Non-existence of point-transitive 2-$(106, 6, 1)$ designs

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Abstract

Let $\mathcal{S}$ be a linear space with 106 points, with lines of size 6, and let G be an automorphism group of $\mathcal{S}$. We prove that G cannot be point-transitive. In other words, there exists no point-transitive 2-$(106, 6, 1)$ designs.

Keywords: linear space; design; point-transitive

1 Introduction

For positive integers v and b satisfying $b \geqslant v \geqslant 2$, a *finite linear space* $\mathcal{S}$ is an incidence structure $(\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ of v points and a collection $\mathcal{L}$ of b distinguished subsets of $\mathcal{P}$ called lines, with sizes $\geqslant 2$ such that any two points are incident with exactly one line. Let α be a point of $\mathcal{P}$, and k be a positive integer. Then r_α^k denotes the number of lines having size k through α, b^k the number of lines of size k, and r_α the number of all lines through α, called the *degree* of α. If all lines have a constant size k, then we say that $\mathcal{S}$ is *regular*, so it is a 2-$(v, k, 1)$ design.

Let Δ be a subset of $\mathcal{P}$ with $|\Delta| \geqslant 2$, $\mathcal{L}_\Delta = \{\lambda \cap \Delta : |\lambda \cap \Delta| \geqslant 2 \text{ for } \lambda \in \mathcal{L}\}$. Then $(\Delta, \mathcal{L}_\Delta)$ forms an incidence structure, and the induced structure is a linear space. We are interested in the case that Δ is $\mathrm{Fix}(g)$ (or $\mathrm{Fix}(H)$), the set of fixed points of $g \in G$ (or $H \leqslant G$) on $\mathcal{P}$. An *automorphism* of $\mathcal{S}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{L}$ invariant. The full automorphism group of $\mathcal{S}$ is denoted by $\mathrm{Aut}(\mathcal{S})$ and any subgroup of $\mathrm{Aut}(\mathcal{S})$ is called an *automorphism group* of $\mathcal{S}$. We say that the automorphism group G of $\mathcal{S}$

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is *point-transitive* if G acts transitively on the set of points. Similarly, G is said to be *line-transitive* if G acts transitively on the set of lines.

Four 2-$(91, 6, 1)$ designs have been found by Mills, McCalla and Colbourn ([6, 7, 12]). All of them have a cyclic automorphism group of order 91, and so they have point-transitive automorphism groups. In 1989, Camina and Di Martino ([3]) proved that any automorphism group of a point-transitive 2-$(91, 6, 1)$ design is the natural split extension of a cyclic group of order 91 by a cyclic group of order d, where $d \mid 12$. Later, Janko and Tonchev ([10]) showed that any cyclic 2-$(91, 6, 1)$ design (i.e. one having an automorphism group of order 91) admitting an automorphism group whose order is larger than 91 is one of the four known designs. Here we are going to discuss the 2-$(106, 6, 1)$ designs, where the number of points is also a product of two distinct primes, and 6 is the largest line-size for a non-trivial regular linear space with 106 points ([8]). The only known 2-$(106, 6, 1)$ design found by Mills ([11]) is not point-transitive, which is one of Miscellaneous Constructions ([8]) and has a cyclic automorphism groups of order 53. It is a question whether there exist point-transitive 2-$(106, 6, 1)$ designs, just like the 2-$(91, 6, 1)$ designs. In this paper, we prove that there is no 2-$(106, 6, 1)$ designs admitting a point-transitive automorphism group.

Theorem 1. *Let $\mathcal{S}$ be a* 2-$(106, 6, 1)$ *design, and G be an automorphism group of $\mathcal{S}$. Then G cannot be transitive on points of $\mathcal{S}$.*

Our paper is organized as follows. Section 2 presents some preliminary results and notation. In Section 3, by considering the number of fixed points of an involutive automorphism, we bound the size of the 2-part of $|\operatorname{Aut}(\mathcal{S})|$. In Section 4, we get a bound on $|\operatorname{Aut}(\mathcal{S})|$ and prove Theorem 1.

2 Preliminary results and notation

Let $\mathcal{S}$ be a linear space with v points, K be a set of positive integers such that $v \geqslant k$ for every $k \in K$ and the set of line-sizes of $\mathcal{S}$ is contained in K. Note that it is not required that there is a line of size k for any $k \in K$. Let α be a point of $\mathcal{P}$, then

$$\sum_{k \in K} (k-1) r_\alpha^k = v - 1 \tag{1}$$

and for each $k \in K$,

$$\sum_{\alpha \in \mathcal{P}} r_\alpha^k = k \cdot b^k. \tag{2}$$

In particular, if $\mathcal{S}$ is a non-trivial finite regular linear space, then the following result is well-known.

