# The proof of the removable pair conjecture for fractional dimension

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### Abstract

In 1971 Trotter (or Bogart and Trotter) conjectured that every finite poset on at least 3 points has a pair whose removal does not decrease the dimension by more than 1. In 1992 Brightwell and Scheinerman introduced fractional dimension of posets, and they made a similar conjecture for fractional dimension. This paper settles this latter conjecture.

## 1 Introduction

### 1.1 Dimension of posets

Let $P$ be a poset. A set of its linear extensions $\{L_1, \ldots, L_d\}$ forms a *realizer*, if $L_1 \cap \cdots \cap L_d = P$. The minimum cardinality of a realizer is called the *dimension* of the poset $P$, denoted by $\dim(P)$. This concept is also sometimes called the *order dimension* or the *Dushnik-Miller dimension of the partial order* as it was introduced in [3].

The dimension is "continuous" in the sense that the removal of a point can never decrease the dimension by more than 1. If the removal of a pair of points decreases the dimension by at most 1, such a pair is called a *removable pair*. The following conjecture has become known as the *Removable Pair Conjecture*.

**Conjecture 1** ([10], pp. 26). *Every poset on at least 3 point has a removable pair.*

The origins of the conjecture are not entirely clear. It appeared in print in a 1975 paper by Trotter [9], but according to Trotter [8], it was probably formulated at the 1971 summer conference on combinatorics held at Bowdoin College, and should be credited either to Trotter or to Bogart and Trotter.

Let $P$ denote a poset. In the following, $x\|y$ denotes that $x$ is incomparable to $y$ in $P$. We will also use the following standard notations.

$$D(x) = \{y \in P : y < x\} \qquad D[x] = D(x) \cup \{x\}$$
$$U(x) = \{y \in P : y > x\} \qquad U[x] = U(x) \cup \{x\}$$
$$I(x) = \{y \in P : y\|x\}$$
$$\min(P) = \{x \in P : D(x) = \emptyset\} \qquad \max(P) = \{x \in P : U(x) = \emptyset\}$$

**Definition 2.** An ordered pair of vertices $(x, y)$ is called a *critical pair*, if $x\|y$, $D(x) \subseteq D(y)$, and $U(y) \subseteq U(x)$. A linear extension $L$ *reverses* the critical pair $(x, y)$ if $y < x$ in $L$. A set of linear extensions *reverses* a critical pair, if the pair is reversed in at least one of the linear extensions.

The following proposition expresses that the critical pairs are the only significant incomparable pairs for constructing realizers.

**Proposition 3** ([7]). *A set of linear extensions is a realizer if and only if it reverses every critical pair.*

## 1.2  Fractional dimension

Determining the dimension of a poset can be regarded as a linear integer programming problem. Let $P$ be a poset and $\{L_1, \dots, L_\ell\}$ the set of its linear extensions and $\{(a_1, b_1), \dots, (a_c, b_c)\}$ the set of its critical pairs. Let $A = [a_{ij}]$ be a $c \times \ell$ binary matrix, where $a_{ij} = 1$ iff $(a_i, b_i)$ is reversed in $L_\ell$. The following integer program gives the dimension of $P$.

$$A\mathbf{x} \geqslant \mathbf{1}$$
$$\mathbf{x} \geqslant \mathbf{0}$$
$$\mathbf{x} \in \mathbb{Z}^\ell$$
$$\min \mathbf{1}^T\mathbf{x}$$

In 1992 Brightwell and Scheinerman [2] introduced the notion of fractional dimension of posets as the optimal solution of the linear relaxation of this integer program. They used the notation fdim($P$) for fractional dimension, but to keep the notation consistent with fractional graph theory, we will use $\dim^*(P)$. A feasible solution of the linear program will be called an *f-realizer*.

One may consider the $\ell$-dimensional vector space generated by the abstract basis $L_1, \ldots, L_\ell$; then an f-realizer is a linear combination $\sum \alpha_i L_i$ with $0 \leqslant \alpha_i \leqslant 1$, where for all critical pairs $(x, y)$, we have $\sum_{i:y<x \text{ in } L_i} \alpha_i \geqslant 1$. In fact, for a linear combination $\sum \alpha_i L_i$, we will say that it reverses the critical pair $(x, y)$ $\alpha$ times, if $\sum_{i:y<x \text{ in } L_i} \alpha_i = \alpha$. If a critical pair is reversed at least once (1 times), we will simply say it is reversed. This way, a linear combination is an f-realizer if and only if it reverses every critical pair. The *weight* of an f-realizer is $\sum_{i=1}^{\ell} \alpha_i$, and the fractional dimension of $P$ is the minimum weight of an f-realizer.

