Minimal crystallizations of 3-manifolds

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Abstract

We have introduced the weight of a group which has a presentation with number of relations is at most the number of generators. We have shown that the number of facets of any contracted pseudotriangulation of a connected closed 3-manifold M is at least the weight of the fundamental group of M. This lower bound is sharp for the 3-manifolds $\mathbb{RP}^3$, $L(3,1)$, $L(5,2)$, $S^1 \times S^1 \times S^1$, $S^2 \times S^1$, $S^2 \times\!\!\!\!\!\times S^1$ and S^3/Q_8, where Q_8 is the quaternion group. Moreover, there is a unique such facet minimal pseudotriangulation in each of these seven cases. We have also constructed contracted pseudotriangulations of $L(kq-1,q)$ with $4(q+k-1)$ facets for $q \geqslant 3$, $k \geqslant 2$ and $L(kq+1,q)$ with $4(q+k)$ facets for $q \geqslant 4$, $k \geqslant 1$. By a recent result of Swartz, our pseudotriangulations of $L(kq+1,q)$ are facet minimal when $kq+1$ are even. In 1979, Gagliardi found presentations of the fundamental group of a manifold M in terms of a contracted pseudotriangulation of M. Our construction is the converse of this, namely, given a presentation of the fundamental group of a 3-manifold M, we construct a contracted pseudotriangulation of M. So, our construction of a contracted pseudotriangulation of a 3-manifold M is based on a presentation of the fundamental group of M and it is computer-free.

Keywords: Pseudotriangulations of manifolds, Crystallizations, Lens spaces, Presentations of groups.

1 Introduction and Results

A *simplicial cell complex* K of dimension d is a poset isomorphic to the face poset $\mathcal{X}$ of a d-dimensional simplicial CW-complex X. The topological space X is called the *geometric carrier* of K and is also denoted by $|K|$. If a topological space M is homeomorphic to $|K|$, then K is said to be a *pseudotriangulation* of M. For $d \geqslant 1$, a $(d+1)$-colored contracted graph $\Gamma = (V, E)$ with an edge coloring $\gamma : E \to \{1, \dots, d+1\}$ determines a

d-dimensional simplicial cell complex $\mathcal{K}(\Gamma)$ whose vertices have one to one correspondence with the colors $1, \dots, d+1$ and the facets have one to one correspondence with the vertices in V. If $\mathcal{K}(\Gamma)$ is a pseudotriangulation of a space M then (Γ, γ) is called a *crystallization* of M. So, if (Γ, γ) is a crystallization of a d-manifold M then the number of vertices in the pseudotriangulation $\mathcal{K}(\Gamma)$ of M is $d+1$. In [15], Pezzana showed the following.

Proposition 1 (Pezzana)**.** *Every connected closed PL-manifold admits a crystallization.*

Thus, every connected closed pl d-manifold has a *contracted* pseudotriangulation, i.e., a pseudotriangulation with $d+1$ vertices. In this article, we are interested in crystallizations of connected closed 3-manifolds with minimum number of vertices.

In [6], Epstein proved that the fundamental group of a 3-manifold has a presentation with the number of relations less than or equal to the number of generators. For such a group G, we define the weight $\psi(G)$ of G in Definition 10 below. The weight of the trivial group is 2 and $\psi(G) \geqslant 8$ for any non-trivial group G as we see later.

Definition 2. For a connected closed 3-manifold M, let $\psi(M)$ be the *weight* $\psi(\pi(M, x))$ of the group $\pi(M, x)$ for some x in M.

If M and N are homeomorphic then clearly $\psi(M) = \psi(N)$. Thus, $\psi(M)$ is a topological invariant. Clearly, $\psi(S^3) = 2$ and, in view of Perelman's theorem (Poincaré conjecture) [14], $\psi(M) \geqslant 8$ for $M \neq S^3$. Here, we have the following.

Lemma 3. *Let $\psi(M)$ be as above and let Q_8 be the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. Then $\psi(\mathbb{RP}^3) = \psi(S^2 \times S^1) = \psi(S^2 \mathbin{\underline{\times}} S^1) = 8$, $\psi(L(3,1)) = 12$, $\psi(L(5,q)) = 16$, $\psi(S^3/Q_8) = 18$, $\psi(S^1 \times S^1 \times S^1) = 24$ for $1 \leqslant q \leqslant 2$.*

For a d-dimensional simplicial cell complex K, let $f_j(K)$ denote the number of j-cells of K for $0 \leqslant j \leqslant d$. Let $g_2(K) := f_1(K) - (d+1)f_0(K) + \binom{d+2}{2}$ and $h_2(K) := f_1(K) - df_0(K) + \binom{d+1}{2}$. For a connected simplicial cell complex K, let $m(K)$ be the minimal number of generators of $\pi(|K|, *)$. For a connected closed pl d-manifold M, let

$$\begin{aligned}\Psi(M) &= \min\{m : M \text{ has a crystallization with } m \text{ vertices}\} \\ &= \min\{f_d(K) : K \text{ is a contracted pseudotriangulation of } M\}.\end{aligned}$$

In [11], Klee proved that $h_2(K) \geqslant \binom{d+1}{2} m(K)$ for any d-dimensional normal pseudomanifold K whose edge graph is $(d+1)$-colorable. Here we have the following.

Theorem 4. *Let M be a connected closed 3-manifold. If (Γ, γ) is a crystallization of M then Γ has at least $\psi(M)$ vertices. Equivalently, if X is a contracted pseudotriangulation of M then $f_3(X) \geqslant \psi(M)$.*

Corollary 5. *Let M be a connected closed 3-manifold M and $\mathbb{F}$ be a field. If X is a contracted pseudotriangulation of M then $g_2(X) = h_2(X) \geqslant \Psi(M) - 2 \geqslant \psi(M) - 2 \geqslant 6m(M) \geqslant 6\beta_1(M; \mathbb{F})$.*

Consider the contracted pseudotriangulation $K_1 := \mathcal{K}(\mathcal{J}_1)$ of $S^2 \times S^1$ corresponding to the crystallization $\mathcal{J}_1$ in Fig. 2 below. Since $f_3(K_1) = 8$, it follows that $f_2(K_1) = 16$ and hence $f_1(K_1) = 12$. Therefore, $g_2(K_1) = 12 - 16 + 10 = 6 = 6\beta_1(S^2 \times S^1; \mathbb{Q})$. Thus, the inequalities in Corollary 5 are equalities and (hence) the lower bound is sharp.

From the complete enumeration (obtained by using high-powered computers) of crystallizations of prime 3-manifolds with at most 30 vertices, we know $\Psi(M)$ for all closed prime 3-manifolds M with $\Psi(M) \leqslant 30$ (cf. [3, 12]). In particular, we know that the minimal crystallizations of several 3-manifolds are unique and there are 3-manifolds which have more than one minimal crystallizations (see Remark 25 below). We have proved the existence and the uniqueness of some crystallizations using presentations of the fundamental groups. Consider a group G which has a presentation with number of relations is at most the number of generators. From Theorem 4 we know that the number of vertices in any crystallization (Γ, γ) of a closed connected 3-manifold M, whose fundamental group is G, is at least $\psi(G)$. We have constructed crystallizations on $\psi(G)$ vertices which yield presentations of G as mentioned at the end of Section 2.4. We have considered the groups $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$, $\mathbb{Z}^3$ and Q_8 and have obtained such crystallizations. Generalizing some of these constructions, we have constructed two infinite families of crystallizations of lens spaces. More explicitly, we have the following.

Theorem 6. (i) *If $M = \mathbb{RP}^3$, $S^2 \times S^1$, $S^2 \underline{\times} S^1$, $L(3,1)$, $L(5,2)$, S^3/Q_8 or $S^1 \times S^1 \times S^1$ then $\Psi(M) = \psi(M)$ and M has a unique contracted pseudotriangulation with $\psi(M)$ facets.*

(ii) *Let X be a contracted pseudotriangulation of a connected closed 3-manifold M. If $f_3(X) \leqslant 8$ then M is (homeomorphic to) S^3, $\mathbb{RP}^3$, $S^2 \times S^1$ or $S^2 \underline{\times} S^1$.*

Corollary 7. *Let X be a contracted pseudotriangulation of a closed 3-manifold M. If M is S^3/Q_8, $S^1 \times S^1 \times S^1$ or $L(p,q)$ for some $p \geqslant 3$ then $h_2(X) > 6m(M)$.*

Theorem 8. (i) *$\Psi(L(kq-1,q)) \leqslant 4(k+q-1)$ for $k,q \geqslant 2$ and*

(ii) *$\Psi(L(kq+1,q)) \leqslant 4(k+q)$ for $k,q \geqslant 1$.*

Remark 9. Recently, Swartz proved that $\Psi(L(kq+1,q)) \geqslant 4(k+q)$ whenever k,q are odd ([16]). Thus, $\Psi(L(kq+1,q)) = 4(k+q)$ for odd positive integers k,q. We found that $\Psi(L(5,1)) = 20 = \Psi(L(7,2))$. So, Swartz's bound is also valid for $L(5,1)$ and $L(7,2)$. We also found that $\psi(\mathbb{Z}_4) = 14$ and $\psi(\mathbb{Z}_6) = \psi(\mathbb{Z}_7) = 18$. Proofs of these are in earlier versions of this article in the arXiv (arXiv:1308.6137). We have omitted these proofs from this version for the sake of brevity.

2 Preliminaries

2.1 Colored Graphs

All graphs considered here are finite multigraphs without loops. If $\Gamma = (V,E)$ is a graph and $U \subseteq V$ then the *induced* subgraph $\Gamma[U]$ is the subgraph of Γ whose vertex set is U and edges are those edges of Γ whose end points are in U. For $n \geqslant 2$, an n-cycle is a closed path with n distinct vertices and n edges. If vertices a_i and a_{i+1} are adjacent in an n-cycle for $1 \leqslant i \leqslant n$ (addition is modulo n) then the n-cycle is denoted by $C_n(a_1, a_2, \dots, a_n)$. A graph Γ is called *h-regular* if the number of edges adjacent to each vertex is h.

An *edge coloring* of a graph $\Gamma = (V, E)$ is a map $\gamma \colon E \to C$ such that $\gamma(e) \neq \gamma(f)$ whenever e and f are adjacent (i.e., e and f are adjacent to a common vertex). The elements of the set C are called the *colors*. If C has h elements then (Γ, γ) is said to be an *h-colored graph*.

Let (Γ, γ) be an h-colored graph with color set C. If $B \subseteq C$ with k elements then the graph $(V(\Gamma), \gamma^{-1}(B))$ is a k-colored graph with coloring $\gamma|_{\gamma^{-1}(B)}$. This colored graph is denoted by Γ_B. Let (Γ, γ) be an h-colored connected graph with color set C. If $\Gamma_{C\setminus\{c\}}$ is connected for all $c \in C$ then (Γ, γ) is called *contracted*.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two disjoint h-regular h-colored graphs with same color set $\{1, \dots, h\}$. For $1 \leqslant i \leqslant 2$, let $v_i \in V_i$. Consider the graph Γ which is obtained from $(\Gamma_1 \setminus \{v_1\}) \sqcup (\Gamma_2 \setminus \{v_2\})$ by adding h new edges $e_1, \dots, e_h$ with colors $1, \dots, h$ respectively such that the end points of e_j are $u_{j,1}$ and $u_{j,2}$, where v_i and $u_{j,i}$ are joined in Γ_i with an edge of color j for $1 \leqslant j \leqslant h$, $1 \leqslant i \leqslant 2$. (Here $\Gamma_i \setminus \{v_i\} = \Gamma_i[V_i \setminus \{v_i\}]$.) The colored graph Γ is called the *connected sum* of Γ_1, Γ_2 and is denoted by $\Gamma_1 \#_{v_1 v_2} \Gamma_2$.

Let $\Gamma = (V, E)$ be a $(d+1)$-regular graph with a $(d+1)$-coloring $\gamma \colon E \to C$. Let $x, y \in V$ be joined by k edges $e_1, \dots, e_k$, where $1 \leqslant k \leqslant d$. Let $B = C \setminus \gamma(\{e_1, \dots, e_k\})$. Let X (resp., Y) be the components of Γ_B containing x (resp., y). If $X \neq Y$ then $\Gamma[\{x, y\}]$ is called a *d-dimensional dipole of type k*. Dipoles of types 1 and d are called *degenerate* dipoles.

Let $\Gamma = (V, E)$ be a $(d+1)$-regular graph with a $(d+1)$-coloring $\gamma \colon E \to C$ and a dipole $\Gamma[\{x, y\}]$ of type k. Let B, X and Y be as above. A $(d+1)$-regular graph (Γ', γ') with same color set C is said to *obtained from* Γ *by cancelling the dipole* $\Gamma[\{x, y\}]$ if (i) Γ'_B is obtained from Γ_B by replacing $X \sqcup Y$ by $X \#_{xy} Y$, and (ii) two vertices u, v of Γ' are joined by an edge of color $c \in B$ if and only if the corresponding vertices of Γ are so (cf. [7]). For standard terminology on graphs see [2].

2.2 Presentation of Groups

Given a set S, let $F(S)$ denote the free group generated by S. So, any element w of $F(S)$ is of the form $w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m}$, where $x_1, \dots, x_m \in S$ and $\varepsilon_i = \pm 1$ for $1 \leqslant i \leqslant m$ and $(x_{j+1}, \varepsilon_{j+1}) \neq (x_j, -\varepsilon_j)$ for $1 \leqslant j \leqslant m-1$. For $R \subseteq F(S)$, let $N(R)$ be the smallest normal subgroup of $F(S)$ containing R. Then the quotient group $F(S)/N(R)$ is denoted by $\langle S \,|\, R \rangle$. So, $\langle S \,|\, T \rangle = \langle S \,|\, R \rangle$ if $N(T) = N(R)$. We write $\langle S_1 \,|\, R_1 \rangle = \langle S_2 \,|\, R_2 \rangle$ only when $F(S_1) = F(S_2)$ and $N(R_1) = N(R_2)$. For $w_1, w_2 \in F(S)$, if $w_1 N(R) = w_2 N(R) \in \langle S \,|\, R \rangle$ then we write $w_1 \equiv w_2 \pmod{R}$. Two elements $w_1, w_2 \in F(S)$ are said to be *independent* (resp., *dependent*) if $N(\{w_1\}) \neq N(\{w_2\})$ (resp., $N(\{w_1\}) = N(\{w_2\})$).

For a finite subset R of $F(S)$, let

$$\overline{R} := \{w \in N(R) \, : \, N((R \setminus \{r\}) \cup \{w\}) = N(R) \text{ for each } r \in R\}. \tag{2.1}$$

Observe that $\overline{\emptyset} = \emptyset$ and if $R \neq \emptyset$ is a finite set then $w := \prod_{r\in R} r \in \overline{R}$ and hence $\overline{R} \neq \emptyset$. Also, $\{wrw^{-1}, wr^{-1}w^{-1} : w \in F(S)\} \subseteq \overline{\{r\}}$ for $r \in F(S)$.

For $w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m} \in F(S)$, $m \geqslant 1$, let

$$\varepsilon(w) := \begin{cases} 0 & \text{if} \quad m = 1, \\ |\varepsilon_1 - \varepsilon_2| + \cdots + |\varepsilon_{m-1} - \varepsilon_m| + |\varepsilon_m - \varepsilon_1| & \text{if} \quad m \geqslant 2. \end{cases}$$

Consider the map $\lambda \colon F(S) \to \mathbb{Z}^+$ define inductively as follows.

$$\lambda(w) := \begin{cases} 2 & \text{if} \quad w = \emptyset, \\ 2m - \varepsilon(w) & \text{if} \quad w = x_1^{\varepsilon_1} \cdots x_m^{\varepsilon_m},\ (x_m, \varepsilon_m) \neq (x_1, -\varepsilon_1), \\ \lambda(w') & \text{if} \quad w = x_1^{\varepsilon_1} w' x_1^{-\varepsilon_1}. \end{cases} \tag{2.2}$$

Since $|\varepsilon_i - \varepsilon_j| = 0$ or 2, $\varepsilon(w)$ is an even integer and hence $\lambda(w)$ is also even. For $w \in F(S)$, $\lambda(w)$ is said to be the *weight* of w. Observe that $\lambda(w_1w_2) = \lambda(w_2w_1)$ for $w_1, w_2 \in F(S)$.