Lemma 2. [5, Lemma 2.1] *Let $\mathcal{S}$ be a non-trivial finite regular linear space. Then*

$$r = \frac{v-1}{k-1}, \quad b = \frac{v(v-1)}{k(k-1)},$$

and

$$k(k-1)+1 \leqslant v,$$

where k is the line size of $\mathcal{S}$, and r is the number of lines through a point.

The following results are very useful for the proof of Theorem 1.

Lemma 3. [4, Lemma 1] *Let $\mathcal{S}$ be a finite regular linear space, G an automorphism group of $\mathcal{S}$, and $H \neq 1$ a subgroup of G. Then $|\mathrm{Fix}(H)| \leqslant r$ unless every point lies on a fixed line and then $|\mathrm{Fix}(H)| \leqslant r+k-3$.*

Lemma 4. [3, Lemma 1] *Let $\mathcal{S}$ be a linear space, α a point of $\mathcal{S}$, and r_α be the degree of α. Then all lines of size $> r_\alpha$ contain α, and for any point β of $\mathcal{P}$, $\beta \neq \alpha$, the number of lines of size $> r_\alpha$ containing β is at most one.*

Throughout this paper, we assume that the following hypothesis holds:

Hypothesis: Let $\mathcal{S}=(\mathcal{P},\mathcal{L})$ be a 2-$(106,6,1)$ design, and G be a point-transitive subgroup of $\mathrm{Aut}(\mathcal{S})$. Let $N:Q$ be the semidirect product of groups N by Q, $N \times Q$ the direct product of groups N and Q, $|G|_p$ the p-part of $|G|$, and $|G|_{p'}$ the p'-part of $|G|$.

3 The 2-part of $|\mathrm{Aut}(\mathcal{S})|$

In this section, our aim is to obtain the maximal size of the 2-part of $|\mathrm{Aut}(\mathcal{S})|$. We begin this section with some information given in [3] about the linear spaces. Assume that $2 \mid |\mathrm{Aut}(\mathcal{S})|$ and t is an involution of $\mathrm{Aut}(\mathcal{S})$. Let $\mathcal{D}=(\mathrm{Fix}(t),\mathcal{L}_{\mathrm{Fix}(t)})$ be the linear space induced by $\mathrm{Fix}(t)$ and $K=\{2,4,6\}$ containing the set of its line sizes. In view of (1), we get

$$r_\alpha^2+3r_\alpha^4+5r_\alpha^6=|\mathrm{Fix}(t)|-1, \tag{3}$$

for each $\alpha \in \mathrm{Fix}(t)$. Since a non-fixed point of t cannot be on two fixed lines of it, all the non-fixed points t on its fixed lines of $\mathcal{S}$ are distinct. Thus

$$4b^2+2b^4 \leqslant 106-|\mathrm{Fix}(t)|. \tag{4}$$

Combing (2) and (4), we obtain

$$2\sum_{\alpha\in\mathrm{Fix}(t)} r_\alpha^2+\frac{1}{2}\sum_{\alpha\in\mathrm{Fix}(t)} r_\alpha^4 \leqslant 106-|\mathrm{Fix}(t)|. \tag{5}$$

Now for each point $\alpha \in \mathrm{Fix}(t)$, define the *weight* ([3]) $\omega(\alpha)$ of α

$$\omega(\alpha)=2r_\alpha^2+\frac{1}{2}r_\alpha^4.$$

So that (5) can be written as

$$\sum_{\alpha\in\mathrm{Fix}(t)} \omega(\alpha) \leqslant 106-|\mathrm{Fix}(t)|. \tag{6}$$

If $r_\alpha^2=x, r_\alpha^4=y$ and $r_\alpha^6=z$, then we say that α is of type (x,y,z).

Lemma 5. *Assume that the* Hypothesis *holds and let* t *be an involution of* $\mathrm{Aut}\,\mathcal{S}$. *Then* $|\mathrm{Fix}(t)| \neq 18, 20$ *or* 22.

Proof. We prove this Lemma by dealing with the three cases separately.

(1) If $|\mathrm{Fix}(t)| = 22$, then according to inequality (6), $\mathrm{Fix}(t)$ has a point α_0 such that $\omega(\alpha_0) \leqslant \frac{106-22}{22} = \frac{42}{11}$. Thus

$$2r^2_{\alpha_0} + \frac{1}{2}r^4_{\alpha_0} \leqslant \frac{42}{11}.$$

Recall that for any $\alpha \in \mathrm{Fix}(t)$, we have $r^2_\alpha + 3r^4_\alpha + 5r^6_\alpha = 21$ from (3). This implies that α_0 is of type $(0,7,0), (0,2,3)$ or $(1,0,4)$.

