Semiarcs with long secants

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Abstract

In a projective plane Π_q of order q, a non-empty point set $\mathcal{S}_t$ is a t-semiarc if the number of tangent lines to $\mathcal{S}_t$ at each of its points is t. If $\mathcal{S}_t$ is a t-semiarc in Π_q, $t < q$, then each line intersects $\mathcal{S}_t$ in at most $q+1-t$ points. Dover proved that semiovals (semiarcs with $t = 1$) containing q collinear points exist in Π_q only if $q \leqslant 3$. We show that if $t > 1$, then t-semiarcs with $q+1-t$ collinear points exist only if $t \geqslant \sqrt{q-1}$. In $\mathrm{PG}(2,q)$ we prove the lower bound $t \geqslant (q-1)/2$, with equality only if $\mathcal{S}_t$ is a blocking set of Rédei type of size $3(q+1)/2$.

We call the symmetric difference of two lines, with t further points removed from each line, a V_t-configuration. We give conditions ensuring a t-semiarc to contain a V_t-configuration and give the complete characterization of such t-semiarcs in $\mathrm{PG}(2,q)$.

Keywords: collineation group; blocking set; semioval

1 Introduction

Semiarcs are natural generalizations of arcs. Let Π_q be a projective plane of order q. A non-empty point set $\mathcal{S}_t \subset \Pi_q$ is called a *t-semiarc* if for every point $P \in \mathcal{S}_t$ there exist exactly t lines $\ell_1, \ell_2, \ldots, \ell_t$ such that $\mathcal{S}_t \cap \ell_i = \{P\}$ for $i = 1, 2, \ldots, t$. These lines are called the *tangents* to $\mathcal{S}_t$ at P. If a line ℓ meets $\mathcal{S}_t$ in $k > 1$ points, then ℓ is called a *k-secant* of $\mathcal{S}_t$. The classical examples of semiarcs are the semiovals (semiarcs with $t = 1$) and point sets of type $(0, 1, n)$ (i.e. point sets meeting each line in either 0, or 1, or n points, in this case $t = q+1-(s-1)/(n-1)$, where s denotes the size of the point set). Arcs, unitals, and subplanes are semiarcs of the latter type. For more examples, see [1], [5] and [10].

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Because of the huge diversity of the geometry of semiarcs, their complete classification is hopeless. In [7] Dover investigated semiovals with a q-secant and semiovals with more than one $(q-1)$-secant. The aim of this paper is to generalize these results and characterize t-semiarcs with long secants.

Many of the known t-semiarcs contain the symmetric difference of two lines, with t further points removed from each line. We will call this set of $2(q-t)$ points a V_t-*configuration*. Recently in [5] it was proved that in $\mathrm{PG}(2,q)$ small semiarcs with a long secant necessarily contain a V_t-configuration or can be obtained from a blocking set of Rédei type. Here we give another condition ensuring a t-semiarc to contain a V_t-configuration and we give the complete characterization of such t-semiarcs in $\mathrm{PG}(2,q)$. To do this we use the classification of perspective point sets in $\mathrm{PG}(2,q)$. This is a result due to Korchmáros and Mazzoca [11] and it is related to Dickson's classification of the subgroups of the affine group on the line $\mathrm{AG}(1,q)$.

Using a result of Weiner and Szőnyi, that was conjectured by Metsch, we prove that t-semiarcs in $\mathrm{PG}(2,q)$ with $q+1-t$ collinear points exist if and only if $t \geqslant (q-1)/2$. The case of equality is strongly related to blocking sets of Rédei type, we also discuss these connections.

If $t = q+1, q$ or $q-1$, then $\mathcal{S}_t$ is single point, a subset of a line or three non-collinear points respectively. To avoid trivial cases, we may assume for the rest of this paper that $t < q-1$.

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If $\mathcal{S}_t$ is a t-semiarc in Π_q, $t < q$, then each line intersects $\mathcal{S}_t$ in at most $q+1-t$ points. In this section we study t-semiarcs containing $q+1-t$ collinear points. The following lemma gives an upper bound for the size of such t-semiarcs.

Lemma 1. *If $\mathcal{S}_t$ is a t-semiarc in Π_q and ℓ is a $(q+1-t)$-secant of $\mathcal{S}_t$, then $|\mathcal{S}_t \setminus \ell| \leqslant q$.*

Proof. Let $U = \mathcal{S}_t \setminus \ell$ and let $D = \ell \setminus \mathcal{S}_t$. Through each point of U there pass exactly t tangents to $\mathcal{S}_t$ and each of them intersects ℓ in D. This implies $t|U| \leqslant q|D|$. Since $|D| = t$, we have $|U| \leqslant q$. □

In [7] Dover proved that semiovals with a q-secant exist in Π_q if and only if $q \leqslant 3$. Our first theorem generalizes this result and shows that if $\mathcal{S}_t$ has a $(q+1-t)$-secant, then t cannot be arbitrary. For related ideas of the proof, see the survey paper by Blokhuis et. al. [3], Theorem 3.2.

Theorem 2. *If $\mathcal{S}_t$ is a t-semiarc in Π_q with a $(q+1-t)$-secant, then $t = 1$ and $q \leqslant 3$ or $t \geqslant \sqrt{q-1}$.*