Let $S = \{x_1, \ldots, x_s\}$ and $R = \{r_1, \ldots, r_t\} \subseteq F(S)$, where $t \leqslant s$. Let r_{t+1} be an element in $\overline{R}$ of minimum weight. Let

$$\varphi(S, R) := \lambda(r_1) + \cdots + \lambda(r_t) + \lambda(r_{t+1}) + 2(s - t). \tag{2.3}$$

For a finitely presented group G and a non-negative integer q, we define

$$\mathcal{P}_q(G) := \{\langle S \,|\, R\rangle \cong G \,:\, \#(R) \leqslant \#(S) \leqslant q\}.$$

For a finitely presented group G, let $m(G)$ be the minimum number of generators of G. Here, we are interested on those groups G for which $\mathcal{P}_q(G) \neq \emptyset$ for some q. Let

$$\mu(G) := \min\{q \,:\, \mathcal{P}_q(G) \neq \emptyset\}, \tag{2.4a}$$

$$\psi(G; q) := \min\{\varphi(S, R) \,:\, \langle S \,|\, R\rangle \in \mathcal{P}_q(G)\} \text{ for } q \geqslant \mu(G). \tag{2.4b}$$

Clearly, $\mu(G) \geqslant m(G)$ and $\psi(G, q) \leqslant \psi(G, \mu(G))$ for all $q \geqslant \mu(G)$. Let

$$\rho(G) := \min\{q \geqslant \mu(G) \,:\, \psi(G; q) \leqslant 6(q + 1)\}. \tag{2.5}$$

So, $\rho(G)$ is the smallest integer q such that $\psi(G; q) \leqslant 6(q + 1)$.

Definition 10. Let G be a group which has a presentation with the number of relations less than or equal to the number of generators. Let $\mu(G)$, $\psi(G; q)$ and $\rho(G)$ be as above. Then $\psi(G) = \max\{\psi(G; \rho(G)), 6\mu(G) + 2\}$ is a positive even integer. The integer $\psi(G)$ is said to be the *weight* of the group G.

Remark 11. Observe that $\min\{\varphi(S, R) \,:\, \langle S \,|\, R\rangle \cong \mathbb{Z}, \#(R) \leqslant \#(S) < \infty\} = 4 = \psi(\mathbb{Z}, \rho(\mathbb{Z})) < 8 = \psi(\mathbb{Z})$ (see the proof of Lemma 3). In general, we have $\min\{\varphi(S, R) \,:\, \langle S \,|\, R\rangle \cong G, \#(R) \leqslant \#(S) < \infty\} = \min\{\min\{\varphi(S, R) \,:\, \langle S \,|\, R\rangle \in \mathcal{P}_q(G)\} \,:\, \mu(G) \leqslant q < \infty\} = \min\{\psi(G; q) \,:\, \mu(G) \leqslant q < \infty\} \leqslant \psi(G; \rho(G)) \leqslant \psi(G)$.

2.3 Lens Spaces

Consider the 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Let p and q be relatively prime integers. Then the action of $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ on S^3 generated by $e^{2\pi i/p} \cdot (z_1, z_2) = (e^{2\pi i/p} z_1, e^{2\pi iq/p} z_2)$ is free and hence properly discontinuous. Therefore the quotient space $L(p, q) := S^3/\mathbb{Z}_p$ is a 3-manifold whose fundamental group is isomorphic to $\mathbb{Z}_p$. The 3-manifolds $L(p, q)$ are called the lens spaces. It is a classical theorem of Reidemeister that $L(p, q')$ is homeomorphic to $L(p, q)$ if and only if $q' \equiv \pm q^{\pm 1} \pmod{p}$.

If T_1, T_2 are two solid tori (i.e., each T_j is homeomorphic to $\{(z, w) \in \mathbb{C}^2 : |z| = 1, |w| \leqslant 1\}$) such that (i) $T_1 \cap T_2 = \partial(T_1) = \partial(T_2) \cong S^1 \times S^1$, (ii) $\pi_1(T_1 \cap T_2, x) = \langle \alpha, \beta \,|\, \alpha\beta\alpha^{-1}\beta^{-1} \rangle$, (iii) $\pi_1(T_1, x) = \langle \alpha \rangle$ and (iv) $\pi_1(T_2, x) = \langle \alpha, \beta \,|\, \alpha\beta\alpha^{-1}\beta^{-1}, \alpha^p\beta^q \rangle$ ($= \langle \alpha^m\beta^n \rangle$, where $m, n \in \mathbb{Z}$ such that $mq - np = 1$), for $x \in T_1 \cap T_2$, then $T_1 \cup T_2$ is homeomorphic to $L(p, q)$.

2.4 Crystallizations

A CW-complex X is said to be *regular* if the attaching maps which define the incidence structure of X are homeomorphisms. Given a regular CW-complex X, let $\mathcal{X}$ be the set of all closed cells of X together with the empty set. Then $\mathcal{X}$ is a poset, where the partial ordering is the set inclusion. This poset $\mathcal{X}$ is said to be the *face poset* of X. Clearly, if X and Y are two finite regular CW-complexes with isomorphic face posets then X and Y are homeomorphic. A regular CW-complex X is said to be *simplicial* if the boundary of each cell in X is isomorphic (as a poset) to the boundary of a simplex of same dimension. A *simplicial cell complex* K of dimension d is a poset isomorphic to the face poset $\mathcal{X}$ of a d-dimensional simplicial CW-complex X. The topological space X is called the *geometric carrier* of K and is also denoted by $|K|$. If a topological space M is homeomorphic to $|K|$, then K is said to be a *pseudotriangulation* of M. A simplicial cell complex K is said to be *connected* if the topological space $|K|$ is path connected (see [1, 13] for more).

Let K be a simplicial cell complex with partial ordering $\leqslant$. If $\beta \leqslant \alpha \in K$ then we say β is a *face* of α. For $\alpha \in K$, the set $\partial\alpha := \{\gamma \in K : \alpha \neq \gamma \leqslant \alpha\}$ is a subcomplex of K with induced partial order and is said to be the *boundary* of α. If $\partial\alpha$ is isomorphic to the boundary complex of an i-simplex then we say that α is an *i-cell* or a cell of *dimension i*. For $\beta \in K$, the set $\{\sigma \in K : \beta \leqslant \sigma\}$ is also simplicial cell complex and is said to be the *link* of α in K and is denoted by $\mathrm{lk}_K(\alpha)$.

If all the maximal cells of a d-dimensional simplicial cell complex K are d-cells then it is called *pure*. Maximal cells in a pure simplicial cell complex K are called the *facets* of K. Clearly, if K is pure of dimension d and α is an i-cell then $\mathrm{lk}_K(\alpha)$ is $(d-i-1)$-dimensional and pure. A pure d-dimensional simplicial cell complex K is said to be a *normal pseudomanifold* if each $(d-1)$-cell is a face of exactly two facets and the link of each cell of dimension $\leqslant d-2$ is connected. Clearly, a pseudotriangulation of a connected manifold is a normal pseudomanifold.

The 0-cells in a simplicial cell complex K are said to be the *vertices* of K. If u is a face of α and u is a vertex then we say u is a *vertex of α*. Clearly, a d-dimensional simplicial cell complex $\mathcal{X}$ has at least $d+1$ vertices. If a d-dimensional simplicial cell complex $\mathcal{X}$ has exactly $d+1$ vertices then $\mathcal{X}$ is called *contracted*.

Let $\mathcal{X}$ be a pure d-dimensional simplicial cell complex. Consider the graph $\Lambda(\mathcal{X})$ whose vertices are the facets of $\mathcal{X}$ and edges are the ordered pairs $(\{\sigma_1, \sigma_2\}, \gamma)$, where σ_1, σ_2 are facets, γ is a $(d-1)$-cell and is a common face of σ_1, σ_2. The graph $\Lambda(\mathcal{X})$ is said to be the *dual graph* of $\mathcal{X}$. Observe that $\Lambda(\mathcal{X})$ is in general a multigraph without loops. On the other hand, for $d \geqslant 1$, if (Γ, γ) is a $(d+1)$-colored graph with color set $C = \{1, \dots, d+1\}$ then we define a d-dimensional simplicial cell complex $\mathcal{K}(\Gamma)$ as follows. For each $v \in V(\Gamma)$ we take a d-simplex σ_v and label its vertices by $1, \dots, d+1$. If $u, v \in V(\Gamma)$ are joined by an edge e and $\gamma(e) = i$, then we identify the $(d-1)$-faces of σ_u and σ_v opposite to the vertices labelled by i, so that equally labelled vertices are identified together. Since there is no identification within a d-simplex, this gives a simplicial CW-complex W of dimension d. So, the face poset (denoted by $\mathcal{K}(\Gamma)$) of W is a pure d-dimensional simplicial cell complex. We say that (Γ, γ) *represents* the simplicial cell complex $\mathcal{K}(\Gamma)$. Clearly, the number of i-labelled vertices of $\mathcal{K}(\Gamma)$ is equal to the number of components of $\Gamma_{C\setminus\{i\}}$ for each $i \in C$. Thus, the simplicial cell complex $\mathcal{K}(\Gamma)$ is contracted if and only if Γ is contracted (cf. [8]).

A *crystallization* of a connected closed d-manifold M is a $(d+1)$-colored contracted graph (Γ, γ) such that the simplicial cell complex $\mathcal{K}(\Gamma)$ is a pseudotriangulation of M. Thus, if (Γ, γ) is a crystallization of a d-manifold M then the number of vertices in $\mathcal{K}(\Gamma)$ is $d+1$. On the other hand, if K is a contracted pseudotriangulation of M then the dual graph $\Lambda(K)$ gives a crystallization of M. Clearly, if (Γ, γ) is a crystallization of a closed d-manifold M then either Γ has two vertices (in which case M is S^d) or the number of edges between two vertices is at most $d-1$. From [5], we know the following.

Proposition 12 (Cavicchioli-Grasselli-Pezzana)**.** *Let (Γ, γ) be a crystallization of an n-manifold M. Then M is orientable if and only if Γ is bipartite.*

For $k \geqslant 2$, let $1, \dots, k$ be the colors of a k-colored graph (Γ, γ). For $1 \leqslant i \neq j \leqslant k$, Γ_{ij} denote the graph $\Gamma_{\{i,j\}}$ and g_{ij} denote the number of connected components of the graph Γ_{ij}. In [9], Gagliardi proved the following.

Proposition 13 (Gagliardi)**.** *Let (Γ, γ) be a contracted 4-colored graph with m vertices. Then (Γ, γ) is a crystallization of a connected closed 3-manifold if and only if*

(i) *$g_{ij} = g_{kl}$ for every permutation $ijkl$ of 1234, and*

(ii) *$g_{12} + g_{13} + g_{14} = 2 + m/2$.*

Let (Γ, γ) be a crystallization (with the color set C) of a connected closed n-manifold M. So, Γ is an $(n+1)$-regular graph. Choose two colors, say, i and j from C. Let $\{G_1, \dots, G_{s+1}\}$ be the set of all connected components of $\Gamma_{C\setminus\{i,j\}}$ and $\{H_1, \dots, H_{t+1}\}$ be the set of all connected components of Γ_{ij}. Since Γ is regular, each H_p is an even cycle. Note that, if $n = 2$, then Γ_{ij} is connected and hence $H_1 = \Gamma_{ij}$. Take a set $\widetilde{S} = \{x_1, \dots, x_s, x_{s+1}\}$ of $s+1$ elements. For $1 \leqslant k \leqslant t+1$, consider the word $\tilde{r}_k$ in $F(\widetilde{S})$

as follows. Choose a vertex v_1 in H_k. Let $H_k = v_1 e_1^i v_2 e_2^j v_3 e_3^i v_4 \cdots e_{2l-1}^i v_{2l} e_{2l}^j v_1$, where e_p^i and e_q^j are edges with colors i and j respectively. Define

$$\tilde{r}_k := x_{k_2}^{+1} x_{k_3}^{-1} x_{k_4}^{+1} \cdots x_{k_{2l}}^{+1} x_{k_1}^{-1}, \tag{2.6}$$

where G_{k_h} is the component of $\Gamma_{C \setminus \{i,j\}}$ containing v_h. For $1 \leqslant k \leqslant t+1$, let r_k be the word obtained from $\tilde{r}_k$ by deleting $x_{s+1}^{\pm 1}$'s in $\tilde{r}_k$. So, r_k is a word in $F(S)$, where $S = \widetilde{S} \setminus \{x_{s+1}\}$. In [10], Gagliardi proved the following.

Proposition 14 (Gagliardi). *For $n \geqslant 2$, let (Γ, γ) be a crystallization of a connected closed n-manifold M. For two colors i, j, let s, t, x_p, r_q be as above. If $\pi_1(M, x)$ is the fundamental group of M at a point x, then*

$$\pi_1(M, x) \cong \begin{cases} \langle x_1, x_2, \dots, x_s \mid r_1 \rangle & \text{if } \ n = 2, \\ \langle x_1, x_2, \dots, x_s \mid r_1, \dots, r_t \rangle & \text{if } \ n \geqslant 3. \end{cases}$$

3 Proofs of Lemma 3, Theorem 4 and Corollary 5

Lemma 3 follows from the next lemma.

Lemma 15. (i) $\psi(\mathbb{Z}) = \psi(\mathbb{Z}_2) = 8$, (ii) $\psi(\mathbb{Z}_3) = 12$, (iii) $\psi(\mathbb{Z}_5) = 16$, (iv) $\psi(Q_8) = 18$ *and* (v) $\psi(\mathbb{Z}^3) = 24$.

Proof. Any presentations of $\mathbb{Z}$ must have at least one generator and $\langle x \rangle$ is a presentation of $\mathbb{Z}$. So, $\mu(\mathbb{Z}) = 1$. If $\langle S | R \rangle \cong \mathbb{Z}$ with $\#(S) = 1$, then $R = \emptyset$ and hence, by the definition (see (2.3)), $\varphi(S, R) = \lambda(\emptyset) + 2(1 - 0) = 2 + 2 = 4 < 12 = 6(\mu(\mathbb{Z}) + 1)$. Therefore, $\psi(\mathbb{Z}; q) \leqslant 4$ for all $q \geqslant 1$. Thus, $\psi(\mathbb{Z}) = \max\{\psi(\mathbb{Z}, \rho(\mathbb{Z})), 6\mu(\mathbb{Z}) + 2\} = \max\{\psi(\mathbb{Z}, \rho(\mathbb{Z})), 8\} = 8$.

Let $p \geqslant 2$ be an integer. Since any presentations of $\mathbb{Z}_p$ must have at least one generator and $\langle x \,|\, x^p \rangle$ is a presentation of $\mathbb{Z}_p$, it follows that $\mu(\mathbb{Z}_p) = 1$. Clearly, if $\langle S = \{x\} \,|\, R = \{r_1\} \rangle$ is a presentation of $\mathbb{Z}_p$, then $r_1 = x^{\pm p}$. Let $r_2 \in \overline{R}$ be of minimum weight. Since $\langle x \,|r_2 \rangle$ is also a presentation of $\mathbb{Z}_p$, $r_2 = x^{\pm p}$. Therefore, by (2.3),

$$\varphi(S, R) = \lambda(r_1) + \lambda(r_2) = (2p - \varepsilon(r_1)) + (2p - \varepsilon(r_2)) = 4p. \tag{3.1}$$

First assume that $p \leqslant 3$. Since, $\langle S \,|\, R \rangle \in \mathcal{P}_1(\mathbb{Z}_p)$ implies (up to renaming) $(S, R) = (\{x\}, \{x^p\})$ or $(\{x\}, \{x^{-p}\})$, it follows that $\psi(\mathbb{Z}_p; 1) = \varphi(\{x\}, \{x^{\pm p}\}) = 4p \leqslant 12 = 6(\mu(\mathbb{Z}_p) + 1)$. This implies that $\rho(\mathbb{Z}_p) = \mu(\mathbb{Z}_p) = 1$. Thus, $\psi(\mathbb{Z}_p; \rho(\mathbb{Z}_p)) = 4p \geqslant 8 = 6\mu(\mathbb{Z}_p) + 2$. Therefore, $\psi(\mathbb{Z}_p) = 4p$. This proves parts (i) and (ii).