If α_0 is of type $(1,0,4)$, then α_0 lies on all lines of size 6 and $r^6_\alpha \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$ by Lemma 4. Precisely, the 20 points on the lines of size 6 through α_0 are of type $(x, y, 1)$, where x and y are non-negative integers. Thus all these 20 points have weight $\geqslant \frac{9}{2}$, and they contribute at least $20 \cdot \frac{9}{2} = 90$ to the total weight, contradicting (6). Similar discussion leads to have a contradiction for type $(0,2,3)$. Therefore, there must be a point $\alpha_0 \in \mathrm{Fix}(t)$ of type $(0,7,0)$, but there is no point of type $(0,2,3)$ or $(1,0,4)$. Moreover, $\omega(\alpha) \geqslant \frac{9}{2}$ for any $\alpha \in \mathrm{Fix}(t)$ with $r^6_\alpha \geqslant 1$.

If all points of $\mathrm{Fix}(t)$ are of type $(0,7,0)$, then $\mathcal{D}$ is a 2-$(22,4,1)$ design. But then $b = \frac{v(v-1)}{k(k-1)}$ is not an integer. Thus, there exists a point $\beta \in \mathrm{Fix}(t)$ that is not of type $(0,7,0)$.

(1.1) Suppose that $r^6_\beta = 0$. Then $r^2_\beta \geqslant 3$ and $\omega(\beta) \geqslant 9$. Let β_1, β_2 and β_3 be three distinct points such that $\{\beta, \beta_i\}$ $(i = 1,2,3)$ are lines of size 2. For $i = 1, 2$ and 3, $\omega(\beta_i) \geqslant 9$ if $r^6_{\beta_i} = 0$, and $\omega(\beta_i) \geqslant \frac{9}{2}$ if $r^6_{\beta_i} \neq 0$. If $r^6_{\beta_i} = 0$ for all $i = 1, 2$ and 3, then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant \omega(\alpha_0) + \omega(\beta) + \sum_{i=1}^{3} \omega(\beta_i) + 17 \cdot \frac{7}{2} = 99,$$

contrary to inequality (6). If there exists one point, say β_1, such that $r^6_{\beta_1} \neq 0$, let λ be a line of size 6 through β_1. Then $\omega(\alpha) \geqslant \frac{9}{2}$ for all $\alpha \in \lambda$. Hence

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant \omega(\alpha_0) + \omega(\beta) + \sum_{\alpha \in \lambda} \omega(\alpha) + 14 \cdot \frac{7}{2} = 88 + \frac{1}{2},$$

which is impossible.

(1.2) Suppose that $r^6_\beta \neq 0$. If $r^2_\beta = 0$, then β is of type $(0,2,3)$, a contradiction. Thus there is at least one line $\{\beta, \beta_1\}$ of size 2 and one line λ of size 6 through β. If $r^6_{\beta_1} \neq 0$, let λ_1 be a line of size 6 through β_1. Then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant \omega(\alpha_0) + \sum_{\alpha \in \lambda \cup \lambda_1} \omega(\alpha) + 10 \cdot \frac{7}{2} > 84,$$

which is a contradiction. If $r^6_{\beta_1} = 0$, then $\omega(\beta_1) \geqslant 9$ and

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant \omega(\alpha_0) + \omega(\beta_1) + \sum_{\alpha \in \lambda} \omega(\alpha) + 14 \cdot \frac{7}{2} = 88 + \frac{1}{2},$$

which is impossible. Therefore, $|\mathrm{Fix}(t)| \neq 22$.

(2) If $|\mathrm{Fix}(t)| = 20$, then $\mathrm{Fix}(t)$ has a point α_0 such that $\omega(\alpha_0) \leqslant \frac{106-20}{20} = \frac{43}{10}$. From (3), we have α_0 is of type $(0, 3, 2)$ or $(1, 1, 3)$.

If α_0 is of type $(0, 3, 2)$, then α_0 lies on all lines of size 6 and $r^6_\alpha \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$ by Lemma 4. Exactly, the 10 points on the lines of size 6 through α_0 are of type $(m, n, 1)$ having weight at least 6, and the 9 points on the lines of size 4 through α_0 are of type $(x, y, 0)$ having weight at least 5, where m, n, x and y are non-negative integers. Then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant \omega(\alpha_0) + 10 \cdot 6 + 9 \cdot 5 = 106 + \frac{1}{2},$$

and we have a contradiction. Similarly, we can prove that α_0 is not of type $(1, 1, 3)$. Therefore, $|\mathrm{Fix}(t)| \neq 20$.

(3) If $|\mathrm{Fix}(t)| = 18$, then $\mathrm{Fix}(t)$ has a point α_0 such that $\omega(\alpha_0) \leqslant \frac{106-18}{18} = \frac{44}{9}$. According to equation (3), α_0 is of type $(0, 4, 1), (1, 2, 2)$ or $(2, 0, 3)$.