Proof. Let ℓ be a line that satisfies $|\mathcal{S}_t \cap \ell| = q+1-t$ and let $U = \mathcal{S}_t \setminus \ell$. The size of U has to be at least $q-t$, otherwise the points of $\ell \cap \mathcal{S}_t$ would have more than t tangents. This and Lemma 1 together yield:

$$q - t \leqslant |U| \leqslant q. \tag{1}$$

Let $q-t+k$ be the size of U, where $0 \leqslant k \leqslant t$. Let δ be the number of lines that do not meet U and denote by $L_1, L_2, \ldots, L_{q^2+q+1-\delta}$ the lines that meet U. For these lines let $e_i = |L_i \cap U|$. The standard double counting argument gives:

$$\sum_{i=1}^{q^2+q+1-\delta} e_i = (q-t+k)(q+1), \tag{2}$$

$$\sum_{i=1}^{q^2+q+1-\delta} e_i(e_i-1) = (q-t+k)(q-t+k-1). \tag{3}$$

If a line ℓ' intersects U in more than one point, then $Q := \ell' \cap \ell$ is in $\mathcal{S}_t$, otherwise the points of $\ell' \cap U$ would have at most $t-1$ tangents. The point $Q \in \mathcal{S}_t$ has at least $q-1-(q-t+k-|\ell' \cap U|) = t-1-k+|\ell' \cap U|$ tangents, hence $|\ell' \cap U| \leqslant k+1$. This implies $e_i \leqslant k+1$, for $i = 1, 2, \ldots, q^2+q+1-\delta$, thus the following holds:

$$\sum_{i=1}^{q^2+q+1-\delta} e_i(e_i-1) \leqslant (k+1) \sum_{i=1}^{q^2+q+1-\delta} (e_i-1) = (k+1)((q-t+k)(q+1)-(q^2+q+1-\delta)). \tag{4}$$

The line ℓ does not meet U and the other lines that do not meet U fall into two classes: there are $(q+1-t)t$ of them passing through $\ell \cap \mathcal{S}_t$ (the tangents to $\mathcal{S}_t$ through the points of $\ell \cap \mathcal{S}_t$) and there are $tq-(q-t+k)t$ of them passing through $\ell \setminus \mathcal{S}_t$ (the lines intersecting $\ell \setminus \mathcal{S}_t$ minus the tangents to $\mathcal{S}_t$ through the points of U). This implies $\delta = t(q+1-k)+1$, hence we can write (4) as:

$$(q-t+k)(q-t+k-1) \leqslant (k+1)((q-t+k)(q+1)-(q^2+q)+t(q+1-k)). \tag{5}$$

Rearranging this inequality we obtain:

$$q^2 - q(2t+1-k+k^2) + k^2t - kt - 2k + t^2 + t \leqslant 0.$$

The discriminant of the left-hand side polynomial is $k^4-2k^3+3k^2+6k+1$. If $k = 0, 1, 2$, then we get $q \leqslant t+1, t+2, t+4$ respectively. Otherwise, we have $k^4-2k^3+3k^2+6k+1 < (k^2-k+3)^2$, which yields $q \leqslant t+k^2-k+1$. The maximum value of k is t, therefor $q \leqslant t^2+1$ follows for $k \geqslant 3$. If $t = 1$, then $k \leqslant 1$, hence $q \leqslant t+2 = 3$. If $t = 2$, then $k \leqslant 2$, hence $q \leqslant t+4 = 6$. Since there is no projective plane of order 6, in this case we get $q \leqslant 5$. If $t \geqslant 3$ and $k < 3$, then $q \leqslant t+4 < t^2+1$ and this completes the proof. □

Before we go further we need some definitions about blocking sets. A *blocking set* of a projective plane is a point set $\mathcal{B}$ that intersects every line in the plane. A blocking set is *minimal* if it does not contain a smaller blocking set and it is *non-trivial* if it does not contain a line. If $\mathcal{B}$ is a non-trivial blocking set, then we have $|\ell \cap \mathcal{B}| \leqslant |\mathcal{B}| - q$ for every line ℓ. If there is a line ℓ such that $|\ell \cap \mathcal{B}| = |\mathcal{B}| - q$, then $\mathcal{B}$ is a blocking set of *Rédei type* and the line ℓ is a *Rédei line* of $\mathcal{B}$.

In $\mathrm{PG}(2,q)$ we can improve the bound in Theorem 2. To do this we use the following result, conjectured by Metsch [13] and proven by Weiner and Szőnyi in [15, 16].

Theorem 3 ([15, 16])**.** *Let U be a point set in $\mathrm{PG}(2,q)$, P a point not in U and assume that there pass exactly r lines through P meeting U. Then the total number of lines meeting U is at most $1+rq+(|U|-r)(q+1-r)$.*

Theorem 4. *Let $\mathcal{S}_t$ be a t-semiarc in $\mathrm{PG}(2,q)$. If $\mathcal{S}_t$ has a $(q+1-t)$-secant, then $t \geqslant (q-1)/2$. In the case of equality, $\mathcal{S}_t$ is a blocking set of Rédei type and its $(q+1-t)$-secants are Rédei lines.*

Proof. Let ℓ be a $(q+1-t)$-secant of $\mathcal{S}_t$ and let $U = \mathcal{S}_t \setminus \ell$. From Lemma 1, we have:

$$|U| \leqslant q. \tag{6}$$

The following statements are easy to check:

- the lines intersecting U in more than one point intersect ℓ in $\ell \cap \mathcal{S}_t$,
- through each point of $\ell \cap \mathcal{S}_t$ there pass exactly $r = q-t$ lines meeting U,
- the total number of lines meeting U is $\delta = |U|t + (q+1-t)(q-t)$.

Applying Theorem 3 for the point set U and for a point $P \in \ell \cap \mathcal{S}_t$, we obtain:

$$\delta = |U|t + (q+1-t)(q-t) \leqslant 1 + (q-t)q + (|U|-q+t)(t+1). \tag{7}$$

After rearranging, we get:

$$2q - 2t - 1 \leqslant |U|. \tag{8}$$

Equations (6) and (8) together imply $t \geqslant (q-1)/2$. If $t = (q-1)/2$, then $|U| = q$ and there are $\delta = (3q^2+2q+3)/4$ lines meeting U and $(q+1-t)t = (q^2+2q-3)/4$ lines meeting $\ell \cap \mathcal{S}_t$ but not U. Together with the line ℓ we get the total number of lines in $\mathrm{PG}(2,q)$, thus $\mathcal{S}_t$ is a blocking set of Rédei type and ℓ is a Rédei line of $\mathcal{S}_t$. □

The following result by Blokhuis yields another connection between blocking sets and semiarcs.