Now, assume $p = 5$. By the similar arguments as for $p \leqslant 3$, $\langle S \,|\, R \rangle \in \mathcal{P}_1(\mathbb{Z}_5)$ implies $\varphi(S, R) = 4p = 20$. Therefore, $\psi(\mathbb{Z}_5; 1) = 20 > 12 = 6(\mu(\mathbb{Z}_5) + 1)$ and hence $\rho(\mathbb{Z}_5) > \mu(\mathbb{Z}_5) = 1$. If we take $S = \{x_1, x_2\}$ and $R = \{r_1 = x_1^2 x_2^{-1}, r_2 = x_2^3 x_1^{-1}\}$ then $\varphi(S, R) \leqslant 16$ (since $r_3 = x_1 x_2^2 \in \overline{R}$ is of weight 6) and $\langle S \,|\, R \rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5)$. Thus, $\psi(\mathbb{Z}_5; 2) \leqslant 16 < 18 = 6(2 + 1)$. Therefore, $\rho(\mathbb{Z}_5) = 2$ and hence $\psi(\mathbb{Z}_5) \leqslant 16$.

Now, let $\langle S \,|\, R \rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5)$ with $\varphi(S, R) \leqslant 16$. Since there is no presentation $\langle S \,|\, R \rangle$ of $\mathbb{Z}_5$ with $(\#(S), \#(R)) = (2, 1)$, it follows that $\#(R) = \#(S) = 2$. Let $S =$

$\{x_1, x_2\}$ and $R = \{r_1, r_2\}$. If $\lambda(r_1) = 2$, then r_1 must be of the form $x_i^{\pm 1}$ or $x_i^{\varepsilon} x_j^{-\varepsilon}$ for some $j \neq i \in \{1, 2\}$ and $\varepsilon = \pm 1$. Since $\langle S \,|\, R\rangle \cong \mathbb{Z}_5$, it follows that $r_2 \equiv x_j^{\pm 5}$ (mod $\{r_1\}$). This implies that $\lambda(r_2) \geqslant \lambda(x_j^{\pm 5}) = 10$. Let $r_3 \in \overline{R}$ be of minimum weight. Then $\langle x_1, x_2 \,|\, r_1, r_3\rangle$ is also a presentation of $\mathbb{Z}_5$ and hence (by the same arguments) $\lambda(r_3) \geqslant 10$. Thus, $\varphi(S, R) = \lambda(r_1) + \lambda(r_2) + \lambda(r_3) \geqslant 2 + 10 + 10 = 22$, a contradiction. So, $\lambda(r_i) \geqslant 4$ for $1 \leqslant i \leqslant 2$. Let $A = \{x_1x_2, x_1^2, x_2^2, x_1^2x_2^{-1}, x_2^2x_1^{-1}, x_1x_2^{-1}x_1x_2^{-1}\}$ and let $A^{-1} = \{w^{-1} : w \in A\}$. Then A is a set of pairwise independent elements of weight 4 in $F(S)$ and $w \in F(S)$ is an element of weight 4 imply that w is dependent with an element of A. Note that $\mathbb{Z}_5$ has no presentation $\langle S \,|\, R\rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5)$ with $R \subseteq A \cup A^{-1}$. So, at most one of r_1, r_2, r_3 has weight 4 and the weights of other two are at least 6. Therefore, $\varphi(S, R) \geqslant 16$. This implies that $\psi(\mathbb{Z}_5) = 16$. This proves part (iii).

Clearly, $\mu(Q_8) = 2$. If we take $S = \{x_1, x_2\}$ and $R = \{x_2x_1x_2x_1^{-1}, x_1x_2x_1x_2^{-1}\}$ then $\langle S \,|\, R\rangle \in \mathcal{P}_2(Q_8)$ and $\varphi(S, R) \leqslant 18$ (since $x_2^2x_1^{-2} \in \overline{R}$ is of weight 6). Thus $\psi(Q_8; 2) \leqslant 18 = 6(2 + 1)$. Therefore, $\rho(Q_8) = 2$ and hence $\psi(Q_8) \leqslant 18$.

Now, let $\varphi(S, R) \leqslant 18$, where $S = \{x_1, x_2\}$ and $\langle S \,|\, R\rangle \in \mathcal{P}_2(Q_8)$. Note that $B = \{x_1x_2, x_1^2, x_2^2, x_1^2x_2^{-1}, x_2^2x_1^{-1}, x_1x_2^{-1}x_1x_2^{-1}, x_2^2x_1, x_1^3x_2^{-1}, x_2^2x_1^{-1}x_2x_1^{-1}, x_1^2x_2, x_2^3x_1^{-1}, x_1^3, x_2^3, x_1^2x_2^{-1}x_1x_2^{-1}, x_1x_2x_1x_2^{-1}, x_2x_1x_2x_1^{-1}, x_1x_2^{-1}x_1x_2^{-1}x_1x_2^{-1}, x_1x_2x_1^{-1}x_2^{-1}, x_2^2x_1^{-2}\}$ is a set of pairwise independent elements of weight 4 or 6 in $F(S)$. It is not difficult to see that $w \in F(S)$ and $4 \leqslant \lambda(w) \leqslant 6$ imply w is dependent with an element of B. Let $B^{-1} = \{w^{-1} : w \in B\}$. Then $R \subseteq B \cup B^{-1}$. Clearly, the only possible choices of $\{r_1^{\pm 1}, r_2^{\pm 1}\}$ are $\{x_2^2x_1^{-2}, x_1x_2x_1x_2^{-1}\}$, $\{x_2^2x_1^{-2}, x_2x_1x_2x_1^{-1}\}$ and $\{x_2x_1x_2x_1^{-1},\ x_1x_2x_1x_2^{-1}\}$. Then $\lambda(r) \geqslant 6$ for $r \in R \cup \overline{R}$. Thus, $\varphi(S, R) \geqslant 18$. Therefore, $\psi(Q_8) = 18$. This proves parts (iv).

Clearly, $\mu(\mathbb{Z}^3) = 3$. If $S_0 = \{x_1, x_2, x_3\}$ and $R_0 = \{x_ix_jx_i^{-1}x_j^{-1} : 1 \leqslant i < j \leqslant 3\}$ then $\langle S_0 \,|\, R_0\rangle \in \mathcal{P}_3(\mathbb{Z}^3)$ and $\varphi(S_0, R_0) \leqslant 24$ (since $x_1x_2^{-1}x_3x_1^{-1}x_2x_3^{-1} \in \overline{R}_0$ is of weight 6). Thus $\psi(\mathbb{Z}^3; 3) \leqslant 24 = 6(3 + 1)$. Therefore, $\rho(\mathbb{Z}^3) = 3$ and hence $\psi(\mathbb{Z}^3) \leqslant 24$.

Claim. If $w \in N(R_0)$ is not the identity then $\lambda(w) \geqslant 6$.

If $w \in N(R_0)$ is not the identity then clearly $\lambda(w) \neq 2$. Observe that, if $w \in F(S_0)$ with $\lambda(w) = 4$, then w is dependent with an element of the set $C = \{x_i^2x_j^{-1}, x_ix_j^{-1}x_ix_j^{-1}, x_i^2, x_ix_j, x_ix_j^{-1}x_ix_k^{-1} : ijk$ is a permutation of $123\}$. Since none of the element in C is in $N(R_0)$, it follows that $N(R_0)$ has no element of weight 4. This proves the claim.

Now, let $\varphi(S, R) \leqslant 24$, where $S = \{x_1, x_2, x_3\}$ and $\langle S \,|\, R\rangle \in \mathcal{P}_3(\mathbb{Z}^3)$. Then $N(R) = N(R_0)$ and hence, by the claim, weight of each element of R is at least 6. This implies $\varphi(S, R) \geqslant 24$ and hence $\varphi(S, R) = 24$. Therefore, $\psi(\mathbb{Z}^3) = 24$. This completes the proof. □

Proof of Theorem 4. Let $G = \pi(M, x)$ for some $x \in M$. To prove the theorem, it is sufficient to show that any crystallization of M needs at least $\psi(M) = \psi(G)$ vertices.

Let (Γ, γ) be a crystallization of M with m vertices and let $\{1, 2, 3, 4\}$ be the color set. Then, by Proposition 14, we know that G has a presentation with $g_{ij} - 1$ generators and $\leqslant g_{ij} - 1$ relations. Therefore, by the definition of $\mu(G)$ (in (2.4a)), $\mu(G) \leqslant g_{ij} - 1$. Then, by part (ii) of Proposition 13,

$$m = 2(g_{12} + g_{13} + g_{14}) - 4 \geqslant 6(\mu(G) + 1) - 4 = 6\mu(G) + 2. \tag{3.2}$$

From the definition of $\rho(G)$ (in (2.5)), $6(\rho(G)+1) \geqslant \psi(G;\rho(G))$. Therefore, $m > 6(\rho(G)+1)$ implies $m > \psi(G;\rho(G))$. Thus, if $m > 6(\rho(G)+1)$ then the result follows from this and Eq. (3.2).

Now, assume that $m \leqslant 6(\rho(G)+1)$. Then, by part (ii) of Proposition 13, $g_{12}+g_{13}+g_{14} \leqslant 2+3(\rho(G)+1)$. This implies, $g_{1j} \leqslant \rho(G)+1$ for some $j \in \{2,3,4\}$. Assume, without loss, that $g_{12} \leqslant \rho(G)+1$.

As in Subsection 2.4, let $G_1,\dots,G_{q+1}$ be the components of Γ_{12} and $H_1,\dots,H_{q+1}$ be the components of Γ_{34}, where $q+1 = g_{34} = g_{12} \leqslant \rho(G)+1$. By Proposition 14, G has a presentation of the form $\langle x_1,x_2,\dots,x_q \,|\, r_1,r_2,\dots,r_q\rangle$, where x_k corresponds to G_k and r_k corresponds to H_k as in Subsection 2.4. Let $S=\{x_1,x_2,\dots,x_q\}$ and $R=\{r_1,\dots,r_q\}$.

For $1 \leqslant i \leqslant q$, let $r_i = x_{i_1}^{\varepsilon_1}\cdots x_{i_n}^{\varepsilon_n}$, where $x_{i_1},\dots,x_{i_n} \in \{x_1,\dots,x_q\}$ and $\varepsilon_j = \pm 1$ for $1 \leqslant j \leqslant n$, $(x_{i_{j+1}},\varepsilon_{j+1}) \neq (x_{i_j},-\varepsilon_j)$ for $1 \leqslant j \leqslant n-1$ and $(x_{i_n},\varepsilon_n) \neq (x_{i_1},-\varepsilon_1)$.

Claim. For $1 \leqslant i \leqslant q$, the length of the cycle H_i is at least $\lambda(r_i)$.

Consider the word $\tilde{r}_i$ (in $F(\{x_1,\dots,x_q,x_{q+1}\})$ which is obtained from r_i by the following rules: if $\varepsilon_j = \varepsilon_{j+1}$ for $1 \leqslant j \leqslant n-1$, then replace $x_{i_j}^{\varepsilon_j}$ by $x_{i_j}^{\varepsilon_j}x_{q+1}^{-\varepsilon_j}$ in r_i and if $\varepsilon_n = \varepsilon_1$, then replace $x_{i_n}^{\varepsilon_n}$ by $x_{i_n}^{\varepsilon_n}x_{q+1}^{-\varepsilon_n}$ in r_i. Observe that $\tilde{r}_i$ is non empty (since r_i is non empty) and the number of letters in $\tilde{r}_i$ is same as $\lambda(r_i)$ (see (2.6) and (2.2)). The claim follows from this.

Let r_{q+1} be a word corresponding to H_{q+1} in Γ_{34}. Then, any q of the relations from the set $\{r_1,r_2,\dots,r_q,r_{q+1}\}$ together with the generators $x_1,x_2,\dots,x_q$ give a presentation of G. This implies, $r_{q+1} \in \overline{R}$. Thus, $m \geqslant \lambda(r_1)+\lambda(r_2)+\cdots+\lambda(r_{q+1}) \geqslant \varphi(S,R) \geqslant \psi(G;\rho(G))$. Therefore, $m \geqslant \max\{\psi(G;\rho(G)), 6a(G)+2\} = \psi(G)$. This proves the theorem. □

Proof of Corollary 5. Let f_i be the number of i-cells in X. So, $f_0 = 4$. Therefore, $g_2(X) = f_1-16+10 = f_1-6 = f_1-12+6 = h_2(X)$. Since $|X|$ is a closed 3-manifold, each 2-cell is a face of two 3-cells and each 3-cell has four 2-dimensional faces. This implies that $2f_2 = 4f_3$. Then, $0 = f_0-f_1+f_2-f_3 = 4-f_1+2f_3-f_3$. Thus, $f_1 = f_3+4$ and hence $g_2(X) = h_2(X) = f_3-2$. Therefore, by Theorem 4, $g_2(X) = h_2(X) = f_3-2 \geqslant \Psi(M)-2 \geqslant \psi(M)-2$.

From the definition of $\psi(G)$, $\psi(G) \geqslant 6\mu(G)+2 \geqslant 6m(G)+2$. So, $\psi(M) = \psi(\pi(M,*)) \geqslant 6m(\pi(M,*))+2$. Since any presentation of $\pi(M,*)$ has at least $\beta_1(M;\mathbb{F})$ generators, it follows that $m(M) = m(\pi(M,*)) \geqslant \beta_1(M;\mathbb{F})$. The corollary now follows. □

Remark 16. If a crystallization (Γ,γ) yields a presentation $\langle S \,|\, R\rangle$ then, from the proof of Theorem 4, we get $\varphi(S,R) \leqslant$ the number of vertices of Γ.

Remark 17. We found that $\rho(\mathbb{Z}^3) = 3$ and $\varphi(S,R) = 24$, where $\langle S \,|\, R\rangle \in \mathcal{P}_3(\mathbb{Z}^3)$. On the other hand, if $S' = \{x_1,\dots,x_5\}$ and $R' = \{x_1x_4^{-1}x_5x_3^{-1}, x_1x_5x_2^{-1}, x_3x_4x_2^{-1}, x_1x_3^{-1}x_5x_4^{-1}, x_5x_1x_2^{-1}\}$ then $\langle S' \,|\, R'\rangle \in \mathcal{P}_5(\mathbb{Z}^3) \setminus \mathcal{P}_4(\mathbb{Z}^3)$ with $\varphi(S',R') = 24$. So, the minimum weight presentation of $\mathbb{Z}^3$ is not unique. This is true for most of the groups.

4 Uniqueness of some crystallizations

Here, we are interested on crystallizations of 3-manifolds M with $\psi(M)$ vertices. For seven 3-manifolds, we show that there exists a unique such crystallization for each of them.

Throughout this section and behind, $1, 2, 3, 4$ are the colors of a 4-colored graph (Γ, γ) and g_{ij} is the number of components of $\Gamma_{ij} = \Gamma_{\{i,j\}}$ for $i \neq j$.

