Remarks on a conjecture of Barát and Tóth

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Abstract

In 2010, Barát and Tóth verified that any r-critical graph with at most $r + 4$ vertices has a subdivision of K_r. Based in this result, the authors conjectured that, for every positive integer c, there exists a bound $r(c)$ such that for any r, where $r \geqslant r(c)$, any r-critical graph on $r + c$ vertices has a subdivision of K_r. In this note, we verify the validity of this conjecture for $c = 5$, and show counterexamples for all $c \geqslant 6$.

Keywords: colour-critical graphs; Hajós conjecture; Albertson conjecture

1   Introduction

In this note, we discuss a few points arising from the interesting paper of Barát and Tóth [2]. Their motivation is Albertson's Conjecture that if the chromatic number $\chi(G)$ of a graph G is r, then the crossing number $\mathrm{cr}(G)$ of G is at least that of K_r. In other words, among all graphs with chromatic number at least r, the one with smallest crossing number is K_r. This is trivial for $r \leqslant 4$ and follows from the Four Colour Theorem for $r = 5$. It was proved by Albertson, Cranston, and Fox [1] for $r \leqslant 12$ (precisely the values of r for which $\mathrm{cr}(K_r)$ is currently known). Barát and Tóth extended this to $r \leqslant 16$ by using their new lower bound on crossing numbers to show that every r-chromatic graph other than K_r has crossing number at least the conjectured value of $\mathrm{cr}(K_r)$.

Albertson's Conjecture is related to Hajós' Conjecture that every r-chromatic graph contains a subdivision of K_r. The Hajós Conjecture obviously implies the Albertson

Conjecture. Unfortunately, the Hajós Conjecture is false for $r \geqslant 7$ [3] and remains open for $r = 5, 6$. There is more detailed information to be found in [8].

Barát and Tóth were interested in graphs that satisfy the Hajós Conjecture. Recall that a graph G is r-*critical* if $\chi(G) = r$ but, for every edge e of G, $\chi(G-e) < r$. Dirac [5, 7] proved that no r-critical graph has $r + 1$ vertices and that $C_5 \vee K_{r-3}$ is the only r-critical graph with $r + 2$ vertices. (Here $G \vee H$ is the *join* of G and H; it is the graph obtained from the disjoint union of G and H by adding all edges having one end in G and one end in H.) Gallai [9] extended this by showing that, for $r \geqslant 4$, there are only two r-critical graphs with $r + 3$ vertices.

An observation of Barát and Tóth is that, for $r \geqslant 6$, there are precisely 22 r-critical graphs with $r + 4$ vertices. In the next section, we prove the following, a straightforward generalization of Barát and Tóth's observation.

Theorem 1. *Let c be a positive integer. There are numbers $n(c)$ and $r(c)$ so that, for any $r \geqslant r(c)$, there are precisely $n(c)$ r-critical graphs with $r + c$ vertices.*

In particular, $r(5) = 7$ and $n(5) = 395$. Moreover, every r-critical graph with $r + 5$ vertices has a subdivision of K_r.

Barát and Tóth also proposed the following conjecture.

Conjecture 2. For every positive integer c, there is a number $r(c)$ so that, if $r \geqslant r(c)$, then any r-critical graph with $r + c$ vertices has a subdivision of K_r.

Theorem 1 shows this conjecture is true for $c = 5$. We shall see in the next section that the methods used to prove Theorem 1 combine with some standard examples to demonstrate that this conjecture is not true for any $c \geqslant 6$.

The note concludes with some remarks on the assertion of Barát and Tóth that the Catlin examples satisfy the Albertson Conjecture. We explain why their proof is not valid, and so it is still open whether the Catlin examples satisfy the Albertson Conjecture.

2   Results

This section contains the proof of Theorem 1 and the proof of Conjecture 2 for $c = 5$, along with its refutation for every $c \geqslant 6$. The main theoretical tool is the following result of Gallai [9]. A vertex v in a graph G is *universal* if v is adjacent to every other vertex in G.

Theorem 3 (Gallai [9]). *Let G be an r-critical, n-vertex graph with $r \geqslant 3$. Then G contains at least $\left\lceil \frac{3}{2}(\frac{5}{3}r - n) \right\rceil$ universal vertices. In particular, if $r > \frac{3}{2}c$, then any r-critical graph with $r + c$ vertices has a universal vertex.*

We are now ready for the proof of Theorem 1.

Proof of Theorem 1. Theorem 3 implies that, for $r > (3c)/2$, an r-critical graph with $r + c$ vertices has a universal vertex. Since it is well-known that $G \vee v$ is $(r + 1)$-critical

if and only if G is r-critical, for $r > (3c)/2$, the number of r-critical graphs with $r + c$ vertices is the same as the number of $(r-1)$-critical graphs with $(r-1)+c$ vertices; thus, $r(c) \leqslant \lfloor (3c)/2 \rfloor$ and $n(c)$ is the number of $(\lfloor (3c)/2 \rfloor)$-critical graphs with $\lfloor (3c)/2 \rfloor + c$ vertices. In particular, $r(5) \leqslant 7$.