If α_0 is of type $(0, 4, 1)$, then α_0 lies on all lines of size 6 and $r^6_\alpha \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$ by Lemma 4. Or rather, the 5 points on the line of size 6 through α_0 are of type $(m, n, 1)$ with weight $\geqslant 2$, and the 12 points on the lines of size 4 through α_0 are of type $(x, y, 0)$ with weight $\geqslant \frac{13}{2}$, where m, n, x and y are non-negative integers. Then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant 6 \cdot 2 + 12 \cdot \frac{13}{2} = 90,$$

a contradiction. Thus there is no point of type $(0, 4, 1)$ and $\omega(\alpha) \geqslant \frac{15}{2}$ for any $\alpha \in \mathrm{Fix}(t)$ with $r^6_\alpha \geqslant 1$.

If α_0 is of type $(1, 2, 2)$, then α_0 lies on all lines of size 6 and $r^6_\alpha \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$ by Lemma 4. In particular, the 10 points on the lines of size 6 through α_0 are of type $(m, n, 1)$ with weight $\geqslant \frac{15}{2}$, and the 7 points which do not lie on the lines of size 6 through α_0 are of type $(x, y, 0)$ with weight $\geqslant \frac{13}{2}$, where m, n, x and y are non-negative integers. This implies

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant 3 + 10 \cdot \frac{15}{2} + 7 \cdot \frac{13}{2} = 123 + \frac{1}{2},$$

which is impossible. Similarly, α_0 cannot be of type $(2, 0, 3)$. Thus $|\mathrm{Fix}(t)| \neq 18$. □

Lemma 6. *Assume that the* HYPOTHESIS *holds and let t be an involution of* $\mathrm{Aut}\,(\mathcal{S})$. *If there is a line λ of $\mathcal{S}$ contained in* $\mathrm{Fix}(t)$, *then* $|\mathrm{Fix}(t)| \neq 12, 14$ *or* 16.

Proof. Since there is a line $\lambda \in \mathcal{S}$ such that $\lambda \subseteq \mathrm{Fix}(t)$, the linear space $\mathcal{D}$ induced by $\mathrm{Fix}(t)$ has at least one line of size 6.

(1) If $|\mathrm{Fix}(t)| = 16$, then $\mathrm{Fix}(t)$ has a point α_0 such that $\omega(\alpha_0) \leqslant \frac{106-16}{16} = \frac{45}{8}$. Thus

$$2r_{\alpha_0}^2 + \frac{1}{2}r_{\alpha_0}^4 \leqslant \frac{45}{8}.$$

Recall that $r_\alpha^2 + 3r_\alpha^4 + 5r_\alpha^6 = 15$ for all $\alpha \in \mathrm{Fix}(t)$ from equation (3). So α_0 is of type $(0,0,3), (0,5,0)$, $(1,3,1)$ or $(2,1,2)$.

Since $\mathcal{D}$ has at least one line of size 6, there is no point of type $(0,5,0)$ by Lemma 4. If α_0 is of type $(0,0,3)$, then α_0 lies on all lines of size 6 and $r_\alpha^6 = 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. More precisely, all the 15 points are of type $(10,0,1)$ having weight 20. Then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) = 0 + 15 \cdot 20 = 300,$$

which is impossible.

If α_0 is of type $(1,3,1)$, then α_0 lies on all lines of size 6 and $r_\alpha^6 \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. In detail, the 5 points on the line of size 6 through α_0 are of type $(x,y,1)$ with weight $\geqslant \frac{7}{2}$, where x and y are non-negative integers. Let $\{\alpha_0, \beta_0\}$ be the line of size 2 through α_0, and so $\omega(\beta_0) \geqslant 8$. If $\alpha \neq \alpha_0$ lies on one of lines of size 4 through α_0, then $r_\alpha^4 \leqslant 4$ since there is no point of type $(0,5,0)$, thus $\omega(\alpha) \geqslant 8$. Hence

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant 6 \cdot \frac{7}{2} + 10 \cdot 8 = 101.$$

It follows that there is no point of type $(1,3,1)$ and $\omega(\alpha) \geqslant 9$ for any $\alpha \in \mathrm{Fix}(t)$ with $r_\alpha^6 \geqslant 1$. Similar analysis as above, we known that α_0 cannot be of type $(2,1,2)$. Therefore, $|\mathrm{Fix}(t)| \neq 16$.

(2) If $|\mathrm{Fix}(t)| = 14$, then $\mathrm{Fix}(t)$ has a point α_0 such that $\omega(\alpha_0) \leqslant \frac{106-14}{14} = \frac{46}{7}$. According to (3), α_0 is of type $(0,1,2), (1,4,0), (2,2,1)$ or $(3,0,2)$.