Let $\mathcal{X}$ be the pseudotriangulation of a connected closed 3-manifold M determined by a crystallization (Γ, γ). So, (Γ, γ) is contracted, i.e., $\Gamma_{\{i,j,k\}}$ is connected for i, j, k distinct. For $1 \leqslant i \leqslant 4$, we denote the vertex of $\mathcal{X}$ corresponding to the color i by v_i. We identify a vertex u of Γ with the corresponding facet σ_u of $\mathcal{X}$. For a facet u ($\equiv \sigma_u$) of $\mathcal{X}$, the 2-face of u not containing the vertex v_i will be denoted by u_i. Similarly, the edge of u not containing the vertices v_i, v_j will be denoted by u_{ij}. Clearly, if $C_{2k}(u^1, u^2, \dots, u^{2k})$ is a $2k$-cycle in Γ with colors i and j alternately, then $u^1_{ij} = u^2_{ij} = \cdots = u^{2k}_{ij}$ in $\mathcal{X}$.

Lemma 18. *Let Γ be a crystallization of a connected closed 3-manifold M with m vertices. If Γ has a 2-cycle, then either M has a crystallization with $m-2$ vertices or $\pi_1(M, x)$ (for $x \in M$) is isomorphic to the free product $\mathbb{Z} * H$ for some group H.*

Proof. Without loss of generality, assume that Γ has a 2-cycle with color 1 and 2, i.e., Γ_{12} has a component of length 2. If this 2-cycle touches two different components of Γ_{34} (say, at vertices v and w, respectively), then $\Gamma[\{v, w\}]$ is a 3-dimensional dipole of type 2. Therefore, the crystallization Γ can be reduced to a crystallization Γ^1 of M with vertex set $V(\Gamma) \setminus \{v, w\}$ so that Γ^1_{12} (resp., Γ^1_{34}) has one less components than Γ_{12} (resp., Γ_{34}) as in Fig. 1 (see [7]). Thus, M has a crystallization (namely, Γ^1) with $m-2$ vertices.

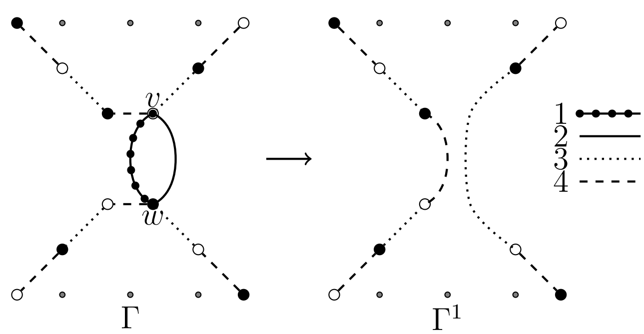

Figure 1: Cancellation of a dipole of type 2

So, assume that the 2-cycle (say G_1) touches only one component (say, H_1) of Γ_{34}. Let $G_1, \dots, G_{q+1}$ be the components of Γ_{12} and $H_1, \dots, H_{q+1}$ be the components of Γ_{34}, where $q + 1 = g_{12} = g_{34}$. Let $x_1, \dots, x_{q+1}$ and $r_1, \dots, r_{q+1}$ be as in Proposition 14. Then, by Proposition 14, $\pi_1(M, x)$ has a presentation of the form $\langle x_1, x_2, \dots, x_q \,|\, r_2, r_3, \dots, r_{q+1} \rangle$. Since G_1 touches only H_1, from the definition of $\tilde{r}_k$ in Eq. (2.6), $\tilde{r}_k$ does not contain $x_1^{\pm 1}$ for $k \neq 1$. Therefore, $\langle x_1, x_2, \dots, x_q \,|\, r_2, \dots, r_{q+1} \rangle = \langle x_1 \rangle * \langle x_2, \dots, x_q \,|\, r_2, \dots, r_{q+1} \rangle$. This proves the lemma. □

Lemma 19. *There exist exactly three 8-vertex crystallizations of non-simply connected, connected, closed 3-manifolds. Moreover, these three are crystallizations of $S^2 \times S^1$, $S^2 \times\!\!\!\!\!\_\, S^1$ and $\mathbb{RP}^3$ respectively.*

Proof. Let (Γ, γ) be an 8-vertex crystallization of a non simply connected, connected, closed 3-manifold M. By Proposition 13, $g_{12} + g_{13} + g_{14} = 8/2 + 2 = 6$ and $g_{ij} = g_{kl}$ for i, j, k, l distinct. Since $\pi_1(M, *)$ has at least one generator, $g_{ij} \geqslant 2$ for $1 \leqslant i \neq j \leqslant 4$. This implies that $g_{ij} = 2$ and hence Γ_{ij} is of the form $C_2 \sqcup C_6$ or $C_4 \sqcup C_4$ for $1 \leqslant i \neq j \leqslant 4$.

Case 1*:* Suppose (Γ, γ) has a 2-cycle. Since M is not simply connected, M has no crystallization with less than 8 vertices. Therefore, by Lemma 18, $\pi_1(M, *)$ must have a torsion free element. Again, $g_{ij} = 2$ implies $\pi_1(M, *)$ is generated by one element and hence isomorphic to $\mathbb{Z}$. Therefore, $M \cong S^2 \times S^1$ or $S^2 \times\!\!\!\!\!\_\, S^1$. Assume, without loss, $\Gamma_{12} = G_1 \sqcup G_2$, where $G_1 = C_2(v_3, v_4)$, $G_2 = C_6(v_1, v_2, v_5, v_6, v_7, v_8)$. Then there is no edge between v_3 and v_4 of color 3 or 4 and (see the proof of Lemma 18), G_1 touches only one component of Γ_{34}. Let $\Gamma_{34} = H_1 \sqcup H_2$, where $G_1 \cap H_1 = \emptyset$. Let x and y be the generators corresponding to the components G_1 and G_2 respectively. If H_2 is a 4-cycle then H_2 represents $xy^{-1}xy^{-1}$ by choosing some v_1, i, j as in Eq. (2.6). But $xy^{-1}xy^{-1}$ does not give identity relation by deleting x or y. Therefore, H_2 is a 6-cycle and hence H_1 is a 2-cycle. Similarly, $G_2 \cap H_2 = \emptyset$. Since the number of edges between any pair of vertices is at most 2, we can assume that $H_1 = C_2(v_1, v_6)$. Assume, without loss, that there is an edge of color 4 between v_2 and v_3. Since Γ_{24} has two components, this implies $\Gamma_{24} = C_4(v_4, v_3, v_2, v_5) \sqcup C_4(v_8, v_1, v_6, v_7)$. So, there exists an edge of color 4 between v_4 and v_5 (resp. v_7 and v_8). Since H_2 is a 6-cycle on the vertex set $\{v_1, \dots, v_8\} \setminus \{v_1, v_6\}$, this implies that $H_2 = C_6(v_2, v_3, v_8, v_7, v_4, v_5)$ or $C_6(v_2, v_3, v_7, v_8, v_4, v_5)$. In the first case, $(\Gamma, \gamma) = \mathcal{J}_1$ and in the second case, $(\Gamma, \gamma) = \mathcal{J}_2$ given in Fig. 2 (a) and (b) respectively. In the first case, Γ is bipartite. Therefore, M is orientable and hence equal to $S^2 \times S^1$. In the second case, Γ is not bipartite. Therefore, M is non-orientable and hence equal to $S^2 \times\!\!\!\!\!\_\, S^1$.

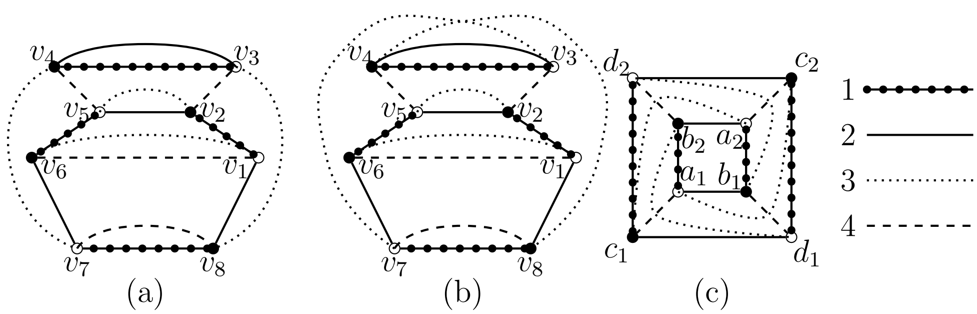

Figure 2: Crystallizations $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{K}_{2,1}$

Case 2*:* Suppose (Γ, γ) has no 2-cycle. So, Γ is a simple graph. Then, $\Gamma_{ij} = C_4 \sqcup C_4$ for $1 \leqslant i \neq j \leqslant 4$. Let $G_1 = C_4(a_1, b_1, a_2, b_2)$ and $G_2 = C_4(c_1, d_1, c_2, d_2)$ be the components of Γ_{12}. If a_1a_2 is an edge of color 3 then (since $\Gamma_{13} = C_4 \sqcup C_4$) b_1b_2 must be an edge of color 3. Then Γ_{123} is disconnected. This is not possible. So, a_1a_2 cannot be an edge of color 3.

Similarly, a_1a_2 cannot be an edge of color 4. These imply, a_1a_2 can not be an edge of Γ. Assume, without loss, a_1c_1 is an edge of color 4. Then a_2c_2, b_1d_1, b_2d_2 are edges of color 4 (since $\Gamma_{i4} = C_4 \sqcup C_4$ for $1 \leqslant i \leqslant 2$). If a_1d_1 is an edge of color 3, then $C_4(b_1, a_1, d_1, c_1)$ would be a component of Γ_{23}. This implies $\Gamma[\{a_1, b_1, c_1, d_1\}]$ would be proper component of $\Gamma_{\{2,3,4\}}$. This is not possible since (Γ, γ) is a contracted graph. Thus, a_1d_1 is not an edge of color 3. Similarly, a_1d_2 is not an edge of color 3. These imply a_1c_2 is an edge of color 3. Similarly, b_1d_2, a_2c_1 and b_2d_1 are edges of color 3. Then, $(\Gamma, \gamma) = \mathcal{K}_{2,1}$ given in Fig. 2 (c). Since $G_1 = C_4(a_1, b_1, a_2, b_2)$ and $H_1 = C_4(d_1, b_2, d_2, b_1)$ is a component of Γ_{34}, $\pi(M, *) = \langle x \,|\, x^2 \rangle \cong \mathbb{Z}_2$. This implies that $M = \mathbb{RP}^3$. This completes the proof. □

Lemma 20. *There exists a unique* 12*-vertex crystallization of* $L(3,1)$.

Proof. By Lemma 15 and Theorem 4, $L(3,1)$ has no crystallization with less than 12 vertices. Let (Γ, γ) be a 12-vertex crystallization of $L(3,1)$. Since $\pi_1(L(3,1), *)$ $(\cong \mathbb{Z}_3)$ has no torsion free element, by Lemma 18, (Γ, γ) has no 2-cycle. So, Γ is a simple graph. This implies that $g_{ij} \leqslant 3$ for $i \neq j$. Also (since $\mathbb{Z}_3$ has at least one generator) $g_{ij} \geqslant 2$. By Proposition 13, $g_{12} + g_{13} + g_{14} = 12/2 + 2 = 8$ and $g_{ij} = g_{kl}$ for i, j, k, l distinct. So, without loss, we can assume that $g_{12} = g_{34} = 2$, $g_{13} = g_{14} = 3$. Then $\Gamma_{ij} = C_4 \sqcup C_4 \sqcup C_4$ for $1 \leqslant i \leqslant 2$, $3 \leqslant j \leqslant 4$. Let G_1, G_2 be the components of Γ_{12} and H_1, H_2 be the components of Γ_{34} such that x_1, x_2 represent the generators corresponding to G_1, G_2 respectively. Since $\langle x_j \,|\, x_j^3 \rangle$ is the only presentation in $\mathcal{P}_1(\mathbb{Z}_3)$, H_i must yield the relations $x_j^{\pm 3}$, for $1 \leqslant i, j \leqslant 2$. Therefore, G_i and H_i are 6-cycles. Let $G_1 = C_6(a_1, b_1, \dots, a_3, b_3)$ and $G_2 = C_6(c_1, d_1, \dots, c_3, d_3)$. Assume, without loss, $a_1c_1 \in \gamma^{-1}(4)$. Then $C_4(b_3, a_1, c_1, d_3) \subseteq \Gamma_{14}$ and hence $b_3d_3 \in \gamma^{-1}(4)$. Similarly, $a_3c_3, b_2d_2, a_2c_2, b_1d_1 \in \gamma^{-1}(4)$. Now, $a_1d_1 \in \gamma^{-1}(3)$ $\Longrightarrow C_4(a_1, d_1, c_1, b_1) \subseteq \Gamma_{23} \Longrightarrow \Gamma[\{a_1, b_1, c_1, d_1\}]$ is a component of $\Gamma_{\{2,3,4\}}$. This is not possible since Γ is a contracted graph. So, $a_1d_1 \notin \gamma^{-1}(3)$. Similarly, $a_1d_3 \notin \gamma^{-1}(3)$. Again, $a_1d_2 \in \gamma^{-1}(3) \Longrightarrow C_4(a_1, d_2, c_2, b_1) \subseteq \Gamma_{23} \Longrightarrow c_2b_1 \in \gamma^{-1}(3) \Longrightarrow C_4(a_2, b_1, c_2, d_1) \subseteq \Gamma_{13} \Longrightarrow$ $\Gamma[\{a_2, b_1, c_2, d_1\}]$ is a component of $\Gamma_{\{1,3,4\}}$, a contradiction. So, $a_1d_2 \notin \gamma^{-1}(3)$. Therefore, up to an isomorphism, $a_1c_2 \in \gamma^{-1}(3)$. Then b_1d_2, a_2c_3, b_2d_3, a_3c_1, $b_3d_1 \in \gamma^{-1}(3)$ and hence $(\Gamma, \gamma) = \mathcal{K}_{3,1}$ given in Fig. 3 (a). Since $H_1 = C_6(d_1, b_1, d_2, b_2, d_3, b_3)$ is one of the two components of Γ_{34}, (Γ, γ) yields $\langle x_1 \,|\, x_1^3 \rangle \cong \mathbb{Z}_3$. So, (Γ, γ) is a crystallization of $L(3,1)$. This completes the proof. □

Lemma 21. *There exists a unique* 16*-vertex* 4*-colored graph* (Γ, γ) *which is a crystallization of a closed connected* 3*-manifold whose fundamental group is* $\mathbb{Z}_5$.

Proof. Let (Γ, γ) be a 16-vertex crystallization of a connected closed 3-manifold M and $\pi(M, *) = \mathbb{Z}_5$. Then M can not have a non-trivial 2-fold cover and hence M is orientable. Also, by Lemma 15, $\psi(M) = 16$ and hence, by Theorem 4, (Γ, γ) is the crystallization of M with minimum number of vertices. Then, by Lemma 18, (Γ, γ) has no 2-cycle. So, Γ is a simple graph. Since M is orientable, Γ is bipartite. By Proposition 14 and Remark 16, (Γ, γ) yields a presentation $\langle S \,|\, R \rangle$ of $\mathbb{Z}_5$ with $\varphi(S, R) = 16$.

Claim 1. If $\langle S = \{x_1, x_2\} \,|\, R = \{r_1, r_2\} \rangle \in \mathcal{P}_2(\mathbb{Z}_5)$, $\varphi(S, R) = 16$ and $r_3 \in \overline{R}$ is of minimum weight then $\{r_1, r_2, r_3\} = \{(x_1^3x_2^{-1})^{\pm 1}, (x_2^2x_1^{-1})^{\pm 1}, (x_1^2x_2)^{\pm 1}\}$ or $\{(x_1^2x_2^{-1}x_1x_2^{-1})^{\pm 1}, (x_1x_2)^{\pm 1}, (x_2^2x_1^{-1}x_2x_1^{-1})^{\pm 1}\}$. (So, the set $\{r_1, r_2, r_3\}$ has 16 choices.)