For the second part of Theorem 1, we consider the special case $c = 5$. To see that $n(5) = 395$, Royle's table [13] of small critical graphs shows there are exactly the following graphs that are, for some r, r-critical, with $r + 5$ vertices, and have no universal vertex:

(i) twenty-one 4-critical graphs on 9 vertices;

(ii) one hundred and forty-one 5-critical graphs on 10 vertices;

(iii) two hundred and thirty-one 6-critical graphs on 11 vertices; and

(iv) two 7-critical graphs on 12 vertices.

Because $r(5) \leqslant 7$, the two 7-critical graphs in this list show $r(5) = 7$. For $r \geqslant 8$, every r-critical graph with $r + 5$ vertices has a universal vertex and so is the join of some K_s with one of the 395 listed graphs. That is, $n(5) = 395$.

Moreover, it suffices to show that each one of the 395 graphs in the list has a subdivision of the appropriate K_r. Dirac [6] proved the Hajós Conjecture for $r = 4$, so each one of the twenty-one 4-critical graphs of the list has a subdivision of K_4. For $r = 5$, Mader's Theorem [11] that any graph with n vertices and at least $3n - 5$ edges has a subdivision of K_5 is helpful. For each one of the one hundred and forty-one 5-critical graphs of the list we counted its edges from the adjacency list provided by Royle's table. We checked by hand those graphs with fewer than $3n - 5$ edges; every one had a subdivision of K_5. Finally, all the two hundred and thirty-one 6-critical graphs were checked by hand, and for each one a K_6-subdivision was found. Most of the 6-critical graphs have approximately the same structure, which made them less difficult to verify. A typical example is shown in Figure 1. This is Graph 391 in the table. (Royle (personal communication) independently verified by computer program that these 395 graphs satisfy the Hajós Conjecture.) ∎

We now turn our attention to showing that Conjecture 2 is false for every $c \geqslant 6$. In order to do this, we use the following subgraphs of Catlin's graphs $L(_k C_5)$ [3].

Family F_c. For $c \geqslant 6$, let F_c be the graph whose vertex set consists of five non-empty pairwise disjoint sets A_1, A_2, C_1, C_2, and C_3 where $|C_1| = |C_2| = |C_3| = 3$ and $|A_1| = |A_2| = c - 4$, such that these sets induce cliques in F_c. The sets A_1, A_2, C_1, C_2, and C_3 are joined in the following way: $A_1 \vee C_1$, $A_1 \vee C_2$, $C_1 \vee A_2$, $A_2 \vee C_3$, and $C_2 \vee C_3$. Figure 2 shows a scheme of the graphs F_c.

In the following, we prove that, for $c \geqslant 6$, F_c is a $(c + 1)$-critical graph that does not contain a subdivision of $K_{(c+1)}$. Since F_c has $2c + 1$ vertices and is $(c+1)$-critical, the join $F_c \vee K_t$ has $2c + t + 1$ vertices, is $(c + t + 1)$-critical, and does not contain a subdivision of $K_{(c+t+1)}$. Thus, Conjecture 2 is false for every $c \geqslant 6$.
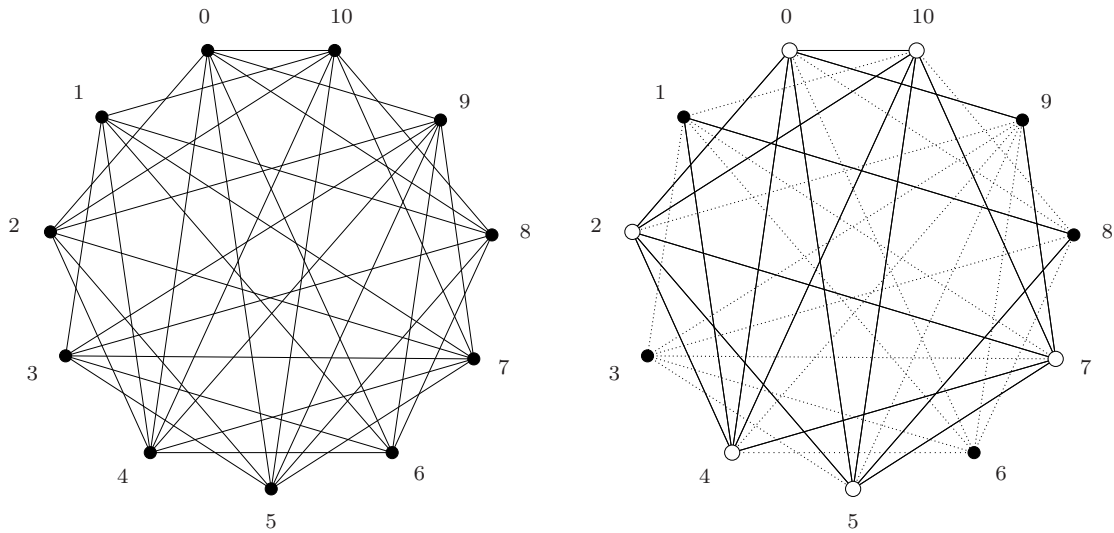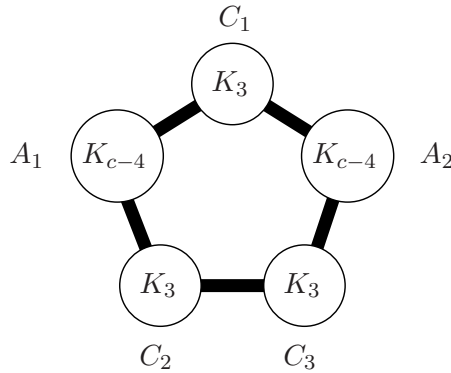
Figure 1: A 6-critical graph and a subdivision of K_6 in it.