Recall that $\mathcal{D}$ has at least one line of size 6. Thus there is no point of type $(1,4,0)$ by Lemma 4. If α_0 is of type $(0,1,2)$, then α_0 lies on all lines of size 6 and $r_\alpha^6 \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. Particularly, the 10 points on the lines of size 6 through α_0 are of type $(8,0,1)$ with weight 16, and the 3 points on the line of size 4 through α_0 are of type $(13,0,0)$ with weight 26. Then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) = \frac{1}{2} + 10 \cdot 16 + 3 \cdot 26 = 238 + \frac{1}{2},$$

which is impossible.

If α_0 is of type $(2,2,1)$, then α_0 lies on all lines of size 6 and $r_\alpha^6 \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. Let α_1 and α_2 be the two points that joined to α_0 form the two lines of size 2. Suppose that there is a line of $\mathcal{D}$ having size 4 not containing α_0. Then this line must contain one point of $\{\alpha_1, \alpha_2\}$. Thus $r_\alpha^4 \leqslant 2$ for each point α which is on the line of size 6 through α_0, and $r_\alpha^4 \leqslant 3$ for each point α which do not lie on the line of size 6 through α_0. This implies

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant 5 + 5 \cdot 5 + 8 \cdot \frac{19}{2} = 106,$$

a contradiction.

If α_0 is of type $(3,0,2)$, then α_0 lies on all lines of size 6 and $r_\alpha^6 \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. Let α_1, α_2 and α_3 be the points each of which forms a line of size 2 with α_0. Suppose that there is a line of $\mathcal{D}$ having size 4. Then this line must contain two points of $\{\alpha_1, \alpha_2, \alpha_3\}$. Thus $r_\alpha^4 \leqslant 1$ for each point $\alpha \in \mathrm{Fix}(t)$. Hence

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant 6 + 10 \cdot \frac{21}{2} + 3 \cdot \frac{41}{2} > 92.$$

Thus $\mathrm{Fix}(t)$ does not have a point of type of $(3,0,2)$. It follows that $|\mathrm{Fix}(t)| \neq 14$.

(3) If $|\mathrm{Fix}(t)| = 12$, then $\mathrm{Fix}(t)$ has a point α_0 such that $\omega(\alpha_0) \leqslant \frac{106-12}{12} = \frac{47}{6}$. So α_0 is of type $(0,2,1), (1,0,2), (2,3,0)$ or $(3,1,1)$ from equation (3).

Since $\mathcal{D}$ has at least one line of size 6, there is no point of type $(2,3,0)$ by Lemma 4. If α_0 is of type $(0,2,1)$, then α_0 lies on all lines of size 6, and $r_\alpha^6 \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. Precisely, the 5 points on the line of size 6 through α_0 are of type $(6,0,1)$ having weight 12, and the 6 points on the lines of size 4 through α_0 are of type $(8,1,0)$ having weight $\frac{33}{2}$. Then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) = 1 + 5 \cdot 12 + 6 \cdot \frac{33}{2} = 160,$$

which is impossible.

If α_0 is of type $(1,0,2)$, then α_0 lies on all lines of size 6 and $r_\alpha^6 \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. The 10 points which lie on the lines of size 6 through α_0 are of type $(6,0,1)$ having weight 12, and the point α_1 such that $\{\alpha_0, \alpha_1\}$ is the line of size 2 is of type $(11,0,0)$ having weight 22. Then

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) = 2 + 10 \cdot 12 + 1 \cdot 22 = 144,$$

a contradiction.

If α_0 is of type $(3,1,1)$, then α_0 lies on all lines of size 6 and $r_\alpha^6 \leqslant 1$ for any other point $\alpha \in \mathrm{Fix}(t)$. Let α_1, α_2 and α_3 be the points that joined to α_0 form the there lines of size 2. Suppose that there is a line of size 4 not containing α_0. Then this line must contain two points of $\{\alpha_1, \alpha_2, \alpha_3\}$. Thus $r_\alpha^4 \leqslant 1$ for the point α which do not lie on one line of size 4 through α_0, and $r_\alpha^4 \leqslant 2$ for the point α which lie on the line of size 4 through α_0. Hence

$$\sum_{\alpha \in \mathrm{Fix}(t)} \omega(\alpha) \geqslant \frac{13}{2} + 5 \cdot \frac{13}{2} + 3 \cdot \frac{33}{2} + 3 \cdot 11 > 94.$$

Thus $\mathrm{Fix}(t)$ does not have a point with type $(3,1,1)$. Therefore $|\mathrm{Fix}(t)| \neq 12$. □

Lemma 7. *Assume that the* Hypothesis *holds and let $T \neq 1$ be a Sylow 2-subgroup of* $\mathrm{Aut}\,(\mathcal{S})$. *Then $|T|$ divides 2^{15}.*