Clearly, for all posets $P$, $\dim^*(P) \leqslant \dim(P)$, but as shown by Brightwell and Scheinerman [2] (with all the other results in this paragraph), the ordinary dimension and the fractional dimension can be arbitrarily far apart. Nevertheless, there exist posets with arbitrarily large fractional dimension. The continuous property translates exactly for fractional dimension: for any element $x \in P$, $\dim^*(P - x) \geqslant \dim^*(P) - 1$.

In their paper Brightwell and Scheinerman conjectured that the fractional version of the Removable Pair Conjecture holds: there is always a pair of points that decreases the fractional dimension by at most one. In 1994, Felsner and Trotter [4] suggested a weakening of the question: is there an absolute constant $\varepsilon > 0$ so that any poset with 3 or more points always contains a pair whose removal decreases the fractional dimension by at most $2 - \varepsilon$? In this paper we prove the Brightwell–Scheinerman conjecture, which is of course equivalent to the Felsner–Trotter conjecture with $\varepsilon = 1$.

We would like to note here that the original definition of fractional dimension by Brightwell and Scheinerman is different from what we introduced above. For completeness, let us give their original, equivalent definition here. A *k-fold realizer* is a multiset of linear extensions such that for any incomparable pair $x, y$ there exists at least $k$ linear extensions (with multiplicity) in the multiset in which $x < y$, and there exists at least $k$ other linear extensions in which $x > y$. Let $t(k)$ be the minimum cardinality of a $k$-fold realizer. The fractional dimension of the poset is $\lim_{k \to \infty} t(k)/k = \inf t(k)/k$.

## 1.3   Interval orders

Let $P$ be a poset, and suppose that there is a map $f$ from $P$ to the set of closed intervals of the real line, so that $x < y$ in $P$ if and only if the right endpoint of $f(x)$ is less than (in the real number system) the left endpoint of $f(y)$. We say that the multiset of intervals $\{f(x) : x \in P\}$ is an *interval representation* of $P$. If a poset has an interval representation, it is an *interval order*.

The poset $\mathbf{2 + 2}$ denotes the poset with ground set $\{a_1, b_1, a_2, b_2\}$, where the only relations are $a_1 < b_1$ and $a_2 < b_2$. If a poset contains $\mathbf{2 + 2}$ it can not be an interval order. In fact this property characterizes interval orders (see Fishburn [5]).

**Theorem 4** ([5])**.** *A poset is an interval order if and only if it does not contain $\mathbf{2 + 2}$.*

An appealing property of interval orders is that they admit a positive answer to the Removable Pair Conjecture.

**Theorem 5** ([11])**.** *Any interval order on at least 3 points contains a removable pair.*

# 2 The existence of a removable pair

The following lemma is the fractional analogue of Theorem 5.

**Lemma 6.** *Let $P$ be an interval order on at least $3$ points. Then there exist points $a, b \in P$ such that $\dim^*(P - a - b) \geqslant \dim^*(P) - 1$.*

*Proof.* If $P$ is an antichain, then $\dim^*(P) = \dim(P) = 2$, so the statement follows. Otherwise let $a < b$ two elements of $P$ such that $a \in \min(P)$, $b \in \max(P)$, and $I(a) \subseteq \min(P)$, $I(b) \subseteq \max(P)$. Such elements always exist: consider a representation, and choose $a$ to be an interval whose right endpoint is the leftmost, and $b$ to be an interval whose left endpoint is the rightmost. The fact that $P$ is not an antichain ensures $a < b$.

Let $I = I(a) \cap I(b)$, and $R = P \setminus (\{a, b\} \cup I(a) \cup I(b))$. Note that $I$ consists of isolated elements. Let $Q = P - \{a, b\}$, and let $\sum \alpha_i L_i$ be an f-realizer of $Q$ of weight $\dim^*(Q)$. For each $i$, we will modify $L_i$ to get a linear extension $L'_i$ of $P$ as follows. Let $L_i$ be such that in $L_i$ we have $I < a < L < b$, where $I$ is ordered the same way for each $i$, and $L$ is an ordered set of elements of $P \setminus (\{a, b\} \cup I)$, such that the ordering preserves that of $L_i$. We construct one additional linear extension $L$ so that $I(a) \setminus I < a < R < b < I(b) \setminus I < I$, and the ordering of $I$ is the reverse of that of the $L_i$'s.

We claim that $\sum \alpha_i L'_i + L$ is an f-realizer of $P$. Critical pairs not involving elements of $I \cup \{a, b\}$ are obviously reversed, and the rest are reversed because $\sum \alpha_i \geqslant 1$. This shows $\dim^*(P) \leqslant \dim^*(Q) + 1$. $\qquad\square$

Note that the lemma above can be proven using more of existing machinery. Proceeding by induction on $|P|$, it is sufficient to prove that the statement holds in case $P$ is not an antichain and $P$ is indecomposable with respect to lexicographic sum. Using this, we may assume that in the above argument $I = \emptyset$, and then $L$ may be defined by $I(a) < a < R < b < I(b)$.