Figure 2: Scheme of a graph F_c. Heavy edges indicate that every vertex in one set is adjacent to every vertex in the other.

Proposition 4. *For $c \geqslant 6$, $\chi(F_c) = c + 1$.*

Proof. Since every independent set in F_c has at most two vertices, we see that

$$\chi(F_c) \geqslant \frac{|V(F_c)|}{2} = \frac{2c+1}{2} = c + \frac{1}{2}.$$

For $c \geqslant 6$, the graph F_c can be assigned a $(c+1)$-coloring such that the vertices in the sets $A_1, A_2, C_1, C_2,$ and C_3 receive the following colors: $C_1 = \{1, 2, 3\}$, $C_2 = \{1, 4, 5\}$, $C_3 = \{2, 6, 7\}$, $A_1 = \{6, 7, 8, 9, \dots, c+1\}$, and $A_2 = \{4, 5, 8, 9, \dots, c+1\}$. ∎

Proposition 5. *For $c \geqslant 6$, F_c is $(c+1)$-critical.*

Proof. For $c \geqslant 6$, we show that, after the removal of an arbitrary edge e, the graph $F_c - e$ is c-colorable. There are 6 different cases to consider.

Case 1. $e \in E(C_1)$.

For $c \geqslant 6$, the vertex subsets of $F_c - e$ can be assigned the following colors: $C_1 = \{1, 2\}$, $C_2 = \{1, 6, 7\}$, $C_3 = \{2, 4, 5\}$, $A_1 = \{4, 5, 8, \ldots, c + 1\}$, and $A_2 = \{6, 7, 8, \ldots, c + 1\}$. Observe that the color 3 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

Case 2. Edge e has one endpoint in the set C_1 and the other endpoint in the set A_1.

For $c \geqslant 6$, the vertex subsets of $F_c - e$ can be assigned the following colors: $C_1 = \{1, 2, 3\}$, $C_2 = \{2, 6, 7\}$, $C_3 = \{1, 3, 5\}$, $A_1 = \{1, 5, 8, \ldots, c + 1\}$, and $A_2 = \{6, 7, 8, \ldots, c + 1\}$. Observe that the color 4 does not appear in this coloring. Therefore, $F_c - e$ is colorable with c colors.

The remaining cases, listed below, are treated analogously.

Case 3. $e \in E(A_1)$.

Case 4. Edge e has one endpoint in the set A_1 and the other endpoint in the set C_2.

Case 5. $e \in E(C_2)$.

Case 6. Edge e has one endpoint in the set C_2 and the other endpoint in the set C_3. ∎

The next proposition completes our analysis of the graph F_c.

Proposition 6. *For $c \geqslant 6$, F_c does not contain a subdivision of $K_{(c+1)}$.*

Proof. Catlin [3] showed that the graphs F_6 and F_7 do not have a subdivision of K_7 and K_8, respectively. Thus, we may suppose $c \geqslant 8$ and F_c has a subdivision of K_{c+1}. Let W be the set of vertices with degree c in a subdivision of K_{c+1}. Within the subdivision, any two vertices of W are joined by c pairwise internally-disjoint paths.

Only if $c = 8$ does $C_1 \cup C_2 \cup C_3$ have enough vertices to contain W. In this case, $W = V(C_1 \cup C_2 \cup C_3)$ and there must be nine disjoint paths through A_1 representing the edges from C_1 to C_2. Since A_1 has only four vertices, this is impossible.