Theorem 5 ([2])**.** *If $\mathcal{B}$ is a minimal non-trivial blocking set in $\mathrm{PG}(2,p)$, $p > 2$ prime, then $|\mathcal{B}| \geqslant 3(p+1)/2$. In the case of equality there pass exactly $(p-1)/2$ tangent lines through each point of $\mathcal{B}$.*

Example 6 ([9], Lemma 13.6)**.** Denote by C the set of non-zero squares in $\mathrm{GF}(q)$, q odd, and let $\mathcal{S}_t = \{(c,0,1),(0,-c,1),(c,1,0)\colon c \in C\} \cup \{(1,0,0),(0,1,0),(0,0,1)\}$. This point set is called projective triangle and it is a t-semiarc with three $(q+1-t)$-secants, where $t = (q-1)/2$. This example shows the sharpness of Theorems 4 and 5.

In $\mathrm{PG}(2,q)$, q prime, Lovász and Schrijver proved that blocking sets of Rédei type of size $3(q+1)/2$ are projectively equivalent to the projective triangle, see [12]. Gács, Lovász, and Szőnyi proved the same if q is a square of a prime, see [8]. These results and Theorem 4 together yield the following:

Corollary 7 ([8, 12])**.** *Let $\mathcal{S}_t$ be a t-semiarc in $\mathrm{PG}(2,q)$ with a $(q+1-t)$-secant. If $t = (q-1)/2$ and $q = p$ or $q = p^2$, p prime, then $\mathcal{S}_t$ is projectively equivalent to the projective triangle.*

3 Semiarcs with two long secants

Throughout the paper, if A and B are two point sets in Π_q, then $A \triangle B$ denotes their symmetric difference, that is $(A \setminus B) \cup (B \setminus A)$.

Definition 8. A V_t-configuration is the symmetric difference of two lines, with t further points removed from both lines. Semiarcs containing a V_t-configuration fall into two types. Let $\mathcal{S}_t$ be a t-semiarc and suppose that there are two lines, ℓ_1 and ℓ_2, such that $(\ell_1 \triangle \ell_2) \cap \mathcal{S}_t$ is a V_t-configuration, then:

- $\mathcal{S}_t$ is of V_t° type if $\ell_1 \cap \ell_2 \notin \mathcal{S}_t$,
- $\mathcal{S}_t$ is of $V_t^\bullet$ type if $\ell_1 \cap \ell_2 \in \mathcal{S}_t$.

For semiovals, Dover proved the following characterization:

Theorem 9 ([7], Lemma 4.1, Theorem 4.2)**.** *Let $\mathcal{S}_1$ be a semioval in Π_q. If $\mathcal{S}_1$ is of V_1° type, then it is contained in a vertexless triangle. If $q > 5$ and $\mathcal{S}_1$ has at least two $(q-1)$-secants, then $\mathcal{S}_1$ is of V_1° type.*

As the above result suggests, the characterization of t-semiarcs with two $(q-t)$-secants works nicely only for semiarcs of V_t° type. In Proposition 11 we generalize the last statement of the above result, but the characterization of V_t° type semiarcs seems to be hopeless in general. In Proposition 12 we consider the case when $t = 2$, but for larger values of t we deal only with the Desarguesian case, see Section 4.

Lemma 10. *Let $\mathcal{S}_t$ be a t-semiarc in Π_q, $t < q$, and suppose that there exist two lines, ℓ_1 and ℓ_2, with their common point in $\mathcal{S}_t$ such that $|\ell_1 \setminus (\mathcal{S}_t \cup \ell_2)| = n$ and $|\ell_2 \setminus (\mathcal{S}_t \cup \ell_1)| = m$. Then $q \leqslant t + 1 + nm/t$ and $|\mathcal{S}_t \setminus (\ell_1 \cup \ell_2)| = q - 1 - t$ in the case of equality.*

Proof. Since $\mathcal{S}_t$ is not contained in a line, we have $n, m \geqslant t$. If one of n or m is equal to q, then $q < q + t + 1 \leqslant t + 1 + nm/t$ and the assertion follows. Thus we can assume that ℓ_1 and ℓ_2 are not tangents to $\mathcal{S}_t$. Let $X = \mathcal{S}_t \setminus (\ell_1 \cup \ell_2)$. Through the point $\ell_1 \cap \ell_2$ there pass exactly t tangents to $\mathcal{S}_t$, hence $q - 1 - t \leqslant |X|$. Through the points of X there pass $|X|t$ tangents to $\mathcal{S}_t$, each of them intersects ℓ_1 and ℓ_2 off $\mathcal{S}_t$, hence $|X|t \leqslant nm$. These two inequalities imply $q \leqslant t + 1 + nm/t$ and $|X| = q - 1 - t$ in the case of equality. □

Proposition 11. *Let $\mathcal{S}_t$ be a t-semiarc in Π_q. If $\mathcal{S}_t$ has at least two $(q-t)$-secants and $q > 2t + 3$, then $\mathcal{S}_t$ is of V_t° type. If $\mathcal{S}_t$ has at least two $(q - t + 1)$-secants, then $\mathcal{S}_t$ is of $V_t^\bullet$ type.*

Proof. If $\mathcal{S}_t$ has at least two $(q-t)$-secants with their common point in $\mathcal{S}_t$, then Lemma 10 implies $q \leqslant t + 1 + (t+1)^2/t = 2t + 3 + 1/t$. If $q > 2t + 3$, then this is only possible when $t = 1$ and $q = 6$, but there is no projective plane of order 6. Hence the common point of the $(q-t)$-secants is not contained in $\mathcal{S}_t$, which means that $\mathcal{S}_t$ is of V_t° type. The proof of the second statement is straightforward. □

Proposition 12. *Let $\mathcal{S}_t$ be a t-semiarc of V_t° type in Π_q. Then the following hold.*

(a) $|\mathcal{S}_t| \neq 2q - 2t + 1$.