Also note that Trotter [11] gave a proof of the Removable Pair Conjecture for interval orders, and that proof can be translated to work with fractional dimension.

**Theorem 7.** *Let $P$ be a poset with at least $3$ elements. There exists a pair whose removal decreases the fractional dimension by at most $1$.*

*Proof.* By Lemma 6 we may assume that $P$ is not an interval order, so by Theorem 4 $P$ has elements $a_1$, $b_1$, $a_2$, and $b_2$, such that $a_1 < b_1$, $a_2 < b_2$, and no other comparabilities between any two of these.

Let $P_1 = P - \{a_1, b_2\}$, $P_2 = P - \{a_2, b_1\}$. Let $\sum \alpha_i L_i$, and $\sum \beta_i M_i$ f-realizers of $P_1$ and $P_2$ of weights $\dim^*(P_1)$ and $\dim^*(P_2)$, respectively. We extend the linear extensions $L_i$ and $M_i$ for each $i$ by inserting the missing elements $a_1, b_2$ or $a_2, b_1$ at arbitrary valid positions to get $L'_i$ and $M'_i$, respectively.

Define $D^-(b_1) = D(b_1) \setminus D[a_1]$, and $D^-(b_2) = D(b_2) \setminus D[a_2]$, and similarly, $U^-(a_1) = U(a_1) \setminus U[b_1]$, and $U^-(a_2) = U(a_2) \setminus U[b_2]$. Let

$$L_1 = D(a_1) < a_1 < D^-(b_1) < b_1 < R_1 < a_2 < U^-(a_2) < b_2 < U(b_2)$$
$$L_2 = D(a_2) < a_2 < D^-(b_2) < b_2 < R_2 < a_1 < U^-(a_1) < b_1 < U(b_1)$$

linear extensions, where $R_1$ and $R_2$ represent the rest of the elements as appropriate. These are exactly the linear extensions that appear in [1] and [6], where their existence is proven. We claim that

$$\frac{1}{2}\sum \alpha_i L_i' + \frac{1}{2}\sum \beta_i M_i' + \frac{1}{2}L_1 + \frac{1}{2}L_2 \tag{1}$$

is an f-realizer of $P$.

Indeed, every critical pair $(x,y)$ is reversed in at least two terms of (1). If neither of $x, y$ is in $\{a_1, a_2, b_1, b_2\}$ then it is reversed in the first and the second term. The critical pair $(a_1, b_2)$ is reversed in the second and fourth term, and the pair $(a_2, b_1)$ is reversed in the first and third term.

It remains to be seen that $(x,y)$ gets reversed if exactly one of $x$ and $y$ is in the set $\{a_1, a_2, b_1, b_2\}$. Up to symmetry, there are four such critical pairs $(x, a_1)$, $(a_1, y)$, $(x, b_2)$, $(b_2, y)$. All of them get reversed in the second term, and they get reversed in $L_1, L_2, L_2, L_1$, respectively.

We have shown that

$$\frac{1}{2}\dim^*(P_1) + \frac{1}{2}\dim^*(P_2) + 1 \geqslant \dim^*(P),$$

which implies that at least one of $P_1$ or $P_2$ has fractional dimension at least $\dim^*(P) - 1$. □

## Acknowledgment

## References

[1] Kenneth P. Bogart, *Maximal dimensional partially ordered sets I. Hiraguchi's theorem*, Discrete Math. **5** (1973), 21–31.

[2] Graham R. Brightwell and Edward R. Scheinerman, *Fractional dimension of partial orders*, Order **9** (1992), no. 2, 139–158.

[3] Ben Dushnik and Edwin W. Miller, *Partially ordered sets*, Amer. J. Math. **63** (1941), 600–610.

[4] Stefan Felsner and William T. Trotter, *On the fractional dimension of partially ordered sets*, Discrete Math. **136** (1994), no. 1-3, 101–117.

[5] Peter C. Fishburn, *Intransitive indifference with unequal indifference intervals*, J. Math. Psych. **7** (1970), 144–149.

[6] Toshio Hiraguchi, *On the dimension of orders*, Sci. Rep. Kanazawa Univ. **4** (1955), no. 1, 1–20.

[7] Issie Rabinovitch and Ivan Rival, *The rank of a distibutive lattice*, Discrete Math. **25** (1979), 275–279.

[8] William T. Trotter, personal communication.

[9] ——, *Inequalities in dimension theory for posets*, Proc. Amer. Math. Soc. **47** (1975), 311–316.

[10] ——, *Combinatorics and partially ordered sets*, The Johns Hopkins University Press, 1992.

[11] ——, *New perspectives on interval orders and interval graphs*, Surveys in combinatorics, 1997 (London) (R. A. Bailey, ed.), London Math. Soc. Lecture Note Ser., vol. 241, Cambridge Univ. Press, Cambridge, 1997, pp. 237–286.