Proof. Since $b = 371$, then there exists a line $\lambda \in \mathcal{L}$ such that T fixes λ setwise. Thus $T/T_{(\lambda)} \leqslant S_6$ and $|T : T_{(\lambda)}|$ divides 2^4. Assuming that $S = T_{(\lambda)} \neq 1$, we obtain

$$100 = \sum_{i=1}^{r} |S : S_{\alpha_i}|,$$

where $\alpha_1, \alpha_2, \dots, \alpha_m$ are all the representatives of the orbits of S on $\mathcal{P} \setminus \lambda$. If S is semiregular on $\mathcal{P} \setminus \lambda$, then $|S|$ divides 100, and hence $|T|$ divides $2^4 \cdot 2^2$. If S is not semiregular on $\mathcal{P} \setminus \lambda$, then $\mathcal{P} \setminus \lambda$ has at least one point $\alpha_j (1 \leqslant j \leqslant m)$ such that $S_{\alpha_j} \neq 1$ and 2^6 is divisible by $|S : S_{\alpha_j}|$. In view of Lemmas 5 and 6, $|\mathrm{Fix}(S_{\alpha_j})| = 8, 10$ or 24.

Let $\lambda_1 \in \mathcal{L}$ be the line containing α_j that has non-empty intersection with λ. Suppose that $1 \neq S_0$ is the kernel of the action of S_{α_j} on the points of λ_1, then $|\mathrm{Fix}(S_0)| = 24$, 2^4 is divisible by $|S_{\alpha_j} : S_0|$ and S_0 acts semiregularly on $\mathcal{P} \setminus \mathrm{Fix}(S_0)$ since $|\mathrm{Fix}(t)| \leqslant 24$ for any involution t of $\mathrm{Aut}(\mathcal{S})$. Thus $|S_0|$ divides 82, and it follows that $|T|$ divides $2^4 \cdot 2^6 \cdot 2^4 \cdot 2 = 2^{15}$. □

4 The $2'$-part of $|\,\mathrm{Aut}(\mathcal{S})|$ and proof of Theorem 1

In this section, we prove a bound on the $2'$-part of $|\,\mathrm{Aut}(\mathcal{S})|$. First, we list a result which is important for our discussion.

Lemma 8. [3] *If $\mathcal{S}$ is a linear space having lines of size 3 and 6 (with at least one line of size 3 and one line of size 6). Then $v = 16$ or 18, provided that $v < 21$.*

Lemma 9. *Assume that the* Hypothesis *holds. Then $|\,\mathrm{Aut}(\mathcal{S})|_{2'}$ divides $3^a \cdot 5^3 \cdot 7 \cdot 53$, where a is a non-negative integer. Furthermore, any element of order 53 in $\mathrm{Aut}(\mathcal{S})$ is fixed-point-free.*

Proof. Let $p \geqslant 5$ be a prime divisor of $|\,\mathrm{Aut}(\mathcal{S})|$, and g an element of order p in $\mathrm{Aut}(\mathcal{S})$. Then $|\mathrm{Fix}(g) \cap \lambda| = 0, 1$ or 6.

Suppose that $\mathrm{Fix}(g)$ is not contained in a line of $\mathcal{S}$, then $\mathrm{Fix}(g)$ induces a regular linear space 2-$(|\mathrm{Fix}(g)|, 6, 1)$. Thus $|\mathrm{Fix}(g)| \geqslant k(k-1)+1 = 31$ by Lemma 2. But $|\mathrm{Fix}(g)| \leqslant k + r - 3 = 24$ according to Lemma 3, which is a contradiction. Hence there exists a line $\lambda \in \mathcal{L}$ such that $\mathrm{Fix}(g) \subseteq \lambda$ and $|\mathrm{Fix}(g)| = 0, 1$ or 6. Therefore, the possible values of p are $5, 7$ and 53 since $106 - |\mathrm{Fix}(g)| \equiv 0 \pmod{p}$.

Let P be a Sylow 5-subgroup of $\mathrm{Aut}(\mathcal{S})$ and $P \neq 1$. Then P fixes one line $\lambda \in \mathcal{L}$ and $|P| \mid 5 \cdot |P_{(\lambda)}|$ since $P/P_{(\lambda)} \leqslant S_6$. If $P_{(\lambda)} \neq 1$, then $P_{(\lambda)}$ acts semiregularly on $\mathcal{P} \setminus \lambda$. Thus $|P_{(\lambda)}| \mid 100$. So $|P_{(\lambda)}| \mid 5^2$, and $|P| \mid 5^3$.

Let P be a Sylow p-subgroup of $\mathrm{Aut}(\mathcal{S})$, where $p = 7$ or 53. Suppose $P \neq 1$, then $|\mathrm{Fix}(P)| = 1$ if $p = 7$, and $|\mathrm{Fix}(P)| = 0$ if $p = 53$. Moreover, P is semiregular on $\mathcal{P} \setminus \mathrm{Fix}(P)$. So $|P| = 7$ or 53. □

Lemma 10. *Assume that the* Hypothesis *holds. Then* $|\mathrm{Aut}(\mathcal{S})|_3$ *divides* 3^7.