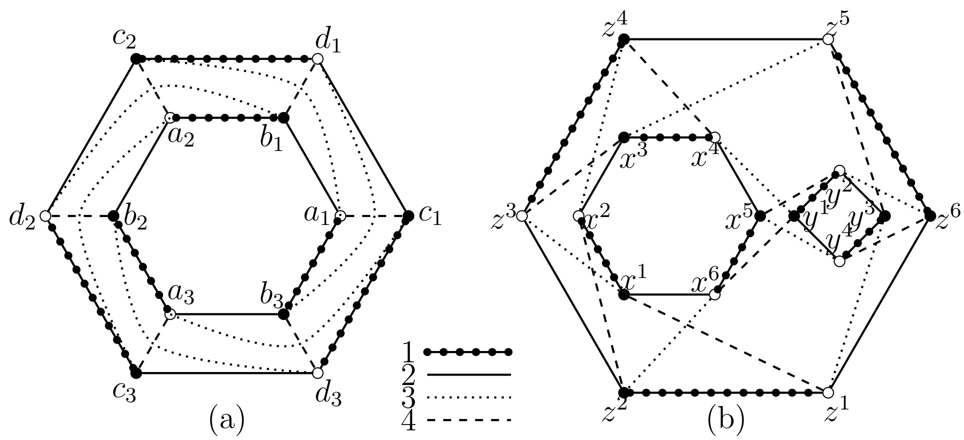

Figure 3: Crystallizations $\mathcal{K}_{3,1}$ and $\mathcal{M}_{2,3}$

Let B be the set as in the proof of Lemma 15. Then $w \in F(S)$ and $4 \leqslant \lambda(w) \leqslant 6$ imply w is dependent with an element of B. Since Γ has no 2-cycle, R has no element of weight less than 4. Since $\varphi(S,R) = 16$, we can assume that $4 \leqslant \lambda(r_1), \lambda(r_2) \leqslant 6$. Since $\langle S \,|\, R\rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5)$, the only possible choices of $\{r_1^{\pm1}, r_2^{\pm1}\}$ are $\{x_1^3x_2^{-1}, x_2^2x_1^{-1}\}$, $\{x_2^2x_1^{-1}, x_1^2x_2\}$, $\{x_1^3x_2^{-1}, x_1^2x_2\}$, $\{x_1^2x_2^{-1}x_1x_2^{-1}, x_1x_2\}$ or $\{x_1^2x_2^{-1}x_1x_2^{-1}, x_2^2x_1^{-1}x_2x_1^{-1}\}$. So, if $\langle S \,|\, R\rangle \in \mathcal{P}_2(\mathbb{Z}_5)$ and $\varphi(S,R) = 16$, then $(r_1^{\pm1}, r_2^{\pm1}, r_3^{\pm1}) = (x_1^3x_2^{-1},\ x_2^2x_1^{-1},\ x_1^2x_2)$ or $(x_1^2x_2^{-1}x_1x_2^{-1},\ x_1x_2,\ x_2^2x_1^{-1}x_2x_1^{-1})$. This proves Claim 1.

If $g_{ij} = 2$ for some $i \neq j$ then (Γ, γ) yields a presentation $\langle S \,|\, R\rangle \in \mathcal{P}_1(\mathbb{Z}_5)$ such that $\varphi(S,R) = 16$ (see Remark 16), which is not possible by Eq. (3.1). Thus, $g_{ij} \geqslant 3$. Since (by Proposition 13) $g_{12} + g_{13} + g_{14} = 16/2 + 2 = 10$, we can assume that $g_{12} = 3 = g_{13}, g_{14} = 4$. In particular, if we choose generators (resp., relations) corresponding to the components of Γ_{12} (resp., Γ_{34}) then (Γ, γ) yields a presentation $\langle S \,|\, R\rangle \in \mathcal{P}_2(\mathbb{Z}_5) \setminus \mathcal{P}_1(\mathbb{Z}_5)$ with $\varphi(S,R) = 16$.

Claim 2. If x_1, x_2 are generators corresponding to two components of Γ_{12} then the relations corresponding to the components of Γ_{34} are $(x_1^3x_2^{-1})^{\varepsilon_1}$, $x_2^2x_1^{-1}$, $(x_1^2x_2)^{\varepsilon_2}$ for some $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$.

Let S, R, r_1, r_2, r_3 be as in Claim 1. Then by choosing $(i,j) = (3,4)$ or $(4,3)$ as in Eq. (2.6), by Claim 1, we can assume $(r_1, r_2, r_3) = \big((x_1^3x_2^{-1})^{\pm1},\ x_2^2x_1^{-1},\ (x_1^2x_2)^{\pm1}\big)$ or $\big((x_1^2x_2^{-1}x_1x_2^{-1})^{\pm1}, (x_1x_2)^{-1}, (x_2^2x_1^{-1}x_2x_1^{-1})^{\pm1}\big)$. In the first case, Claim 2 trivially holds. In the second case, $\tilde{r}_2 = (x_1x_3^{-1}x_2x_3^{-1})^{-1}$, where x_3 corresponds to the third component of Γ_{12} (see Eq. (2.6)). By deleting x_2 and renaming x_3 by x_2 in $\tilde{r}_2^{\pm1}$, we get the new relation $x_2^2x_1^{-1}$. Claim 2 now follows from Claim 1.

To construct $\tilde{r}_i$ as in Eq. (2.6), we can choose, without loss, $(i,j) = (4,3)$. Since $g_{23} = g_{14} = 4$, Γ_{14} and Γ_{23} are of the form $C_4 \sqcup C_4 \sqcup C_4 \sqcup C_4$. Again, $g_{12} = g_{34} = g_{24} = g_{13} = 3$ implies Γ_{13}, Γ_{24}, Γ_{12} and Γ_{34} are of the form $C_4 \sqcup C_6 \sqcup C_6$. Let G_1, G_2, G_3 be the components of Γ_{12} and H_1, H_2, H_3 be the components of Γ_{34} such that x_1, x_2, x_3 represent the generators corresponding to G_1, G_2, G_3 respectively and $(x_1^3x_2^{-1})^{\varepsilon_1}$, $x_2^2x_1^{-1}$, $(x_1^2x_2)^{\varepsilon_2}$ represent the relations corresponding to H_1, H_2, H_3 respectively.

Let $G_1 = C_6(x^1, \dots, x^6)$, $G_2 = C_4(y^1, \dots, y^4)$ and $G_3 = C_6(z^1, \dots, z^6)$. Then to form the relations $(x_1^3x_2^{-1})^{\varepsilon_1}$, $x_2^2x_1^{-1}$, $(x_1^2x_2)^{\varepsilon_2}$, we need to add the following: (i) two edges of color 4 between G_1 and G_2, (ii) four edges of colors 4 between G_1 and G_3, (iii) two edges of color 4 between G_2 and G_3. These give all the 8 edges in $\gamma^{-1}(4)$. Therefore, we must have the following: (a) two 4-cycles between G_1 and G_3 in Γ_{14}, (b) one 4-cycle between G_1 and G_2 in Γ_{14}, (c) one 4-cycle between G_2 and G_3 in Γ_{14}. So, the 4-cycle in Γ_{24} is in between G_1 and G_3. Thus, up to an isomorphism, $\gamma^{-1}(4)$ is unique. In particular, we can assume that Γ_{124} is as in Fig. 3 (b). Now, y^2x^5 is an edge of color 4 between G_1 and G_2. Thus, y^1 (resp., y^2) is in H_1 or H_2. Assume, without loss, $y^1 \in H_1$. Then $y^2 \in H_2$. Since Γ is bipartite and H_2 represents $x_2x_3^{-1}x_2x_1^{-1}$, taking $v_1 = x^5$ as in Eq. (2.6), $H_2 = C_4(x^5, y^2, z^6, y^4)$. Since $\Gamma_{23} = C_4 \sqcup C_4 \sqcup C_4 \sqcup C_4$ and $\Gamma_{13} = C_4 \sqcup C_6 \sqcup C_6$, we have x^4y^1, x^3z^5, x^2z^4, x^1z^3, y^3z^1, $x^6z^2 \in \gamma^{-1}(3)$. Then $(\Gamma, \gamma) = \mathcal{M}_{2,3}$ of Fig. 3 (b). This completes the proof. □

Lemma 22. *There exists a unique* 18*-vertex crystallization of* S^3/Q_8.

Proof. Let (Γ, γ) be an 18-vertex crystallization of S^3/Q_8. By Lemma 15 and Theorem 4, (Γ, γ) is the crystallization of S^3/Q_8 with minimum number of vertices. So, by Lemma 18, (Γ, γ) has no 2-cycle. Thus, Γ is a simple graph. Since S^3/Q_8 is orientable, Γ is bipartite. By Proposition 14 and Remark 16, (Γ, γ) yields a presentation $\langle S \,|\, R\rangle$ of Q_8 with $\varphi(S, R) = 18$. Again, (Γ, γ) has no 2-cycle implies $g_{ij} \leqslant 4$ for $i \neq j$. By Proposition 13, $g_{12} + g_{13} + g_{14} = 18/2 + 2 = 11$. Assume, without loss, that $g_{12} = 3$ and $g_{13} = g_{14} = 4$. Therefore, if we choose generators (resp., relations) correspond to the components of Γ_{12} (resp., Γ_{34}) then (Γ, γ) yields a presentation $\langle S \,|\, R\rangle \in \mathcal{P}_2(Q_8) \setminus \mathcal{P}_1(Q_8)$ with $\varphi(S, R) = 18$. Then by the proof of part (v) in Lemma 15, $R = \{(x_2^2x_1^{-2})^{\varepsilon_1}, (x_1x_2x_1x_2^{-1})^{\varepsilon_2}, (x_2x_1x_2x_1^{-1})^{\varepsilon_3}\}$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}$. Then, by choosing $(i, j) = (3, 4)$ or $(4, 3)$ as in Eq. (2.6), we can assume that the three relations correspond to components of Γ_{34} are $(x_2^2x_1^{-2})^{\varepsilon_1}, (x_1x_2x_1x_2^{-1})^{\varepsilon_2}, x_2x_1x_2x_1^{-1}$, for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Since Γ has no 2-cycle, $\Gamma_{ij} = C_4 \sqcup C_4 \sqcup C_4 \sqcup C_6$ for $1 \leqslant i \leqslant 2$ and $3 \leqslant j \leqslant 4$. Let G_1, G_2, G_3 be the components of Γ_{12} and H_1, H_2, H_3 be the components of Γ_{34} such that x_1, x_2, x_3 represent the generators corresponding to G_1, G_2, G_3 and $(x_2^2x_1^{-2})^{\varepsilon_1}, x_2x_1x_2x_1^{-1}, (x_1x_2x_1x_2^{-1})^{\varepsilon_2}$ represent the relations corresponding to H_1, H_2, H_3 respectively. Then G_i, H_i are 6-cycles for $1 \leqslant i \leqslant 3$. Let $G_1 = C_6(a_1, \dots, a_6)$, $G_2 = C_6(b_1, \dots, b_6)$, $G_3 = C_6(c_1, \dots, c_6)$. Again, to form these relations, there are exactly three edges with color 4 between G_i and G_j for $i \neq j$. Since each of Γ_{14} and Γ_{24} has three 4-cycles, the three edges with color 4 between G_i and G_j for $i \neq j$, yield two 4-cycles. Then, up to an isomorphism, Γ_{124} is as in Fig. 4. Same arguments hold for color 3.

To construct $\tilde{r}_k$ as in Eq. (2.6), choose $(i, j) = (4, 3)$. Since H_2 presents the relation $x_2x_1x_2x_1^{-1}$, up to isomorphism, the starting vertex v_1 (as in Eq. (2.6)) is a_2 or a_3. If $v_1 = a_2$ then $H_2 = C_6(a_2, b_3, c_4, a_5, c_2, b_5)$ or $C_6(a_2, b_3, c_4, a_5, c_6, b_1)$. In the first case, if $b_4c_3 \in \gamma^{-1}(3)$, then b_4c_3 lies in a cycle of size at least 8 in Γ_{23}, which is not possible. Then the 4-cycle in Γ_{13} between G_2 and G_3 must be $C_4(b_1, b_2, c_5, c_6)$. But this is not possible since $b_1c_6 \in \gamma^{-1}(4)$. In the second case, if $b_2c_5 \in \gamma^{-1}(3)$, then b_2c_5 lies in a cycle of size at least 8 in Γ_{13}, which is not possible. Then the 4-cycle in Γ_{23} between G_2 and G_3 must be

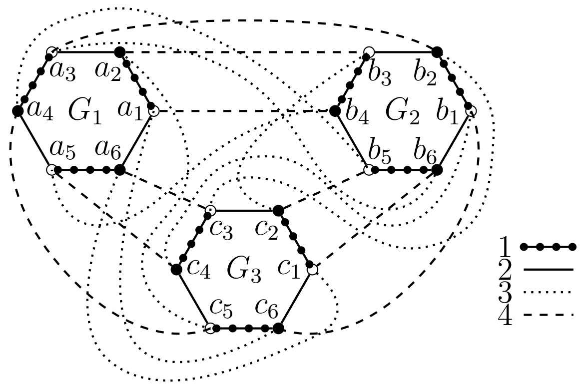

Figure 4: Crystallization $\mathcal{J}_3$ of $S^3/\mathbb{Q}_8$

$C_4(b_4, b_5, c_2, c_3)$. Again, this is not possible since $b_5c_2 \in \gamma^{-1}(4)$. Thus, $v_1 = a_3$. Now, if b_2c_5 is an edge of color 3 then a_4c_1 and a_3b_6 must be edges of color 3. Then b_5c_2 must be an edge of color 3 to make a 6-cycle in Γ_{13}, which is a contradiction (since b_5c_2 is already an edge of color 4). Thus, $H_2 = C_6(a_3, b_2, c_3, a_6, c_1, b_6)$. Since the three edges with color 3 between G_2 and G_3 yield two 4-cycles (in Γ_{13} and Γ_{23}), b_1c_4, b_3c_2 must be edges of color 3 between G_2 and G_3. To make a 6-cycle in Γ_{13}, a_5b_4 must be an edge of color 3. By similar arguments, $a_1c_6, a_2c_5, a_4b_5 \in \gamma^{-1}(3)$. Then, $(\Gamma, \gamma) = \mathcal{J}_3$ of Fig. 4.

Now, the components $H_1 = C_6(a_2, b_3, c_2, b_5, a_4, c_5)$ and $H_3 = C_6(b_4, a_1, c_6, b_1, c_4, a_5)$ of Γ_{34} yield the relations $x_2^2x_1^{-2}$ and $x_1x_2x_1x_2^{-1}$ respectively. Thus (Γ, γ) yields the presentation $\langle x_1, x_2 \,|\, x_2^2x_1^{-2}, x_1x_2x_1x_2^{-1}\rangle \cong Q_8$. This completes the proof. □

Lemma 23. *There exists a unique* 24*-vertex crystallization of* $S^1 \times S^1 \times S^1$.

Proof. Let (Γ, γ) be a 24-vertex crystallization of $(S^1)^3$. By Lemma 15 and Theorem 4, (Γ, γ) is the crystallization of $(S^1)^3$ with minimum number of vertices. So, by Lemma 18, (Γ, γ) has no 2-cycle. Thus, Γ is a simple graph. Since $(S^1)^3$ is orientable, Γ is bipartite. By Proposition 14 and Remark 16, (Γ, γ) yields a presentation $\langle S \,|\, R\rangle$ of $\mathbb{Z}^3$ with $\varphi(S, R) = 24$. Since any presentation of $\mathbb{Z}^3$ has at least three generators, $g_{ij} \geqslant 4$ for $i \neq j$. By Proposition 13, $g_{12} + g_{13} + g_{14} = 14$ and $g_{ij} = g_{kl}$ for i, j, k, l distinct.

Claim. (Γ, γ) does not yield a presentation $\langle S \,|\, R\rangle \in \mathcal{P}_4(\mathbb{Z}^3) \setminus \mathcal{P}_3(\mathbb{Z}^3)$ with $\varphi(S, R) = 24$.