(b) *If* $t = 2$*, then* $\mathcal{S}_t$ *is a* V_2*-configuration or* $|\mathcal{S}_t| = 2q - 2$ *and* $\mathcal{S}_t = (\ell_1 \cup \ell_2) \triangle \Pi_2$*, where* ℓ_1 *and* ℓ_2 *are two lines in* Π_2*, that is a Fano subplane contained in* Π_q*.*

(c) *If* $t > 1$*, then* $|\mathcal{S}_t| \leqslant 2q - t$*.*

Proof. Let $\mathcal{S}_t$ be a t-semiarc of V_t° type and let ℓ_1 and ℓ_2 be two $(q-t)$-secants of $\mathcal{S}_t$ such that $P := \ell_1 \cap \ell_2$ is not contained in $\mathcal{S}_t$. Denote the points of $\ell_1 \setminus (\mathcal{S}_t \cup P)$ by $A_1, \ldots, A_t$, the points of $\ell_2 \setminus (\mathcal{S}_t \cup P)$ by $B_1, \ldots, B_t$. Let $X = \mathcal{S}_t \setminus (\ell_1 \cup \ell_2)$ and define the line set $\mathcal{L} := \{A_iB_j : 1 \leqslant i, j \leqslant t\}$ of size t^2. Through each point $Q \in X$ there pass exactly t lines of $\mathcal{L}$, otherwise there would be an index $i \in \{1, 2, \ldots, t\}$ for which the line QA_i meets ℓ_2 in $\mathcal{S}_t$. But then there would be at most $t - 1$ tangents to $\mathcal{S}_t$ through the point $QA_i \cap \ell_2$, a contradiction.

Suppose, contrary to our claim, that X consists of a unique point denoted by Q. Then Q would have $t + 1$ tangents: the t lines of $\mathcal{L}$ that pass through Q and the line PQ.

If $t = 2$, then exactly two of the points of $\Pi_q \setminus (\ell_1 \cup \ell_2)$ are contained in two lines of $\mathcal{L}$. These are $Q_1 := A_1B_1 \cap A_2B_2$ and $Q_2 := A_1B_2 \cap A_2B_1$. Since $|X| > 1$, we have $X = \{Q_1, Q_2\}$. If P were not collinear with Q_1 and Q_2, then PQ_i would be a third tangent to $\mathcal{S}_t$ at Q_i, for $i = 1, 2$. It follows that the point set $\Pi_2 := \{P, A_1, A_2, B_1, B_2, Q_1, Q_2\}$ is a Fano subplane in Π_q.

To prove (c), define $Y \subseteq X$ as $Y := \{A : A \in X, |AP \cap \mathcal{S}_t| = 1\}$. The line set $\mathcal{L}$ contains $|Y|(t-1)$ tangents through the points of Y and $(|X| - |Y|)t$ tangents through the points of $X \setminus Y$, hence

$$|X|(t-1) \leqslant |X|t - |Y| = |\mathcal{L}| - \delta \leqslant t^2, \tag{9}$$

where δ denotes the number of non-tangent lines in $\mathcal{L}$. Because of (b), we may assume $t > 2$, hence $|X| \leqslant t^2/(t-1) < t + 2$ follows. To obtain a contradiction, suppose that $|X| = t + 1$. If this is the case, then (9) implies $t \leqslant |Y|$. If $|Y| = t$, then $X \setminus Y$ consists of a unique point, but this contradicts the definition of Y. If $|Y| = t + 1$, then $X = Y$ and through each point of X there pass a non-tangent line, which is in $\mathcal{L}$. Thus if $\delta = 1$, then the points of X are contained in a line $\ell \in \mathcal{L}$. We may assume that $\ell = A_tB_t$. Then we can find $2(t-1)$ other non-tangent lines in $\mathcal{L}$, these are A_tB_i and B_tA_i for $i = 1, 2, \ldots, t-1$. On the other hand $\delta > 1$ contradicts (9) and this contradiction proves $|X| \leqslant t$. □

The following result shows some kind of stability of semiarcs containing a V_t-configuration.

Theorem 13. *Let* $\mathcal{S}_t$ *be a* t*-semiarc in* Π_q*,* $t < q$*, and suppose that there exist two lines,* ℓ_1 *and* ℓ_2*, such that* $|\ell_1 \setminus (\mathcal{S}_t \cup \ell_2)| = n$ *and* $|\ell_2 \setminus (\mathcal{S}_t \cup \ell_1)| = m$*.*

1. *If* $\ell_1 \cap \ell_2 \notin \mathcal{S}_t$*,* $t > 1$ *and* $q > \min\{n, m\} + 2nm/(t-1)$*, then* $\mathcal{S}_t$ *is of* V_t° *type.*

2. *If* $\ell_1 \cap \ell_2 \in \mathcal{S}_t$ *and* $q > \min\{n, m\} + nm/t$*, then* $t = (q-1)/2$*,* $|\mathcal{S}_t| = 3(q+1)/2$ *and* $\mathcal{S}_t$ *is of* $V_t^\bullet$ *type.*

We have $n = m = t$ in both cases.

Proof. We may assume $m \geqslant n$. In part 1, we have $n \geqslant t-1$, with equality only if ℓ_2 is not a secant of $\mathcal{S}_t$, i.e. when $m \in \{q-1, q\}$. The assumption $q > \min\{n, m\} + 2nm/(t-1)$ implies $n, m < q-1$, hence this is not the case. It follows that $n, m \geqslant t$ holds. In part 2, we have $n, m \geqslant t$, hence the assumption implies $n, m < q$ or, equivalently, the lines ℓ_1 and ℓ_2 are secants of $\mathcal{S}_t$. First we show $n = m = t$ in both cases. From this, part 1 follows immediately.

Suppose, contrary to our claim, that $m \geqslant t+1$. Denote by P the intersection of ℓ_1 and ℓ_2. Let $\mathcal{N} = \{N_1, N_2, \ldots N_{q-n}\}$ be the set of points of $(\ell_1 \setminus P) \cap \mathcal{S}_t$ and $\mathcal{M} = \{M_1, M_2, \ldots M_m\}$ be the set of points of $\ell_2 \setminus (\mathcal{S}_t \cup P)$. Let $X = \mathcal{S}_t \setminus (\ell_1 \cup \ell_2)$. Through each point $N_j \in \mathcal{N}$ there pass exactly $m-t$ non-tangent lines that intersect ℓ_2 in $\mathcal{M}$. Each of these lines contains at least one point of X. Denote the set of these points by $X(N_j)$. Then we have the following:

- $|X(N_i)| \geqslant m-t$, for $i = 1, 2, \ldots, q-n$,
- $X \supseteq \cup_{i=1}^{q-n} X(N_i)$,
- if $P \notin \mathcal{S}_t$, then each point of X is contained in at most $m-t+1$ point sets of $\{X(N_1), \ldots, X(N_{q-n})\}$,
- if $P \in \mathcal{S}_t$, then each point of X is contained in at most $m-t$ point sets of $\{X(N_1), \ldots, X(N_{q-n})\}$.