Proof. If $1 \neq T \in Syl_3(G)$, then $\mathcal{S}$ has a line $\lambda \in \mathcal{L}$ such that $T \leqslant G_\lambda$ since $b = 371$. Thus $T/T_{(\lambda)} \leqslant S_6$ and $|T|$ divides $3^2 \cdot |T_{(\lambda)}|$. Let $S = T_{(\lambda)}$, then $|\mathrm{Fix}(S)| \equiv 1 \pmod 3$.

If $S \neq 1$, then $\mathrm{Fix}(S)$ induces a linear space $\mathcal{D}$ with sizes of lines from $K = \{3, 6\}$. Since $|\mathrm{Fix}(S)| \leqslant 24$ by Lemma 3 and $v \geqslant 31$ for a 2-$(v, 6, 1)$ design by Lemma 2, $\mathcal{D}$ has at least one line of size 3 and one line of size 6. It follows that $|\mathrm{Fix}(S)| = 16$ or 22 by Lemma 8.

Suppose first that $|\mathrm{Fix}(S)| = 16$. Then

$$90 = \sum_{i=1}^{r} |S : S_{\alpha_i}|,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are all the representatives of the orbits of S on $\mathcal{P} \setminus \mathrm{Fix}(S)$. If S is semiregular on $\mathcal{P} \setminus \mathrm{Fix}(S)$, then $|S| \mid 90$, this implies that $|S|$ divides 3^2. If there exists $\alpha_j \in \mathcal{P} \setminus \mathrm{Fix}(S) (1 \leqslant j \leqslant m)$ such that $S_{\alpha_j} \neq 1$, then $|\mathrm{Fix}(S_{\alpha_j})| = 22$ and S_{α_j} acts semiregularly on $\mathcal{P} \setminus \mathrm{Fix}(S_{\alpha_j})$. Hence $|S_{\alpha_j}|$ divides 3, so $|S|$ divides 3^5 since 3^4 is divisible by $|S : S_{\alpha_j}|$.

Now suppose that $|\mathrm{Fix}(S)| = 22$. Then S is semiregular on $\mathcal{P} \setminus \mathrm{Fix}(S)$. Hence $|S|$ divides 3. Therefore, $|T|$ divides 3^7. □

According to the discussion above, we have $|G|$ divides $2^{15} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 53$.

Lemma 11. *Assume that the* Hypothesis *holds. If* N *is a minimal normal subgroup of* G, *then* $N = \mathrm{Soc}(G) \cong Z_{53}$.

Proof. Since $N \trianglelefteq G$ and G is point-transitive, N is $\frac{1}{2}$-transitive on $\mathcal{P}$, and the common length of orbits is $2, 53$ or 106. Assume that N is not elementary abelian. Then $N \cong T^\ell$ is a direct product of $\ell \geqslant 1$ copies of non-abelian simple groups T. So $|N|$ is divisible by 3 or 5 ([9, Remarks 3.7 a)]). Suppose that the orbit-length of N on the points is 2 and let $g \in N$ be of order 3 or 5. Then g fixes every orbit of N on $\mathcal{P}$, which implies $|\mathrm{Fix}(g)| = 106$, a contradiction. Thus the common length must be divisible by 53, so 53 divides $|N|$ and $N = T = \mathrm{Soc}(G)$ is simple since $53^2 \nmid |G|$.

Using the list of non-abelian simple groups, it is easy to check that there is no non-abelian simple group N such that 53 divides $|N|$ and $|N|$ divides $2^{15} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 53$. Hence N is abelian and $N \cong Z_{53}$. □

Lemma 12. *Assume that the* Hypothesis *holds. Then* G *is solvable and* $G \cong Z_{53} : Z_4$.

Proof. According to Lemma 11, $N = \mathrm{Soc}(G) \cong Z_{53}$.

If $N \neq C_G(N)$, then by Schur-Zassenhaus'theorem, $C_G(N)$ has a normal subgroup M such that $C_G(N) = M \times N$, since N is a normal Hall-subgroup of $C_G(N)$. Since M is characteristic in $C_G(N)$ and $C_G(N) \trianglelefteq G$, then there exists another minimal normal subgroup of G contained in M, but this is impossible. Therefore $N = C_G(N)$, and $G/C_G(N) \lesssim \mathrm{Aut}(N) \cong Z_{52}$. This implies that G is solvable and $G \lesssim Z_{53} : Z_4$ since $13 \nmid |G|$.