Assume $\langle S \,|\, R\rangle \in \mathcal{P}_4(\mathbb{Z}^3) \setminus \mathcal{P}_3(\mathbb{Z}^3)$, where $S = \{x_1, x_2, x_3, x_4\}$. Then $A := \{(x_k^2x_l^{-1})^{\pm 1}, (x_jx_k^{-1}x_jx_l^{-1})^{\pm 1}, x_ix_j^{-1}x_kx_l^{-1}, (x_jx_kx_l^{-1})^{\pm 1}, (x_kx_l^{-1})^2, x_l^{\pm 2} \,:\, i, j, k, l \text{ are distinct}\}$ is the set of all relations of weight four. Since Γ has no 2-cycle, R has no element of weight two. This implies that R has at least three elements of weights four. Since $\mathbb{Z}^3$ has no torsion element, $x_l^{\pm 2}, (x_kx_l^{-1})^2 \notin R$. Consider an element $w \in R \cap A$. Assume, without loss, $w = (w_1x_4^{-1})^{\pm 1}$. Then $\langle S_1 \,|\, R_1\rangle \in \mathcal{P}_3(\mathbb{Z}^3)$, where $S_1 = \{x_1, x_2, x_3\}$ and R_1 consists of the elements $\bar{r}$, where $\bar{r}$ can be obtained from a relation $r \in R$ by replacing x_4 by w_1. Let $A(w) := \{\bar{r} \,:\, r \in A \setminus \{w^{\pm 1}\}\}$. Observe that the weights of the elements in $A(w)$ are 6 or 8.

Since $\langle S_1 \,|\, R_1\rangle \in \mathcal{P}_3(\mathbb{Z}^3) \setminus \mathcal{P}_2(\mathbb{Z}^3)$, we have $N(R_1) = N(R_0)$, where $R_0 = \{x_1x_2x_1^{-1}x_2^{-1}, x_1x_3x_1^{-1}x_3^{-1}, x_2x_3x_2^{-1}x_3^{-1}\}$ and hence $N(R_1)$ has no element of weight less than 6 (see the proof of part (v) of Lemma 15). Again, since $\#(R_1 \cap A(w)) \geqslant 2$, R_1 has at least two elements of weights 6 or 8. Observe that $D := \{x_ix_jx_i^{-1}x_j^{-1},\ x_ix_j^{-1}x_kx_i^{-1}x_jx_k^{-1},\ x_ix_jx_k^{-1}x_i^{-1}x_kx_j^{-1},\ x_ix_jx_i^{-1}x_kx_j^{-1}x_k^{-1}\ :\ \{i,j,k\} = \{1,2,3\}\}$ is the set of all relations of weights at most 8 in $N(R_0)$. So, R_1 has at least two independent relations from $D \cap A(w)$. But $D \cap A(w)$ does not contain two such elements, a contradiction. This proves the claim.

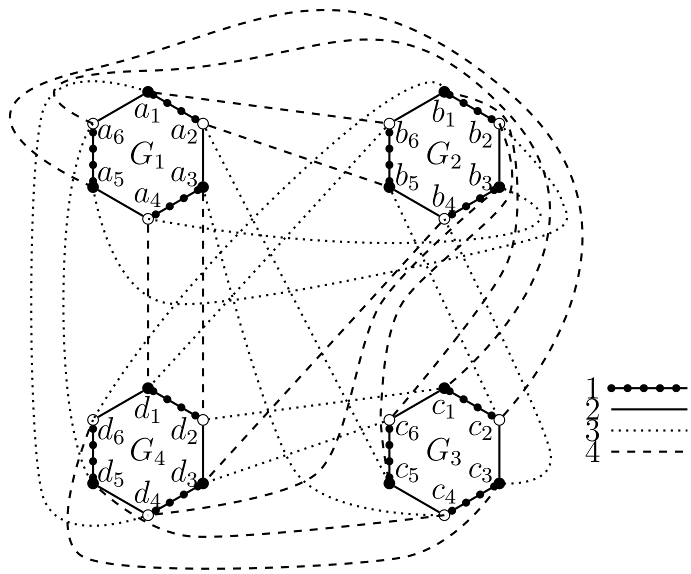

Figure 5: Crystallization $\mathcal{J}_4$ of $S^1 \times S^1 \times S^1$

By the claim, $g_{ij} \neq 5$ for all $1 \leqslant i \neq j \leqslant 4$. So, we can assume that $g_{12} = g_{13} = 4$ and $g_{14} = 6$. Then all the components of Γ_{14} and Γ_{23} are 4-cycles. Let $\Gamma_{12} = G_1 \sqcup \cdots \sqcup G_4$ and $\Gamma_{34} = H_1 \sqcup \cdots \sqcup H_4$ such that $x_1, \dots, x_4$ represent the generators corresponding to $G_1, \dots, G_4$ respectively and $r_1, \dots, r_4$ represent the relations corresponding to $H_1, \dots, H_4$ respectively. To construct $\tilde{r}_k$ as in Eq. (2.6), choose $(i,j) = (4,3)$. Thus (Γ, γ) yields a presentation $\langle S = \{x_1, x_2, x_3\} \,|\, R = \{r_1, r_2, r_3\} \rangle \in \mathcal{P}_3(\mathbb{Z}^3) \setminus \mathcal{P}_2(\mathbb{Z}^3)$ with $\varphi(S,R) = 24$. Then R contains three independent relations of weight 6 from the set $\{x_ix_jx_i^{-1}x_j^{-1},\ x_ix_j^{-1}x_kx_i^{-1}x_jx_k^{-1}\ :\ \{i,j,k\} = \{1,2,3\}\}$ (see the proof of part (v) of Lemma 15). Without loss of generality, we can assume that $R = \{x_1x_2x_1^{-1}x_2^{-1},\ (x_2x_3x_2^{-1}x_3^{-1})^{\varepsilon_1},\ (x_1x_3x_1^{-1}x_3^{-1})^{\varepsilon_2}\}$ for some $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$. Then, all the components of Γ_{12} and Γ_{34} are 6-cycles. Similarly, all the components of Γ_{13} and Γ_{24} are 6-cycles. Let $G_1 = C_6(a_1, \dots, a_6)$, $G_2 = C_6(b_1, \dots, b_6)$, $G_3 = C_6(c_1, \dots, c_6)$ and $G_4 = C_6(d_1, \dots, d_6)$. To form the relations, there are exactly two edges of color 3 (resp., 4) between G_i and G_j for $1 \leqslant i \neq j \leqslant 4$. Then, up to an isomorphism, Γ_{124} is as in Fig. 5. Now for the relation $x_1x_2x_1^{-1}x_2^{-1}$, we can choose $v_1 = b_6$ as in Eq. (2.6). Then the cycle for $x_1x_2x_1^{-1}x_2^{-1}$ is $H_1 = C_6(b_6, a_1, d_4, b_3, a_4, d_1)$. Since Γ_{23} consists of 4-cycles, it follows that $a_6d_5,\ a_5b_2,\ b_1d_6 \in \gamma^{-1}(3)$. Then the cycle for the relation $x_2x_3x_2^{-1}x_3^{-1}$ is $H_2 = C_6(c_6, b_1, d_6, c_3, b_4, d_3)$. Again (since Γ_{23} is union of 4-cycles and Γ_{13} is union of 6-cycles), $d_2c_1,\ b_5c_2,\ a_3c_4,\ a_2c_5 \in \gamma^{-1}(3)$. Then $(\Gamma, \gamma) = \mathcal{J}_4$ of Fig. 5.

Now, the components H_1, H_2 and $H_3 = C_6(c_1, a_6, d_5, c_4, a_3, d_2)$ yield the relations $x_1x_2x_1^{-1}x_2^{-1}$, $x_2x_3x_2^{-1}x_3^{-1}$ and $x_1x_3x_1^{-1}x_3^{-1}$ respectively. Thus (Γ, γ) yields the presentation $\langle x_1, x_2, x_3 \,|\, x_1x_2x_1^{-1}x_2^{-1},\, x_2x_3x_2^{-1}x_3^{-1}, x_1x_3x_1^{-1}x_3^{-1}\rangle \cong \mathbb{Z}^3$. This completes the proof. □

Remark 24. The crystallizations $\mathcal{K}_{2,1}$, $\mathcal{K}_{3,1}$ and $\mathcal{M}_{3,2}$ (in Figures 2 and 3) were originally found by Gagliardi *et al.* ([8, 10]). The first two have the following natural generalization: Consider the bipartite graph Γ consists of two disjoint $2p$-cycles $G_1 = C_{2p}(a_1, b_1, \dots, a_p, b_p)$, $G_2 = C_{2p}(c_1, d_1, \dots, c_p, d_p)$ together with $4p$ edges a_ic_i, b_id_i, a_ic_{i+q}, b_id_{i+q} for $1 \leqslant i \leqslant p$. Consider the edge-coloring γ with colors $1, 2, 3, 4$ of Γ as: $\gamma(b_ia_{i+1}) = \gamma(d_ic_{i+1}) = 1$, $\gamma(a_ib_i) = \gamma(c_id_i) = 2$, $\gamma(a_ic_{i+q}) = \gamma(b_id_{i+q}) = 3$ and $\gamma(a_ic_i) = \gamma(b_id_i) = 4$, $1 \leqslant i \leqslant p$. (Summations in the subscripts are modulo p.) Then, $\mathcal{K}_{p,q} = (\Gamma, \gamma)$ is a $4p$-vertex crystallization of $L(p, q)$, for $p \geqslant 2$ and $q \geqslant 1$. This series is more or less known in the literature. In the next section, we present some generalizations of $\mathcal{M}_{3,2}$.

5 Two series of crystallizations of lens spaces

Generalizing the construction of $\mathcal{M}_{3,2}$ (Fig. 3 (b)) we have constructed the following series of crystallizations.

5.1 A $4(k+q-1)$-vertex crystallization of $L(kq-1, q)$

Let $q \geqslant 3$. For each $k \geqslant 2$, we construct a $4(k+q-1)$-vertex 4-colored simple graph $\mathcal{M}_{k,q} = (\Gamma^k, \gamma^k)$ with the color set $\{1, 2, 3, 4\}$ inductively which yields the presentation $\langle x, y \,|\, x^qy^{-1}, y^kx^{-1}\rangle$. For this, we want $g^k_{12} = g^k_{34} = 3$. Then, without loss, $g^k_{13} = g^k_{24} = k+q-2$ and $g^k_{14} = g^k_{23} = k+q-1$, where g^k_{ij} is the number of components of Γ^k_{ij} for $i \neq j$. These imply, Γ^k_{14} and Γ^k_{23} must be union of 4-cycles and Γ^k_{13} (resp., Γ^k_{24}) has two 6-cycles and $(k+q-4)$ 4-cycles. Then, by Proposition 13, $\mathcal{M}_{k,q}$ would be a crystallization of some connected closed 3-manifold M_k.

***k* = 2 case:** The crystallization $\mathcal{M}_{2,q}$ is given in Fig. 6. Then, the components of Γ^2_{12} are $G_1 = C_{2q}(x^1, \dots, x^{2q})$, $G_2 = C_4(y^1, \dots, y^4)$, $G_3 = C_{2q}(z^1, \dots, z^{2q})$ and the components of Γ^2_{34} are $H_1 = C_{2q}(y^1, x^{2q}, z^2, x^2, \dots, z^{2q-2},\, x^{2q-2})$, $H_2 = C_4(x^{2q-1}, y^2, z^{2q}, y^4)$, $H_3 = C_{2q}(z^{2q-1}, y^3, z^1, x^1, \dots, z^{2q-3}, x^{2q-3})$. Let x, y be the generators corresponding to G_1 and G_2 respectively. To construct $\tilde{r}_1$ (resp., $\tilde{r}_2$) as in Eq. (2.6), choose $(i, j) = (4, 3)$ and $v_1 = y^1$ (resp., $v_1 = x^{2q-1}$). Then H_1 and H_2 represent the relations x^qy^{-1} and y^2x^{-1} respectively. Therefore, by Proposition 14, $\pi(M_2, *) \cong \langle x, y \,|\, x^qy^{-1}, y^2x^{-1}\rangle \cong \mathbb{Z}_{2q-1}$.

Let T and T_2 be the 3-dimensional simplicial cell complexes represented by the color graphs $\Gamma^2|_{\{x^1,x^2,x^3,z^3\}}$ and $\Gamma^2|_{V(\Gamma^2)\setminus\{x^1,x^2,x^3,z^3\}}$ respectively. Then $|T|$ and $|T_2|$ are solid tori and the facets (2-cells) of $T \cap T_2$ are $x^1_2, x^1_4, x^2_3, x^2_4,\, x^3_1, x^3_3, z^3_1, z^3_2$. Thus, $|T \cap T_2|$ is a torus (see Fig. 7 (b)) with $\pi_1(|T \cap T_2|, v_1) = \langle \alpha = [a], \beta = [b] \,|\, \alpha\beta\alpha^{-1}\beta^{-1}\rangle$, where $a = x^2_{34}x^3_{34}$ and $b = x^3_{34}z^3_{13}z^3_{23}$. Then $b = x^1_{34}x^1_{13}x^1_{23} = \partial(x^1_3) \sim 1$ in $|T|$. Therefore, $\pi_1(|T|, v_1) = \langle \alpha, \beta \,|\, \beta\rangle$.

Since $\alpha\beta = \beta\alpha$ in $|T| \cap |T_2|$, $ab \sim ba$ in $|T_2|$. Now $ab = (x^2_{34}x^1_{34})(x^1_{34}x^1_{13}x^1_{23}) \sim x^2_{34}x^1_{13}x^1_{23} = x^2_{34}x^2_{13}x^1_{23} \sim x^2_{23}x^1_{23}$. Therefore, $a^2b \sim aba \sim x^2_{23}x^1_{23}x^2_{34}x^1_{34} = x^2_{23}x^{2q}_{23}x^{2q}_{34}x^1_{34} \sim x^2_{23}x^{2q}_{13}x^1_{34} = x^2_{23}z^1_{13}z^1_{34} \sim x^2_{23}z^1_{23}$. Thus, $a^3b \sim x^2_{34}x^1_{34}x^2_{23}z^1_{23} = x^2_{34}x^3_{34}x^3_{23}z^1_{23} \sim x^2_{34}x^3_{13}z^1_{23} =$