In part 1, we have the following lower bound for the size of X:

$$\frac{(q-n)(m-t)}{m-t+1} \leqslant |X|. \tag{10}$$

On the other hand, through each point of X there pass at least $t-1$ tangents that intersect both $\ell_1 \setminus (\mathcal{S}_t \cup P)$ and $\mathcal{M}$. Hence we have:

$$|X| \leqslant \frac{nm}{t-1}. \tag{11}$$

Summarizing these two inequalities we get:

$$q \leqslant n + \frac{nm}{t-1} + \frac{nm}{(m-t)(t-1)} \leqslant n + \frac{2nm}{t-1},$$

that is a contradiction.

In part 2, observe that Lemma 10 and $q > \min\{m, n\} + nm/t$ together imply $n = t$. If $m \geqslant t+1$, then similarly to (10) and (11), we get $(q-t)(m-t)/(m-t) \leqslant |X|$ and $|X| \leqslant mt/t$ respectively. These two inequalities imply $q \leqslant t+m$, contradicting our assumption $q > \min\{n, m\} + nm/t = t+m$, hence $m = t$ follows. If $n = m = t$, then Lemma 10 implies $q \leqslant 2t+1$ while our assumption yields $q > 2t$, thus $t = (q-1)/2$. Since in this case there is equality in Lemma 10, we have $|\mathcal{S}_t| = 3q - 3t = 3(q+1)/2$. $\square$

Let $\alpha_{n,m}$ and $\beta_{n,m}$ denote the lower bounds on q in part 1 and in part 2 of Theorem 13, respectively. The following example shows that the weaker assumptions $\alpha_{n,m} < 3q$ and $\beta_{n,m} < 2q$, respectively, do not imply the existence of a V_t-configuration contained in the semiarc.

Example 14. We give two examples for t-semiarcs, $\mathcal{S}_t$, such that they do not contain a V_t-configuration and there exist two lines, ℓ_1 and ℓ_2, with $\ell_1 \backslash (\ell_2 \cup \mathcal{S}_t) = t$ and $\ell_1 \backslash (\ell_2 \cup \mathcal{S}_t) = t+1$. To do this, choose a conic $\mathcal{C}$ in Π_s, that is a projective plane of order $s > 3$. Let Q_1 and Q_2 be two points of $\mathcal{C}$ and proceed as follows.

1. Let ℓ_i be the tangent of $\mathcal{C}$ at the point Q_i, for $i = 1, 2$, and denote $\ell_1 \cap \ell_2$ by P. Take a point $Z \in Q_1Q_2$ such that PZ is a secant of $\mathcal{C}$. Then $\mathcal{S}_0 := (\ell_1 \cup \ell_2 \cup \mathcal{C} \cup \{Z\}) \backslash \{P, Q_2\}$ is a point set without tangents. Now, if Π_s is contained in Π_q, then $\mathcal{S}_0 \subset \Pi_s$ is a t-semiarc in Π_q, with $t = q - s$. We have $\ell_1 \cap \ell_2 \notin \mathcal{S}_t$ and

$$\alpha_{t,t+1} = (q-s) + 2\frac{(q-s+1)(q-s)}{q-s-1} < 3q.$$

2. Let ℓ_1 be the tangent of $\mathcal{C}$ at Q_1 and let ℓ_2 be the line Q_1Q_2. Take a point $Z \in \ell \backslash (\ell_1 \cup \ell_2)$, where ℓ denotes the tangent of $\mathcal{C}$ at Q_2. Then $\mathcal{S}_0 := (\ell_1 \cup \ell_2 \cup \mathcal{C} \cup \{Z\}) \backslash \{Q_2\}$ is a point set without tangents. As before, if Π_s is contained in Π_q, then $\mathcal{S}_0 \subset \Pi_s$ is a t-semiarc in Π_q, with $t = q - s$. We have $\ell_1 \cap \ell_2 \in \mathcal{S}_t$ and

$$\beta_{t,t+1} = (q-s) + \frac{(q-s+1)(q-s)}{q-s} < 2q.$$

The next example is due to Suetake and it shows that when $t = 1$, then there is no analogous result for part 1 of Theorem 13.

Example 15 ([14], Example 3.3)**.** Let A be a proper, not empty subset of $\mathrm{GF}(q) \setminus \{0\}$, such that $A = -A := \{-a \colon a \in A\}$ and $|A| \geqslant 2$. Let $B = \mathrm{GF}(q) \setminus (A \cup \{0\})$ and define the following set of points in $\mathrm{PG}(2, q)$:

$$\mathcal{S}_1 := \{(0,a,1), (b,0,1), (c,c,1), (m,1,0) \colon a \in A,\, b \in B,\, c,m \in \mathrm{GF}(q) \setminus \{0\},\, m \neq 1\}.$$

Then $\mathcal{S}_1$ is a semioval with a $(q-1)$-secant, $X = Y$, and a $(q-2)$-secant, $Z = 0$, intersecting each other not in $\mathcal{S}_1$. Also, $\mathcal{S}_1$ is not of V_1° type.

When $A = \mathrm{GF}(q) \setminus \{0\}$ in the above example, then $\mathcal{S}_1$ is a vertexless triangle with one point deleted from one of its sides. This example exists also in non-Desarguesian planes, but it is a semioval of V_1° type.

Semiarcs that properly contain a V_t-configuration exist in Π_q whenever Π_q contains a subplane. Some of the following examples were motivated by an example due to Korchmáros and Mazzocca (see [11], pg. 64).