If $|G| = 106$, then G is regular on $\mathcal{P}$. Thus any involution of G has no fixed point. This implies that $k \mid v$, a contradiction. Therefore, $G \cong Z_{53} : Z_4$. □

Proof of Theorem 1. According to Lemma 12, $G \cong Z_{53} : Z_4$ which is a metacyclic group. Let $G = \langle x, y | x^{53} = 1, y^4 = x^h, x^y = x^\ell \rangle$, where h and ℓ are non-negative integers satisfying

$$\ell^4 \equiv 1 \pmod{53}, h(\ell - 1) \equiv 0 \pmod{53}$$

by [1]. It follows that $\ell \equiv 1, 23, 30 \,\text{or}\, 52 \pmod{53}$. If $\ell \equiv 1 \pmod{53}$, then $h \equiv 1 \pmod{53}$ and if $\ell \equiv 23, 30 \,\text{or}\, 52 \pmod{53}$, then $h \equiv 0 \pmod{53}$. Suppose first $\ell \equiv 1 \,\text{or}\, 52 \pmod{53}$, then G has a normal subgroup of order 2, a contradiction.

For the cases $\ell \equiv 23 \,\text{and}\, 30 \pmod{53}$, the groups are the same up to isomorphic. Without loss of generality we assume that $\ell \equiv 23 \pmod{53}$ in the following. Since G has only one conjugacy class of involutions, G has exactly one transitive permutation representation on 106 points, which is equivalent to the permutation representation of G on $H = \langle y^2 \rangle$.

Since $G = \{x^m y^n | 0 \leqslant m \leqslant 52, 0 \leqslant n \leqslant 3\}$, the right cosets of H in G are H, Hx, Hx^2,..., Hx^{52}, Hy, Hxy, ..., $Hx^{52}y$. Let $Hx^m = i_{m+1}, Hx^m y = j_{m+1}$, where $0 \leqslant m \leqslant 52$. Then the permutation representation of G on $\mathcal{P} = \{i_1, i_2, \cdots, i_{53}, j_1, j_2, \cdots, j_{53}\}$ is $P(G) = \langle g_1, g_2 \rangle$, where

$$g_1 = (i_1 i_2 \cdots i_{53})(j_{53} j_{52} \cdots j_1)$$

and

$$g_2 = (i_1 j_1)(i_2 j_2 i_{53} j_{53})(i_3 j_3 i_{52} j_{52}) \cdots (i_{27} j_{27} i_{28} j_{28}).$$

Assume that λ is the line through i_1 and j_1. Then $\lambda^{g_2} = \lambda$. It follows that λ is a union of orbits of g_2 on $\mathcal{P}$. Thus there exists an integer e $(2 \leqslant e \leqslant 27)$ such that

$$\lambda = \{i_1, j_1, i_e, j_e, i_{55-e}, j_{55-e}\}.$$

However, $\lambda^{g_1^{54-e}} = \{i_{55-e}, j_e, i_1, j_{2e-1}, i_{56-2e}, j_1\}$, which is impossible. This completes the proof of Theorem 1. □

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References

[1] B G Basmaji. On the isomorphisms of two metacyclic groups. Pro. Amer. Math. Soc., 22: 175-182, 1969.

[2] T Beth, D Jungnickel, and H Lenz. Design Theory. Bibliographisches Institut Mannheim, Wien, Zurich, 1985.

[3] A R Camina, and L Di Martino. The group of automorphisms of a transitive 2-$(91, 6, 1)$ design. Geometriae Dedicata, 31: 151-163, 1989.

[4] A R Camina, and J Siemons. Block transitive automorphism groups of 2-$(v, k, 1)$ block designs. J. Combinatorial Theory Ser. A, 51(2): 268-276, 1989.

[5] A R Camina, P M Neumann, and C E Praeger. Alternating groups acting on finite linear spaces. London Math. Soc., 87(3): 29-53, 2003.

[6] C J Colbourn, and M J Colbourn. Cyclic steiner systems having multiplier automorphisms. Utilitas Math., 17: 127-149, 1980.

[7] C J Colbourn, and M J Colbourn. On cyclic Steiner systems S(2, 6, 91). Abstracts Amer. Math. Soc., 2: 463, 1981.

[8] C J Colbourn, J H Dinitz. Handbook of Combinatorial Designs (2nd edition). Boca Raton, FL: CRC press, 2006

[9] B Hupper, N Blackburn. Endliche Gruppen III, Springer-Verlag, Berlin, 1982.

[10] Z Janko, and V D Tonchev. Cyclic 2-$(91, 6, 1)$ designs with multiplier automorphisms. Discrete Mathematics, 97(1-3): 265-268, 1991.

[11] W H Mills. A new block design. Congr. Numer, 14: 461-465, 1975.

[12] W H Mills. Two new block designs. Utilitas Math., 7: 73-75, 1975.