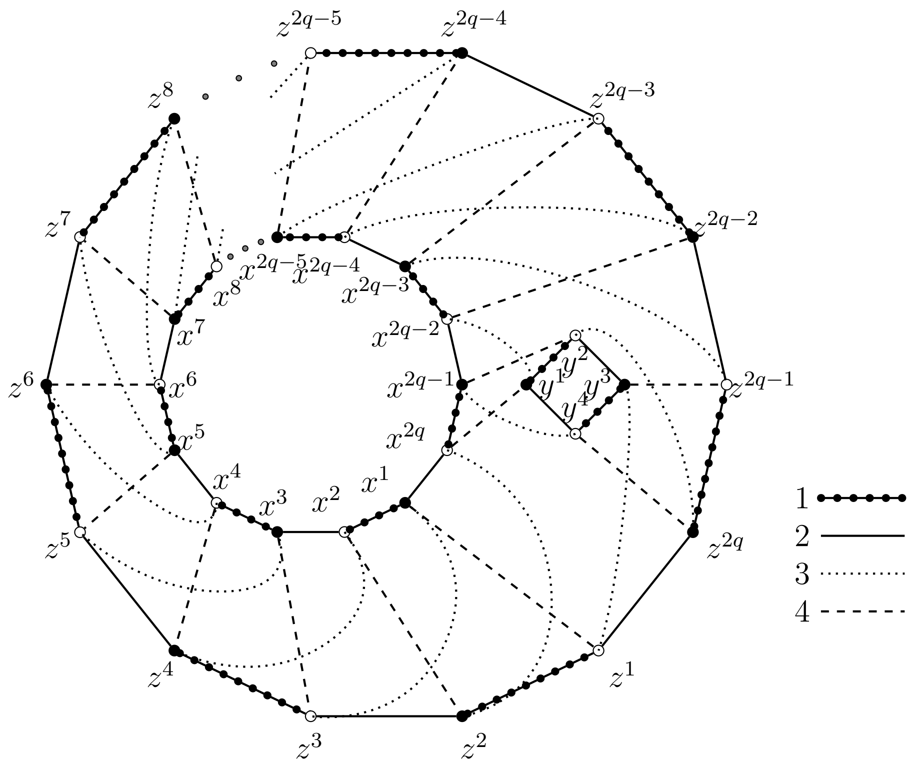

Figure 6: Crystallization $\mathcal{M}_{2,q}$ of L$(2q-1,q)$

$x_{34}^4 x_{13}^4 z_{23}^1 \sim x_{23}^4 z_{23}^1$. Similarly, $a^{q-1}b \sim x_{23}^{2q-4} z_{23}^1$. Therefore, $a^q b \sim x_{34}^2 x_{34}^1 x_{23}^{2q-4} z_{23}^1 = x_{34}^2 x_{34}^{2q-3} x_{23}^{2q-3} z_{23}^1 \sim x_{34}^2 x_{13}^{2q-3} z_{23}^1 = x_{34}^{2q-2} x_{13}^{2q-2} z_{23}^1 \sim x_{23}^{2q-2} z_{23}^1 = x_{23}^{2q-1} z_{23}^1 \sim x_{34}^{2q-1} x_{13}^{2q-1} z_{23}^1 = x_{34}^{2q-1} z_{13}^1 z_{23}^1 \sim x_{34}^{2q-1} z_{34}^1 = x_{34}^{2q-1} z_{34}^{2q-1}$. Since $k = 2$, we have $z_{13}^{2q-1} = z_{13}^{2q}$. This implies, $a^{2q-1}b^2 \sim x_{34}^{2q-1} z_{34}^{2q-1} x_{23}^{2q-4} z_{23}^1 = x_{34}^{2q-1} z_{34}^{2q-1} z_{23}^{2q-1} z_{23}^1 \sim x_{34}^{2q-1} z_{13}^{2q-1} z_{23}^1 = z_{34}^{2q} z_{13}^{2q} z_{23}^{2q} = \partial(z_3^{2q}) \sim 1$ in $|T_2|$. Thus $\pi_1(|T_2|, v_1) = \langle \alpha, \beta \,|\, \alpha^{2q-1}\beta^2, \alpha\beta\alpha^{-1}\beta^{-1} \rangle$. This implies that (see the second paragraph of Subsection 2.3) $|T| \cup |T_2| = L(2q-1,2)$. Therefore, $\mathcal{M}_{2,q}$ is a crystallization of $L(2q-1,2) \cong L(2q-1,q)$.

***k* = 3 case:** Here $z_{13}^{2q-1} \neq z_{13}^{2q}$. Let

$$\Gamma^3 = (V(\Gamma^2) \cup \{y^5, y^6, z^{2q+1}, z^{2q+2}\}, E(\Gamma^2) \setminus \{y^2z^{2q}, y^3z^1, y^3y^4, z^{2q-1}z^{2q}\} \cup \{y^3y^5, y^5y^6, y^6y^4, z^{2q-1}z^{2q+1}, z^{2q+1}z^{2q+2}, z^{2q+2}z^{2q}, y^5z^{2q+1}, y^6z^{2q+2}, y^2z^{2q+1}, y^3z^{2q+2}, y^5z^{2q}, y^6z^1\}).$$

Consider the following coloring γ^3 on the edges of Γ^3: same colors on the old edges as in $\mathcal{M}_{2,q}$, color 1 on the edges y^3y^5, y^6y^4, $z^{2q-1}z^{2q+1}$, $z^{2q+2}z^{2q}$, color 2 on the edges y^5y^6, $z^{2q+1}z^{2q+2}$, color 3 on the edges y^2z^{2q+1}, y^3z^{2q+2}, y^5z^{2q}, y^6z^1 and color 4 on the edges y^5z^{2q+1}, y^6z^{2q+2} (see Fig. 7 (a)). Let T be as in the case $k = 2$ and T_3 be the cell complex represented by the colored graph $\Gamma^3|_{V(\Gamma^3)\setminus\{x^1,x^2,x^3,z^3\}}$.

Then, $a^{2q-1}b^2 \sim z_{34}^{2q} z_{13}^{2q-1} z_{23}^{2q} = z_{34}^{2q+1} z_{13}^{2q+1} z_{23}^{2q} \sim z_{23}^{2q+1} z_{23}^{2q} = z_{23}^{2q+2} z_{23}^{2q}$. This implies, $a^{3q-1}b^3 = (a^q b)(a^{2q-1}b^2) \sim (x_{34}^{2q-1} z_{34}^{2q-1})(z_{23}^{2q+2} z_{23}^{2q}) = z_{34}^{2q} z_{34}^{2q+2} z_{23}^{2q+2} z_{23}^{2q} \sim z_{34}^{2q} z_{13}^{2q+2} z_{23}^{2q} = z_{34}^{2q} z_{13}^{2q} z_{23}^{2q} = \partial(z_3^{2q}) \sim 1$ in $|T_3|$. Thus, $\pi_1(|T_3|, v_1) = \langle \alpha, \beta \,|\, \alpha^{3q-1}\beta^3, \alpha\beta\alpha^{-1}\beta^{-1} \rangle$ and hence $|T| \cup |T_3| = L(3q-1,3)$. Therefore, $\mathcal{M}_{3,q}$ is a crystallization of $L(3q-1,3) \cong L(3q-1,q)$.

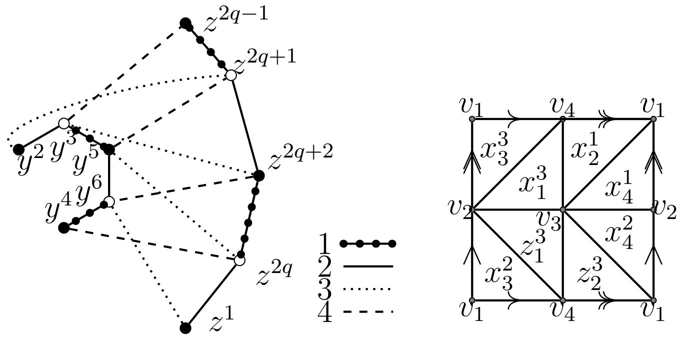

Figure 7: (*a*) Crystallization $\mathcal{M}_{3,q}$ of L$(3q-1,q)$ (*b*) $|T_1 \cap T_k|$

***k* ⩾ 4 case:** Consider the graph

$$\Gamma^k = (V(\Gamma^{k-1}) \cup \{y^{2k-1}, y^{2k}, z^{2q+2k-5}, z^{2q+2k-4}\}, E(\Gamma^{k-1}) \setminus \{y^{2k-3}z^{2q}, y^{2k-2}z^1, y^{2k-2}y^4, z^{2q+2k-6}z^{2q}\} \cup \{y^{2k-2}y^{2k-1}, y^{2k-1}y^{2k}, y^{2k}y^4, z^{2q+2k-6}z^{2q+2k-5}, z^{2q+2k-5}z^{2q+2k-4}, z^{2q+2k-4}z^{2q}, y^{2k-1}z^{2q+2k-5}, y^{2k}z^{2q+2k-4}, y^{2k-3}z^{2q+2k-5}, y^{2k-2}z^{2q+2k-4}, y^{2k-1}z^{2q}, y^{2k}z^1\}).$$

Also, consider the following coloring γ^k on the edges of Γ^k: same colors on the old edges as in $\mathcal{M}_{k-1,q}$, color 1 on the edges $y^{2k-2}y^{2k-1}$, $y^{2k}y^4$, $z^{2q+2k-6}z^{2q+2k-5}$, $z^{2q+2k-4}z^{2q}$, color 2 on the edges $y^{2k-1}y^{2k}$, $z^{2q+2k-5}z^{2q+2k-3}$, color 3 on the edges $y^{2k-3}z^{2q+2k-5}$, $y^{2k-2}z^{2q+2k-4}$, $y^{2k-1}z^{2q}$, $y^{2k}z^1$ and color 4 on the edges $y^{2k-1}z^{2q+2k-5}$, $y^{2k}z^{2q+2k-4}$. Let T be as in the case $k=2$ and T_k be the cell complex represented by the colored graph $\Gamma^k|_{V(\Gamma^k)\setminus\{x^1,x^2,x^3,z^3\}}$.

Claim. $a^{kq-1}b^k \sim z_{34}^{2q} z_{13}^{2q+2k-4} z_{23}^{2q}$ in $|T_k|$.

We prove the claim by induction. It is true for $k=3$. Assume that $a^{(k-1)q-1}b^{k-1} \sim z_{34}^{2q} z_{13}^{2q+2(k-1)-4} z_{23}^{2q}$ in $|T_{k-1}|$. Now, $a^q b \sim z_{34}^{2q} z_{34}^1 = z_{34}^{2q} z_{34}^{2q+2k-4}$ and $a^{(k-1)q-1}b^{(k-1)} \sim z_{34}^{2q} z_{13}^{2q+2k-6} z_{23}^{2q} = z_{34}^{2q+2k-5} z_{13}^{2q+2k-5} z_{23}^{2q} \sim z_{23}^{2q+2k-5} z_{23}^{2q} = z_{23}^{2q+2k-4} z_{23}^{2q}$. Thus, $a^{kq-1}b^k \sim (a^q b)(a^{(k-1)q-1}b^{(k-1)}) \sim z_{34}^{2q} z_{34}^{2q+2k-4} z_{23}^{2q+2k-4} z_{23}^{2q} = z_{34}^{2q} z_{13}^{2q+2k-4} z_{23}^{2q}$ in $|T_k|$. The claim now follows by induction.

Since $z_{13}^{2q} = z_{13}^{2q+2k-4}$ in T_k, by the claim we get $a^{kq-1}b^k \sim 1$ in $|T_k|$. Thus, $\pi_1(|T_k|, v_1) = \langle \alpha, \beta \,|\, \alpha^{kq-1}\beta^k, \alpha\beta\alpha^{-1}\beta^{-1}\rangle$ and hence $|T| \cup |T_k| = L(kq-1,k) \cong L(kq-1,q)$. Therefore, $\mathcal{M}_{k,q}$ is a crystallization of $L(kq-1,q)$.

5.2 A $4(k+q)$-vertex crystallization of $L(kq+1,q)$

Let $q \geqslant 4$. For each $k \geqslant 1$, we construct a $4(k+q)$-vertex 4-colored simple graph $\mathcal{N}_{k,q} = (\Gamma^k, \gamma^k)$ with the color set $\{1,2,3,4\}$ inductively which yields the presentation $\langle x, y \,|\, x^q y^{-1}, xy^k\rangle$. For this, we want $g_{12}^k = g_{34}^k = 3$. Then, without loss, $g_{13}^k = g_{24}^k = k+q-1$ and $g_{14}^k = g_{23}^k = k+q$, where g_{ij}^k is the number of components of Γ_{ij}^k for $i \neq j$. These imply, Γ_{14}^k and Γ_{23}^k must be union of 4-cycles and Γ_{13}^k (resp., Γ_{24}^k) has two 6-cycles and $(k+q-4)$ 4-cycles. Then, by Proposition 13, $\mathcal{N}_{k,q}$ would be a crystallization of some connected closed 3-manifold M_k.

$k = 1$ case: The crystallization $\mathcal{N}_{1,q}$ is given in Fig. 8. Then, the components of Γ^1_{12} are $G_1 = C_{2q}(x^1, \dots, x^{2q})$, $G_2 = C_4(y^1, \dots, y^4)$, $G_3 = C_{2q}(z^1, \dots, z^{2q})$ and the components of Γ^1_{34} are $H_1 = C_{2q}(y^3, x^2, z^2, x^4, \dots, z^{2q-2}, x^{2q})$, $H_2 = C_4(z^{2q-1}, x^1, z^1, y^1)$, $H_3 = C_{2q}(x^3, y^2, z^{2q}, y^4, x^{2q-1}, z^{2q-3}, x^{2q-3}, \dots, z^4, x^4, z^3)$. Let x, y be the generators corresponding to G_1 and G_2 respectively. To construct $\tilde{r}_1$ (resp., $\tilde{r}_2$) as in Eq. (2.6), choose $(i, j) = (4, 3)$ and $v_1 = y^3$ (resp., $v_1 = z^{2q-1}$). Then H_1 and H_2 represent the relations $x^q y^{-1}$ and xy respectively. Therefore, by Proposition 14, $\pi(M_1, *) \cong \langle x, y \,|\, x^q y^{-1}, xy \rangle \cong \mathbb{Z}_{q+1}$.

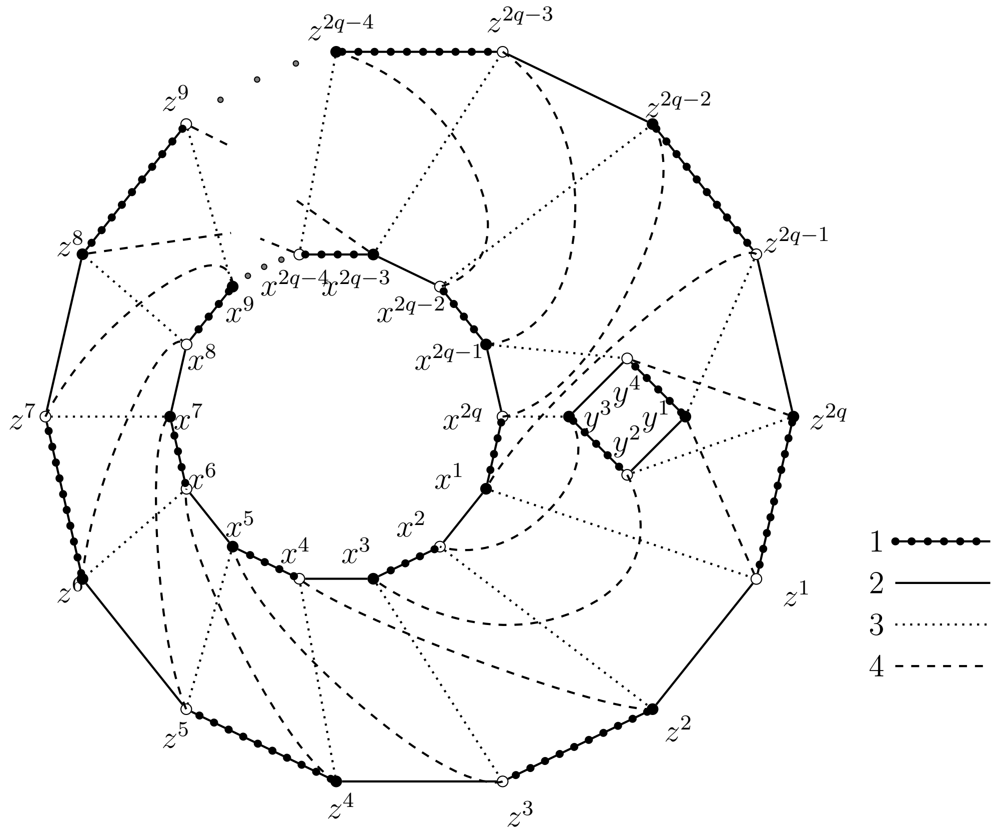

Figure 8: Crystallization $\mathcal{N}_{1,q}$ of $L(q+1, q)$

Let T and T_1 be the 3-dimensional simplicial cell complexes represented by the color graphs $\Gamma^1|_{\{x^5,x^4,x^3,z^3\}}$ and $\Gamma^1|_{V(\Gamma^1)\setminus\{x^5,x^4,x^3,z^3\}}$ respectively. Then $|T|$ and $|T_1|$ are solid tori and the facets (2-cells) of $T \cap T_1$ are x^5_2, x^5_3, x^4_3, x^4_4, x^3_1, x^3_4, z^3_1, z^3_2. Thus, $|T \cap T_1|$ is a torus (see Fig. 9 (b)) with $\pi_1(|T \cap T_1|, v_1) = \langle \alpha = [a], \beta = [b] \,|\, \alpha\beta\alpha^{-1}\beta^{-1} \rangle$, where $a = x^4_{34}x^3_{34}$ and $b = x^3_{34}x^3_{13}x^4_{23}$. Then $b = x^3_{34}x^3_{13}x^3_{23} = \partial(x^3_1) \sim 1$ in $|T|$. Therefore, $\pi_1(|T|, v_1) = \langle \alpha, \beta \,|\, \beta \rangle$.