Example 16. Let $\Pi^0, \Pi^1, \ldots, \Pi^{s-1}$ be subplanes of $\Pi^s := \Pi_q$ such that $\Pi^{i-1} \subset \Pi^i$ for $i = 1, \ldots, s$. Denote by r the order of Π^0 and let ℓ_1 and ℓ_2 be two lines in this plane. Let $P = \ell_1 \cap \ell_2$ and set

$$S(0) := (\ell_1 \cup \ell_2) \cap (\Pi^0 \setminus P),\ S(j) := (\ell_1 \cup \ell_2) \cap (\Pi^j \setminus \Pi^{j-1}), \text{ for } j = 1, \ldots, s.$$

By I we denote a subset of $\{1, 2, \ldots, s\}$. We give four examples.

1. Let ℓ be a line in Π^0 passing through P and let Z be a subset of $(\ell \cap \Pi^0) \setminus \{P\}$ of size at least two. If I is not empty, then $\mathcal{S}_t := \cup_{j \in I} S(j) \cup Z$ is a t-semiarc of V_t° type with $t = q - \frac{1}{2}\sum_{j \in I} |S(j)|$.

2. Let ℓ be a line in Π^0 that does not pass through P and let Z be a subset of $(\ell \cap \Pi^0) \setminus (\ell_1 \cup \ell_2)$ of size at least two. If I is not empty, then $\mathcal{S}_t := \cup_{j \in I} S(j) \cup Z$ is a t-semiarc of V_t° type with $t = q - \frac{1}{2}\sum_{j \in I} |S(j)|$.

3. Let Z be a subset of $\Pi^0 \setminus (\ell_1 \cup \ell_2)$ such that there is no line in Π^0 passing through P and meeting Z in exactly one point. If I is a proper subset of $\{1, 2, \ldots, s\}$, then $\mathcal{S}_t := \cup_{j \in I} S(j) \cup Z \cup S(0)$ is a t-semiarc of V_t° type with $t = q - r - \frac{1}{2}\sum_{j \in I} |S(j)|$.

4. Let Z be a subset of $\Pi^0 \setminus (\ell_1 \cup \ell_2)$ such that for each line $\ell \neq \ell_1, \ell_2$ through P, ℓ is a line in Π^0, we have $|\ell \cap Z| \geqslant 1$. Then $\mathcal{S}_t := \{P\} \cup S(0) \cup Z$ is a t-semiarc of $V_t^\bullet$ type with $t = q - r$.

4 Semiarcs containing a V_t-configuration in $\mathrm{PG}(2, q)$

In this section our aim is to characterize t-semiarcs containing a V_t-configuration in $\mathrm{PG}(2, q)$. We will need the following definition.

Definition 17. Let ℓ_1 and ℓ_2 be two lines in a projective plane and let P denote their common point. We say that $X_1 \subseteq \ell_1 \setminus P$ and $X_2 \subseteq \ell_2 \setminus P$ are two perspective point sets if there is a point Q such that each line through Q intersects both X_1 and X_2 or intersects none of them. In other words, there is a perspectivity which maps X_1 onto X_2.

Lemma 18. *Let $\mathcal{S}_t$ be a t-semiarc in Π_q and suppose that $(\ell_1 \triangle \ell_2) \cap \mathcal{S}_t$ is a V_t-configuration for some lines ℓ_1 and ℓ_2. If $\mathcal{S}_t \not\subseteq \ell_1 \cup \ell_2$, then $\mathcal{S}_t \cap (\ell_1 \setminus \ell_2)$ and $\mathcal{S}_t \cap (\ell_2 \setminus \ell_1)$ are perspective point sets and each point of $\mathcal{S}_t \setminus (\ell_1 \cup \ell_2)$ is the centre of a perspectivity which maps $\mathcal{S}_t \cap (\ell_1 \setminus \ell_2)$ onto $\mathcal{S}_t \cap (\ell_2 \setminus \ell_1)$.*

Proof. Let $X = \mathcal{S}_t \setminus (\ell_1 \cup \ell_2)$ and $X_i = \mathcal{S}_t \cap (\ell_i \setminus \ell_j)$, for $\{i, j\} = \{1, 2\}$. For each $Q \in X$, if there were a line ℓ through Q intersecting X_i but not X_j, then the point $\ell \cap X_i \in \mathcal{S}_t$ would have at most $t - 1$ tangents. This shows that each point of X is the centre of a perspectivity which maps X_1 onto X_2. If $\mathcal{S}_t \not\subseteq \ell_1 \cup \ell_2$, then X is not empty, hence X_1 and X_2 are perspective point sets. □

The following theorem characterizes perspective point sets in $\mathrm{PG}(2,q)$. This result was first published by Korchmáros and Mazzoca in [11] but we will use the notation of [4] by Bruen, Mazzocca and Polverino.

Theorem 19 ([4], Result 2.2, Result 2.3, Result 2.4, see also [11])**.** *Let ℓ_1 and ℓ_2 be two lines in $\mathrm{PG}(2,q)$, $q=p^r$, and let P denote their common point. Let $X_1 \subseteq \ell_1 \setminus P$ and $X_2 \subseteq \ell_2 \setminus P$ be two perspective point sets. Denote by U the set of all points which are centres of a perspectivity mapping X_1 onto X_2. Using a suitable projective frame in $\mathrm{PG}(2,q)$, there exist an additive subgroup B of $\mathrm{GF}(q)$ and a multiplicative subgroup A of $\mathrm{GF}(q)$ such that:*

(a) B is a subspace of $\mathrm{GF}(q)$ of dimension h_1 considered as a vectorspace over a subfield $\mathrm{GF}(q_1)$ of $\mathrm{GF}(q)$ with $q_1=p^d$ and $d|r$. This implies that B is an additive subgroup of $\mathrm{GF}(q)$ of order p^h with $h=dh_1$.

(b) A is a multiplicative subgroup of $\mathrm{GF}(q_1)$ of order n, where $n|(p^d-1)$. In this way, B is invariant under A, i.e. $B=AB:=\{ab\colon a\in A,\, b\in B\}$.

