 5$ or $d(v_8)\geq 4$, then $ch^{'}(v)\geq ch(v)-1-\frac{3}{4}-3\times\frac{5}{4}-\frac{3}{2}-1-\max\{\frac{3}{2}+\frac{1}{3},\frac{5}{4}+\frac{3}{4}\}=0$.

Suppose $f_3(v)=4$. Then there is at least one 3-face in $\{f_2,f_7\}$, without loss of generality, let $d(f_2)=d(f_i)=d(f_j)=d(f_t)=3$, where $2<i<j<t$ and $t\in\{6,7\}$. If $f_{5^+}(v)\geq 2$, then $ch^{'}(v)\geq ch(v)-1-f_{5^+}(v)\times\frac{1}{3}-4\times\frac{3}{2}-(4-f_{5^+}(v))\times 1=\frac{2}{3}f_{5^+}(v)-1\geq 0$. Then $f_{5^+}(v)\leq 1$. We need to consider two cases. First, suppose there is one 5^+-face in $\{f_1\}\cup\{f_x|t+1\leq x\leq 8\}$, then at least two faces in $\{f_3,f_4,f_5,f_6\}$ are good by Fig. 1(5) and Lemma 5. So $ch^{'}(v)\geq ch(v)-1-\frac{1}{3}-\max\{2\times\frac{3}{2}+2\times\frac{5}{4}+3\times 1, 3\times\frac{3}{2}+\frac{5}{4}+2\times 1+\frac{3}{4}, 4\times\frac{3}{2}+1+2\times\frac{3}{4}\}=\frac{1}{6}>0$. Second, suppose $d(f_1)=d(f_x)=4$ for all $t+1\leq x\leq 8$. If $d(v_2)=3$ or $d(v_y)=3$ for all $t+1\leq y\leq 8$, then v is incident with at least three good 3-faces and one good 4-face by Lemma 6. So $ch^{'}(v)\geq ch(v)-1-\frac{3}{2}-3\times\frac{5}{4}-3\times 1-\frac{3}{4}=0$. Otherwise, $d(v_2)\geq 4$ and $\max\{d(v_y)|t+1\leq y\leq 8\}\geq 4$, that is, there are at least two good 4-faces in $\{f_1\}\cup\{f_x|t+1\leq x\leq 8\}$. Then $f_{5^+}(v)=1$ or at least two faces in $\{f_3,f_4,f_5,f_6\}$ are good. So $ch^{'}(v)\geq ch(v)-1-\max\{4\times\frac{3}{2}+1+2\times\frac{3}{4}+\frac{1}{3}, 2\times\frac{3}{2}+2\times\frac{5}{4}+2\times 1+2\times\frac{3}{4}, 3\times\frac{3}{2}+\frac{5}{4}+1+3\times\frac{3}{4}, 4\times\frac{3}{2}+4\times\frac{3}{4}\}=0$.

Suppose $f_3(v)=3$. If $f_{5^+}(v)\geq 1$, then $ch^{'}(v)\geq ch(v)-1-f_{5^+}(v)\times\frac{1}{3}-3\times\frac{3}{2}-(5-f_{5^+}(v))\times 1=\frac{2}{3}f_{5^+}(v)-\frac{1}{2}\geq 0$. Otherwise, at least two faces incident with v are good by Lemma 5 and Lemma 6. So $ch^{'}(v)\geq ch(v)-1-\max\{2\times\frac{3}{2}+\frac{5}{4}+\frac{3}{4}+4\times 1, 3\times\frac{3}{2}+2\times\frac{3}{4}+3\times 1\}=0$. Suppose $f_3(v)\leq 2$. Then $ch^{'}(v)\geq ch(v)-1-f_3(v)\times\frac{3}{2}-(8-f_3(v))\times 1=1-\frac{1}{2}f_3(v)\geq 0$.