Since $\alpha\beta = \beta\alpha$ in $|T \cap T_1|$, it follows that $ab \sim ba$ in $|T_1|$. Now, $ab = (x^4_{34}x^3_{34})(x^3_{34}x^3_{13}x^3_{23}) \sim x^4_{34}x^3_{13}x^3_{23} = z^2_{34}z^2_{13}x^3_{23} \sim z^2_{23}x^3_{23} = z^1_{23}x^3_{23}$. Thus, $a^2b \sim aba \sim (z^1_{23}x^3_{23})(x^4_{34}x^3_{34}) = z^1_{23}x^4_{23}x^4_{34}\ x^3_{34} \sim z^1_{23}x^4_{13}x^3_{34} = z^1_{23}x^5_{13}x^5_{34} \sim z^1_{23}x^5_{23} = z^1_{23}x^6_{23}$. Therefore, $a^3b \sim aba^2 \sim z^1_{23}x^6_{23}x^4_{34}x^3_{34} = z^1_{23}x^6_{23}x^6_{34}x^3_{34} \sim z^1_{23}x^6_{13}x^3_{34} = z^1_{23}x^7_{13}x^7_{34} \sim z^1_{23}x^7_{23} = z^1_{23}x^8_{23}$. Similarly we get, $a^{q-2}b \sim z^1_{23}x^{2q-2}_{23}$. Thus, $a^{q-1}b \sim a^{q-2}ba \sim z^1_{23}x^{2q-2}_{23}x^4_{34}x^3_{34} = z^1_{23}x^{2q-2}_{23}x^{2q-2}_{34}x^3_{34} \sim z^1_{23}x^{2q-2}_{13}x^3_{34} = z^1_{23}x^{2q-1}_{13}x^{2q-1}_{34} \sim z^1_{23}x^{2q-1}_{23} = z^1_{23}x^{2q}_{23}$. Therefore, $a^q b \sim z^1_{23}x^{2q}_{23}x^4_{34}x^3_{34} =$

$z^1_{23}x^{2q}_{23}x^{2q}_{34}\ x^3_{34} \sim z^1_{23}x^{2q}_{13}x^3_{34} = z^1_{23}y^2_{13}y^2_{34} \sim z^1_{23}y^2_{23} \sim z^1_{34}z^1_{13}y^2_{23} = z^1_{34}y^2_{13}y^2_{23} \sim z^1_{34}y^2_{34} = z^1_{34}z^{2q}_{34}$. Again, $a = x^4_{34}x^3_{34} = y^3_{34}y^2_{34} \sim y^3_{24}y^3_{14}y^2_{34} = y^3_{24}y^2_{14}y^2_{34} \sim y^3_{24}y^2_{24} = y^3_{24}z^1_{24}$. Therefore, $a^{q+1}b \sim aa^qb \sim y^3_{24}z^1_{24}z^1_{34}z^{2q}_{34} \sim z^{2q}_{24}z^1_{14}z^{2q}_{34} = z^{2q}_{24}z^{2q}_{14}z^{2q}_{34} = \partial(z^{2q}_4) \sim 1$ in $|T_1|$. Thus $\pi_1(|T_1|, v_1) = \langle \alpha, \beta \,|\, \alpha^{q+1}\beta, \alpha\beta\alpha^{-1}\beta^{-1}\rangle$. This implies that $|T| \cup |T_1| = L(q+1, 1)$. Therefore, $\mathcal{N}_{1,q}$ is a crystallization of $L(q+1,1) \cong L(q+1,q)$.

***k* = 2 case:** Here $z^1_{14} \neq z^{2q}_{14}$. Let

$$\Gamma^2 = (V(\Gamma^1) \cup \{y^5, y^6, z^{2q+1}, z^{2q+2}\}, E(\Gamma^1) \setminus \{y^2z^{2q}, y^1z^{2q-1}, y^1y^4, z^1z^{2q}\} \cup \{y^1y^6, y^5y^6, y^4y^5, z^1z^{2q+2}, z^{2q+1}z^{2q+2}, z^{2q}z^{2q+1}, y^5z^{2q+1}, y^6z^{2q+2}, y^2z^{2q+2}, y^1z^{2q+1}, y^6z^{2q}, y^5z^{2q-1}\}).$$

To construct $\mathcal{N}_{2,q}$, consider the following coloring γ^2 on the edges of Γ^2: same colors on the old edges as in $\mathcal{N}_{1,q}$, color 1 on the edges y^1y^6, y^4y^5, $z^1z^{2q+2}, z^{2q}z^{2q+1}$ color 2 on the edges y^5y^6, $z^{2q+1}z^{2q+2}$, color 3 on the edges y^2z^{2q+2}, y^1z^{2q+1}, y^6z^{2q}, y^5z^{2q-1} and color 4 on the edges y^5z^{2q+1}, y^6z^{2q+2} (see Fig. 9 (*a*)). Let T be as in the case $k = 2$ and T_2 be the cell complex represented by the colored graph $\Gamma^2|_{V(\Gamma^2)\setminus\{x^5,x^4,x^3,z^3\}}$.

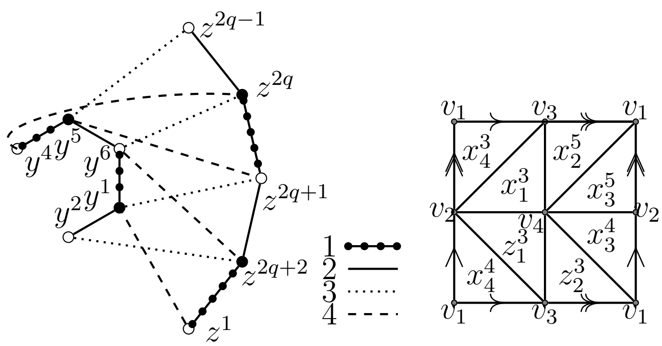

Figure 9: (a) Crystallization $\mathcal{N}_{2,q}$ of $L(2q+1,q)$ (b) $|T| \cap |T_k|$

Then, $a^{q+1}b \sim z^{2q}_{24}z^1_{14}z^{2q}_{34} = z^{2q}_{24}z^{2q+2}_{14}z^{2q+2}_{34} \sim z^{2q}_{24}z^{2q+2}_{24} = z^{2q}_{24}z^{2q+1}_{24}$. This implies, $a^{2q+1}b^2 \sim a^{q+1}ba^qb \sim z^{2q}_{24}z^{2q+1}_{24}z^1_{34}z^{2q}_{34} = z^{2q}_{24}z^{2q+1}_{24}z^{2q+1}_{34}z^{2q}_{34} \sim z^{2q}_{24}z^{2q+1}_{14}z^{2q}_{34} = z^{2q}_{24}z^{2q}_{14}z^{2q}_{34} = \partial(z^{2q}_4) \sim 1$ in $|T_2|$. Thus, $\pi_1(|T_2|, v_1) = \langle \alpha, \beta \,|\, \alpha^{2q+1}\beta^2, \alpha\beta\alpha^{-1}\beta^{-1}\rangle$ and hence $|T| \cup |T_2| = L(2q+1, 2)$. Therefore, $\mathcal{N}_{2,q}$ is a crystallization of $L(2q+1, 2) \cong L(2q+1, q)$.

***k* ⩾ 3 case:** Let

$$\Gamma^k = (V(\Gamma^{k-1}) \cup \{y^{2k+1}, y^{2k+2}, z^{2q+2k-3}, z^{2q+2k-2}\}, E(\Gamma^{k-1}) \setminus \{y^{2k}z^{2q}, y^{2k-1}z^{2q-1}, y^{2k-1}y^4, z^{2q+2k-5}z^{2q}\} \cup \{y^{2k-1}y^{2k+2}, y^{2k+1}y^{2k+2}, z^{2q+2k-5}z^{2q+2k-2}, z^{2q+2k-3}z^{2q+2k-2}, z^{2q}z^{2q+2k-3}, y^{2k+1}z^{2q+2k-3}, y^{2k+2}z^{2q+2k-2}, y^{2k}z^{2q+2k-2}, y^{2k-1}z^{2q+2k-3}, y^{2k+2}z^{2q}, y^{2k+1}z^{2q-1}, y^4y^{2k+1}\}).$$

To construct $\mathcal{N}_{k,q}$, consider the following coloring γ^k on the edges of Γ^k: same colors on the old edges as in $\mathcal{N}_{k-1,q}$, color 1 on the edges $y^{2k-1}y^{2k+2}$, y^4y^{2k+1}, $z^{2q+2k-5}z^{2q+2k-2}$, $z^{2q}z^{2q+2k-3}$, color 2 on the edges $y^{2k+1}y^{2k+2}$, $z^{2q+2k-3}z^{2q+2k-2}$, color 3 on the edges $y^{2k}z^{2q+2k-2}$, $y^{2k-1}z^{2q+2k-3}$, $y^{2k+2}z^{2q}$, $y^{2k+1}z^{2q-1}$ and color 4 on the edges $y^{2k+1}z^{2q+2k-3}$, $y^{2k+2}z^{2q+2k-2}$. Let T be as in the case $k = 1$ and T_k be the cell complex represented by the colored graph $\Gamma^k|_{V(\Gamma^k)\setminus\{x^5,x^4,x^3,z^3\}}$.

Claim. $a^{kq+1}b^k \sim z_{24}^{2q} z_{14}^{2q+2k-3} z_{34}^{2q}$ in $|T_k|$.

We prove the claim by induction. It is true for $k = 2$. Assume that $a^{(k-1)q+1}b^{k-1} \sim z_{24}^{2q} z_{14}^{2q+2(k-1)-3} z_{34}^{2q}$ in $|T_{k-1}|$. Now $a^q b \sim z_{34}^{2q+2k-3} z_{34}^{2q}$ and $a^{q(k-1)+1}b^{(k-1)} \sim z_{24}^{2q} z_{24}^{2q+2k-4} = z_{24}^{2q} z_{24}^{2q+2k-3}$. So, $a^{qk+1}b^k \sim (a^{q(k-1)+1}b^{(k-1)}(a^q b) \sim z_{24}^{2q} z_{24}^{2q+2k-3} z_{34}^{2q+2k-3} z_{34}^{2q} = z_{24}^{2q} z_{14}^{2q+2k-3}$ in $|T_k|$. The claim now follows by induction.

Since $z_{14}^{2q} = z_{14}^{2q+2k-3}$ in T_k, by the claim we get $a^{kq+1}b^k \sim 1$ in $|T_k|$. Thus, $\pi_1(|T_k|, v_1) = \langle \alpha, \beta \,|\, \alpha^{kq+1}\beta^k, \alpha\beta\alpha^{-1}\beta^{-1} \rangle$ and hence $|T| \cup |T_k| = L(kq+1, k) \cong L(kq+1, q)$. Therefore, $\mathcal{N}_{k,q}$ is a crystallization of $L(kq+1, q)$.

A few days after we posted the first version of this article (arXiv:1308.6137) in the arXiv, Casali and Cristofori posted an article on complexity of lens spaces [4] in the arXiv (arXiv:1309.5728). In that paper, the authors constructed crystallizations of $L(p, q)$ with $4S(p, q)$ vertices, where $S(p, q)$ denotes the sum of all partial quotients in the expansion of q/p as a regular continued fraction. In particular, they have constructed $L(kq-1, q)$ with $4(k+q-1)$ vertices for $k, q \geqslant 2$ and $L(kq+1, q)$ with $4(k+q)$ vertices for $k, q \geqslant 1$. Their constructions are different from ours.

Remark 25. From the enumeration of crystallizations of prime 3-manifolds with at most 30 vertices (see [3, 12]), we know that $\Psi(L(9, 4)) = 24$ and $\Psi(L(13, 4)) = 28$. From our constructions in Subsections 5.1 and 5.2, we know $\mathcal{M}_{2,5}$ and $\mathcal{N}_{2,4}$ are 24-vertex crystallizations of $L(9, 4)$. The induced subgraphs of $\mathcal{M}_{2,5}$ on 2-colored edges are of the form $2C_{10} \sqcup C_4$, $2C_6 \sqcup 3C_4$ or $6C_4$ and such subgraphs of $\mathcal{N}_{2,4}$ are of the form $C_{10} \sqcup C_8 \sqcup C_6$, $2C_6 \sqcup 3C_4$ or $6C_4$. So, $\mathcal{M}_{2,5}$ and $\mathcal{N}_{2,4}$ are non-isomorphic. Thus, $L(9, 4)$ has more than one (non-isomorphic) crystallizations with minimum number of vertices. The constructions in [4] give a 28-vertex of crystallization of $L(13, 4)$ with $\{g_{12}, g_{13}, g_{14}\} = \{4, 5, 7\}$. Observe that $\mathcal{N}_{3,4}$ is also a 28-vertex of crystallization of $L(13, 4)$ with $\{g_{12}, g_{13}, g_{14}\} = \{3, 6, 7\}$. Thus, these two crystallizations of $L(13, 4)$ are non-isomorphic. So, the minimal crystallization $\mathcal{N}_{3,4}$ of $L(13, 4)$ is not unique. Also, from the list of crystallizations in [12], we know that there are several 3-manifolds having more than one crystallizations with minimum number of vertices.

6 Proofs of Theorems 6, 8 and Corollary 7

Proof of Theorem 6. Let $\mathcal{M}_{2,3}$ be as in Subsection 5.1. Then, $\mathcal{M}_{2,3}$ is a 16-vertex crystallization of $L(5, 3) = L(5, 2)$. Part (i) now follows from Lemmas 15, 19, ..., 23.

If $f_3(X) < 8$ then, by Theorem 4, $\psi(M) < 8$ and hence $\psi(M) = 2$. Therefore $\pi(M, *) = \{0\}$ and hence, by Perelman's theorem (Poincaré conjecture), $M = S^3$. Part (ii) now follows from Lemma 19. □

Proof of Corollary 7. From the proof of Lemma 22, $m(Q_8) = 2$. Therefore, if X is a pseudotriangulation of S^3/Q_8 then, by Corollary 5 and Lemma 3, $h_2(X) \geqslant \psi(S^3/Q_8) - 2 = 18 - 2 > 12 = 6m(S^3/Q_8)$.

Again, if X is a pseudotriangulation of $S^1 \times S^1 \times S^1$ then, by Corollary 5 and Lemma 3, $h_2(X) \geqslant \psi(S^1 \times S^1 \times S^1) - 2 = 24 - 2 > 6 \times 3 = 6m(S^1 \times S^1 \times S^1)$.

For p, q relatively prime and $p \geqslant 3$, let X be a pseudotriangulation of $L(p,q)$. Then, by Theorem 6 (ii) and Corollary 5, $h_2(X) \geqslant \psi(L(p,q)) - 2 > 8 - 2 = 6 \times 1 = 6m(L(p,q))$ for $p \geqslant 3$. This completes the proof. □

Proof of Theorem 8. Let $\mathcal{K}_{p,q}$ be as in Remark 24. Then $\mathcal{K}_{3,1}$ is a 12-vertex crystallization of $L(3,2)$. Part (a) now follows by the constructions in Subsection 5.1.

Again, $\mathcal{K}_{q+1,q}$ is a $4(q+1)$-vertex crystallization of $L(q+1,q)$ for $1 \leqslant q \leqslant 3$. Part (b) now follows by the constructions in Subsection 5.2. □

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