(c) If G_i denotes the full group of affinities of $\ell_i \setminus P$ preserving the set X_i, $i=1,2$, then $G_1 \cong G_2 \cong G = G(A,B) = \{g\colon g(y)=ay+b,\, a\in A,\, b\in B\} \leqslant \Sigma$, where Σ is the full affine group on the line $\mathrm{AG}(1,q)$.

(d) X_i is a union of orbits of G_i on $\ell_i \setminus P$, $i=1,2$, and $|U|=|G|=np^h$.

(e) For every two integers n, h, such that $n|(p^d-1)$ and $d|\gcd(r,h)$, there exists in Σ a subgroup of type $G=G(A,B)$ of order np^h, where A and B are multiplicative and additive subgroups of $\mathrm{GF}(q)$ of order n and p^h, respectively.

(f) G has one orbit of length p^h on $\mathrm{AG}(1,q)$, namely B, and G acts regularly on the remaining orbits, say $O_1, O_2, \ldots, O_m$, where

$$m = \frac{q-p^h}{np^h} = \frac{p^{r-h}-1}{n}.$$

In the sequel we denote by B^i the orbit of G_i on $\ell_i \setminus P$ corresponding to B and by $O^i_1, O^i_2, \ldots, O^i_m$ the remaining orbits, for $i=1,2$. With this notation B^1 is the image of B^2 under the perspectivities with centre in U and also O^2_j is the image of O^1_j for $j=1,2,\ldots,m$ and vice versa.

(g) $B^1 \subseteq X_1$ if and only if $B^2 \subseteq X_2$ and the same holds for the other orbits, i.e. $O^1_j \subseteq X_1$ if and only if $O^2_j \subseteq X_2$, for $j=1,2,\ldots,m$.

(h) If a line ℓ not through P meets U in at least two points, then ℓ intersects both B^1 and B^2.

Exactly one of the following cases must occur.

1. *Both A and B are trivial. Then U consists of a singleton.*

2. *A is trivial and B is not trivial. Then U is a set of p^h points all collinear with the point P.*

3. *B is trivial and A is not trivial. Then U is a set of n points on a line not through P.*

4. *A and B are the multiplicative and the additive group, respectively, of a subfield $\mathrm{GF}(p^h)$ of $\mathrm{GF}(q)$. Then*

$$U \cup B^1 \cup B^2 \cup \{P\} = \mathrm{PG}(2, p^h).$$

5. *None of the previous cases occur. Then U is a point set of size np^h and of type $(0, 1, n, p^h)$, i.e. 0, 1, n, p^h are the only intersection numbers of U with respect to the lines in $\mathrm{PG}(2, q)$. In addition, using the fact that $|U| = np^h$,*

   - *there are exactly n lines intersecting U in exactly p^h points and they are all concurrent at the common point P of ℓ_1 and ℓ_2,*
   - *each line intersecting U in exactly n points meets both B^1 and B^2.*

Lemma 20 ([6], Proposition 3.1)**.** *If $\mathcal{S}_t$ is a $(q-2)$-semiarc in Π_q, then it is one of the following three configurations: four points in general position, the six vertices of a complete quadrilateral, or a Fano subplane.*

In the next theorems we will use the notation of Theorem 19.

Theorem 21. *Let $\mathcal{S}_t$ be a t-semiarc in $\mathrm{PG}(2, q)$, $q = p^r$, and suppose that $(\ell_1 \triangle \ell_2) \cap \mathcal{S}_t$ is a V_t-configuration for some lines ℓ_1 and ℓ_2. To avoid trivial cases, suppose that $\mathcal{S}_t \nsubseteq \ell_1 \cup \ell_2$.*

Let $X_i = \ell_i \cap \mathcal{S}_t$, for $i = 1, 2$, and let $X = \mathcal{S}_t \setminus (\ell_1 \cup \ell_2)$. Also let $P = \ell_1 \cap \ell_2$. Because of Lemma 18 we have that X_1 and X_2 are perspective point sets and $X \subseteq U$, where U is the set of all points which are centres of a perspectivity mapping X_1 onto X_2. Choose a suitable coordinate system as in Theorem 19 and suppose that the size of $G = G(A, B)$ is np^h, i.e. $|A| = n$ and $|B| = p^h$, where A and B are the multiplicative and the additive subgroup of $\mathrm{GF}(q)$ associated to the perspective point sets X_1 and X_2.

(I) If $P \notin \mathcal{S}_t$, i.e. $\mathcal{S}_t$ is of V_t° type, then one of the following holds.

- *(i) X is contained in a line through P that meets U in p^h points, $h \geqslant 1$, and we have $2 \leqslant |X| \leqslant p^h$,*
- *(ii) X is contained in a line not through P that meets U in $n \geqslant 2$ points and we have $2 \leqslant |X| \leqslant n$,*
- *(iii) $|X| \geqslant 2$ and X is a subset of U such that there is no line through P that meets X in exactly one point.*

In the first two cases $X_i = \cup_{j\in I} O^i_j$ *for some not empty subset* $I \subseteq \{1, 2, \dots, m\}$ *and for* $i = 1, 2$. *We have* $t = q - knp^h$, *where* $k = |I|$ *and* $1 \leqslant k \leqslant m$, *where* $m = (p^{r-h} - 1)/n$.

In the third case $X_i = \cup_{j\in I} O^i_j \cup B^i$ *for some proper subset* $I \subset \{1, 2, \dots, m\}$ *and for* $i = 1, 2$. *We have* $t = q - knp^h - p^h$, *where* $k = |I|$, $h \geqslant 1$ *and* $0 \leqslant k \leqslant m - 1$.