Thus, in all cases, at least one vertex of W is in $A_1 \cup A_2$. Since $V(C_1 \cup C_2)$ is a cut-set in F_c of size $6 < c$, there cannot be vertices of W in both $V(A_1)$ and $V(A_2)$. Therefore, we may assume $W \cap V(A_1) \neq \emptyset$ and $W \cap V(A_2) = \emptyset$.

Since $V(C_1 \cup C_2)$ also separates A_1 from C_3, we deduce that $W \cap V(C_3) = \emptyset$. Therefore, $W \subseteq V(C_1 \cup A_1 \cup C_2)$.

Since $|W| = c + 1$ and $|V(C_1 \cup A_1 \cup C_2)| = c + 2$, exactly one vertex v in $C_1 \cup A_1 \cup C_2$ is not in W. It follows that there are at least 6 (5 is enough) internally-disjoint $C_1 C_2$-paths in K_{c+1} representing the edges from $W \cap V(C_1)$ to $W \cap V(C_2)$. Since at most one vertex of A_1 is not in W, at most one of the 6 $C_1 C_2$-paths can go through A_1. In the other direction, at most three can go through C_3. Thus there are too few paths for F_c to have a subdivision of K_{c+1}.

To conclude the proof, there is a subdivision of K_c consisting of the $c - 4$ vertices in A_1, three vertices in C_1, and one vertex in C_2. This uses only three internally disjoint $C_1 C_2$-paths through $C_3 \cup A_2$. ∎

3   Catlin graphs and Albertson's Conjecture

In their paper, Barát and Tóth claim (Lemma 16) that Catlin's graphs satisfy Albertson's Conjecture. In the proof, they use (asymptotic versions of) the not-yet proved conjectures on the values of the crossing numbers of K_{2k}, K_k, and $K_{k,k}$. We prove the following, without any assumption about the crossing number of K_k and $K_{k,k}$.

Theorem 7. *The graph F_c has crossing number at least that of K_{c+1}.*

Proof.   We have already exhibited a subdivision of K_c in F_c. This subdivision uses three internally disjoint C_2C_1-paths through A_2, but is otherwise disjoint from A_2. It is known [10, 12] that the crossing number of K_c, denoted $\operatorname{cr}(K_c)$, is at most

$$f(c) = \frac{1}{4} \left\lfloor \frac{c}{2} \right\rfloor \left\lfloor \frac{c-1}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor \left\lfloor \frac{c-3}{2} \right\rfloor .$$

Since, $\operatorname{cr}(K_c) \leqslant f(c)$, there is a vertex v of K_c so that the edges incident with v are involved in at most $\lfloor 4f(c)/c \rfloor$ crossings.

Let D be a drawing of K_c with at most $f(c)$ crossings. We obtain a drawing D' of K_{c+1} by adding a new vertex v' close to v and joined to v with no crossings. For each other vertex w of K_c, we add the edge $v'w$ alongside vw. These latter edges cross the same number of edges to $K_c - v$ as those incident with v do, which is to say at most $\lfloor 4f(c)/c \rfloor$ crossings. In addition, Woodall [14] observes that the edges at v' can be drawn so as to cross the edges incident with v at most $\lfloor (c-1)/2 \rfloor \lfloor (c-2)/2 \rfloor$ times. Thus,

$$\operatorname{cr}(K_{c+1}) \leqslant \operatorname{cr}(K_c) + \left\lfloor \frac{4f(c)}{c} \right\rfloor + \left\lfloor \frac{c-1}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor .$$

On the other hand, the crossing number of K_k is known [4] to be at least $0.8594\, f(k)$. The three internally disjoint paths through A_2 are all incident with a common node of the subdivision of K_c, so no crossing between two of them is counted in the crossings of this K_c. Therefore, all the crossings in the K_{c-1} induced by $A_2 \cup C_3$ are additional. We deduce that the crossing number of F_c is at least $\operatorname{cr}(K_c) + 0.8594\, f(c-1)$.

It follows that, if

$$.8594\, f(c-1) \geqslant \left\lfloor \frac{4f(c)}{c} \right\rfloor + \left\lfloor \frac{c-1}{2} \right\rfloor \left\lfloor \frac{c-2}{2} \right\rfloor ,$$

then the crossing number of F_c is at least that of K_{c+1}. We note that, since the left side is degree 4 in c while the right is degree 3 in c ($f(c)$ is degree 4 in c), this certainly is true for c large enough. In fact, this holds for $c \geqslant 12$. For $c < 12$, already [1] shows that Albertson's Conjecture is true, so in these cases also, $\operatorname{cr}(F_c) \geqslant \operatorname{cr}(K_{c+1})$. ∎

Unfortunately, a straightforward analogue of this argument does not show that the Catlin graphs satisfy the Albertson Conjecture.

Acknowledgments

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