Next, we consider the case that v_1 is incident with a 3-face. Then $f_3(v)\leq 6$, and the other 3-faces incident with v are good by Fig. 1(2). If $f_3(v)=6$, then $d(f_1)=d(f_2)=d(f_3)=d(f_5)=d(f_6)=d(f_7)=3$, v sends at most $\frac{1}{2}$ to f_4, and v sends at most $\frac{3}{4}$ to f_8. So $ch^{'}(v)\geq ch(v)-1-\frac{3}{2}-5\times\frac{5}{4}-\frac{1}{2}-\frac{3}{4}=0$. Suppose $f_3(v)\leq 5$. If $f_{5^+}(v)\geq 1$, then $ch^{'}(v)\geq ch(v)-1-\frac{3}{2}-4\times\frac{5}{4}-f_{5^+}(v)\times\frac{1}{3}-(3-f_{5^+}(v))\times 1=\frac{2}{3}f_{5^+}(v)-\frac{1}{2}\geq 0$. Otherwise, $f_{5^+}(v)=0$. If $f_3(v)=5$, then at least two 4-faces incident with v are good. So $ch^{'}(v)\geq ch(v)-1-\frac{3}{2}-4\times\frac{5}{4}-1-2\times\frac{3}{4}=0$. If $f_3(v)\leq 4$, then at least one 4-face incident with v is good. So $ch^{'}(v)\geq ch(v)-1-\frac{3}{2}-(f_3(v)-1)\times\frac{5}{4}-(8-f_3(v)-1)\times 1-\frac{3}{4}=1-\frac{1}{4}f_3(v)\geq 0$.

Case 9. $n_2(v)=0$. Note that $f_3(v)\leq 6$. If $f_3(v)\leq 4$, then $ch^{'}(v)\geq ch(v)-f_3(v)\times\frac{3}{2}-(8-f_3(v))\times 1=2-\frac{1}{2}f_3(v)\geq 0$. Suppose $f_3(v)=5$. Then without loss of generality, let $d(f_1)=d(f_2)=d(f_4)=d(f_5)=d(f_i)=3$ ($i\in\{6,7\}$). If $\min\{d(v_2),d(v_5)\}=3$, then v is incident with at most four bad 3-faces by Lemma 7. So $ch^{'}(v)\geq ch(v)-4\times\frac{3}{2}-\frac{5}{4}-2\times 1-\frac{3}{4}=0$. Otherwise, $\min\{d(v_2),d(v_5)\}\geq 4$, then $d(f_3)\geq 5$ or $\max\{d(v_1),d(v_3),d(v_4)\}\geq 4$ by

Fig .1(6), so $ch'(v) \geq ch(v) - 4 \times \frac{3}{2} - \frac{5}{4} - 2 \times 1 - \frac{3}{4} = 0$. Suppose $f_3(v) = 6$. Then without loss of generality, let $d(f_1) = d(f_2) = d(f_3) = d(f_5) = d(f_6) = d(f_7) = 3$. If $\min\{d(v_2), d(v_3), d(v_6), d(v_7)\} = 3$, then v is incident with at most four bad 3-faces by Lemma 7. So $ch'(v) \geq ch(v) - 4 \times \frac{3}{2} - 2 \times \frac{5}{4} - 2 \times \frac{3}{4} = 0$. Otherwise, $\min\{d(v_2), d(v_3), d(v_6), d(v_7)\} \geq 4$. If $\max\{d(v_1), d(v_4), d(v_5), d(v_8)\} \geq 4$, then $ch'(v) \geq ch(v) - 3 \times \frac{3}{2} - 3 \times \frac{5}{4} - 1 - \frac{3}{4} = 0$. If $d(v_1) = d(v_4) = d(v_5) = d(v_8) = 3$, then $\min\{d(v_2), d(v_3), d(v_6), d(v_7)\} \geq 7$, so $ch'(v) \geq ch(v) - 4 \times \frac{3}{2} - 2 \times 1 - 2 \times 1 = 0$.

Hence we complete the proof of the theorem.

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