(II) If $P \in \mathcal{S}_t$, *i.e.* $\mathcal{S}_t$ *is of* $V_t^\bullet$ *type, then one of the following holds.*

(i) $\mathcal{S}_t$ *consists of the six vertices of a complete quadrilateral or* $\mathcal{S}_t$ *is a Fano subplane. We have* $t = q - 2$ *in both cases.*

(ii) ℓ_1 *and* ℓ_2 *are lines in the subplane* $\mathrm{PG}(2, p^h)$ *and*

$$\mathcal{S}_t = \mathrm{PG}(2, p^h) \cap (\ell_1 \cup \ell_2) \cup X,$$

where X *is a subset of* $\mathrm{PG}(2, p^h) \setminus (\ell_1 \cup \ell_2)$ *such that for each line* $\ell \neq \ell_1, \ell_2$ *through* P, ℓ *is a line in* $\mathrm{PG}(2, p^h)$, *we have* $|\ell \cap X| \geqslant 1$. *In this case* $t = q - p^h$.

(iii) $\mathcal{S}_t$ *is projectively equivalent to the following set of* $3(n + 1)$ *points:*

$$\mathcal{S}_t := \{(a, 0, 1), (0, -a, 1), (a, 1, 0) \colon a \in A\} \cup \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\},$$

In this case $t = q - 1 - n$, *where* $n \mid q - 1$.

The converse is also true, if X_1 *and* X_2 *are perspective point sets and* X *is as in one of the three cases in (I), then* $X \cup X_1 \cup X_2$ *is a* t*-semiarc of* V_t° *type. If* $\mathcal{S}_t$ *is as in one of the three cases in (II), then* $\mathcal{S}_t$ *is a* t*-semiarc of* $V_t^\bullet$ *type.*

Proof. We begin by proving (I). First assume $B^1 \subseteq \ell_1 \setminus X_1$. Then Theorem 19 (g) implies $B^2 \subseteq \ell_2 \setminus X_2$. Suppose that there exist three non-collinear points in X, say L, M and N. Then between the lines LM, LN and MN there are at least two, say LM and LN, not through P. Theorem 19 (h) and $X \subseteq U$ imply that these two lines intersect both B^1 and B^2. But then through L there pass at most $t - 1$ tangents, a contradiction. It follows that X is contained in a line and hence it is as in one of our first two cases. The condition $|X| \geqslant 2$ comes from Proposition 12 (a).

Now asume $B^1 \subseteq X_1$ and hence $B^2 \subseteq X_2$. In this case for every two points $M, N \in X$, the line MN intersects ℓ_i in X_i, for $i = 1, 2$. Thus the number of tangents through a point $L \in X$ is t if and only if the line LP contains at least one other point of X. Case 3 of Theorem 19 shows that this is not possible when B is trivial, i.e. when $h = 0$. Hence X is as in our third case.

Now we prove (II). First assume $B^1 \subseteq \ell_1 \setminus X_1$ and hence $B^2 \subseteq \ell_2 \setminus X_2$. Suppose that there exist two points in X, say M and N, not collinear with P. Then the line MN intersects ℓ_1 and ℓ_2 not in $\mathcal{S}_t$. But then the number of tangents through M is at most $t - 1$, a contradiction. Thus X is contained in a line through P and through P there pass exactly $q - 2$ tangents. So $\mathcal{S}_t$ is a $(q - 2)$-semiarc. According to Lemma 20, $\mathcal{S}_t$ is as in (II)(i).

Now assume $B^1 \subseteq X_1$ and hence $B^2 \subseteq X_2$. In this case $t = q - knp^h - p^h$ for some $k \in \{0, 1, \dots, m - 1\}$, where m is the number of orbits of G of size np^h on $\mathrm{AG}(1, q) \setminus B$.

Since P has exactly t tangents, there are $q+1-t$ non-tangent lines through P. According to Theorem 19, we have $q-1-t \leqslant n$ and hence $knp^h + p^h - 1 \leqslant n$. We distinguish two subcases.

If $h > 0$, then $n|p^h - 1$ implies $n \leqslant p^h - 1$ and hence $knp^h = 0$. This occurs only if $k = 0$ and $n = p^h - 1$. But n divides $p^d - 1$, where $d|h$ and B is a subspace over the field $\mathrm{GF}(p^d)$. This implies $d = h$, thus B is a subfield and U is as in case 4 of Theorem 19. This is only possible if $\mathcal{S}_t$ is as in our second case.

If $h = 0$, then $kn \leqslant n$ and U is as in case 3 of Theorem 19. If $k = 0$, then $t = q - 1$, which we excluded. Thus we have $k = 1$ and $t = q - n - 1$. This occurs only if $\mathcal{S}_t$ is as in our third case (see [4], pg. 56–57). □

Theorem 22. *Let $\mathcal{S}_t$ be a t-semiarc of V_t° type in $\mathrm{PG}(2,q)$, $q = p^r$. Then the following hold.*

(a) If $\gcd(q,t) = 1$ and $\gcd(q-1,t-1) = 1$, then $\mathcal{S}_t$ is a V_t-configuration.

(b) If $\gcd(q,t) = 1$, then $\mathcal{S}_t$ is contained in a vertexless triangle.

(c) If $\gcd(q-1,t) = 1$, then $\mathcal{S}_t$ is contained in a vertexless triangle or in the union of three concurrent lines without their common point.

Proof. We have $p^h|t$ in all three cases of Theorem 21 (I), where p^h is the size of B. Hence $\gcd(q,t) = 1$ implies $p^h = 1$, i.e. $h = 0$. This occurs only in the second case of Theorem 21 (I) and this proves (b).

In the first two cases of Theorem 21 (I) we have $n|(t-1)$ and hence also $n|\gcd(q-1,t-1)$, where n is the size of A. We have seen previously that $\gcd(q,t) = 1$ can hold only in the second case of Theorem 21 (I). But in that case we have $n \geqslant 2$, which is a contradiction when $\gcd(q-1,t-1) = 1$. This proves (a).

If $\mathcal{S}_t$ is as in one of the first two cases of Theorem 21 (I), then we are done. So to prove (c), it is enough to consider Theorem 21 (I)(iii). In this case $t = (q-1) - nkp^h - (p^h - 1)$ and hence $n|\gcd(q-1,t)$. If $\gcd(q-1,t) = 1$, then $n = 1$, i.e. A is trivial. If this happens, then case 2 of Theorem 19 implies that $\mathcal{S}_t$ is contained in the union of three concurrent lines without their common point. □

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