A group action on derangements*

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Submitted: Sep 18, 2013; Accepted: Mar 14, 2014; Published: Mar 24, 2014
Mathematics Subject Classifications: 05A15, 05A05, 05E18, 26C99

Abstract

In this paper we define a cyclic analogue of the MFS-action on derangements, and give a combinatorial interpretation of the expansion of the n-th derangement polynomial on the basis $\{q^k(1+q)^{n-1-2k}\}, k = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$.

Keywords: derangement polynomials; group action

1 Introduction

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ denote the set of all permutations of $[n]$. For $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$ and $x \in [n]$, we write π as the concatenation $\pi = w_1w_2xw_3w_4$, where w_2 is the maximal contiguous subword immediately to the left of x whose letters are all smaller than x, and w_3 is the maximal contiguous subword immediately to the right of x whose letters are all smaller than x. Following Foata and Strehl [4, 5], this concatenation is called the *x-factorization* of π. For example, let $\pi = 714358296$ and $x = 5$. Then $w_1 = 7$, $w_2 = 143$, $w_3 = \emptyset$ and $w_4 = 8296$.

Foata and Strehl [4, 5] defined an involution acting on $\mathfrak{S}_n$ by $\varphi_x(\pi) = w_1w_3xw_2w_4$ for $x \in [n]$ and $\varphi_S(\pi) = \prod_{x\in S} \varphi_x(\pi)$ for $S \subseteq [n]$. The group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the functions φ_S for $S \subseteq [n]$.

Definition 1. Let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathfrak{S}_n$ and denote $\pi_0 = \pi_{n+1} = n + 1$. The entry π_k is called a *valley* if $\pi_{k-1} > \pi_k < \pi_{k+1}$; a *peak* if $\pi_{k-1} < \pi_k > \pi_{k+1}$; a *double ascent* if $\pi_{k-1} < \pi_k < \pi_{k+1}$; a *double descent* if $\pi_{k-1} > \pi_k > \pi_{k+1}$.

*Supported by the National Natural Science Foundation of China (Nos. 11071030, 11371078) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20110041110039).

Let $Val(\pi)$, $Peak(\pi)$, $Dasc(\pi)$, $Ddes(\pi)$ denote the set of all valley, peaks, double ascents and double descents of π, respectively. The corresponding cardinalities are $val(\pi)$, $peak(\pi)$, $dasc(\pi)$ and $ddes(\pi)$, respectively. Shapiro *et al.* [6] modified the Foata-Strehl action in the following way. For $x \in [n]$, let

$$\varphi'_x(\pi) = \begin{cases} \varphi_x(\pi) & \text{if } x \text{ is a double ascent or a double descent,} \\ \pi & \text{if } x \text{ is a valley or a peak.} \end{cases} \tag{1}$$

For any subset $S \subseteq [n]$, define $\varphi'_S(\pi) = \prod_{x \in S} \varphi'_x(\pi)$. From the definition, if x is a double ascent (double descent, resp.) of π, then x is a double descent (double ascent, resp.) of $\varphi'_x(\pi)$. The group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the functions $\varphi'_S, S \in [n]$ and call this action the *MFS-action*.

By the theory of symmetric functions, Brenti [2] showed that derangement polynomials are symmetric and unimodal polynomials. Using the method of continued fractions, Shin and Zeng [7] gave a combinatorial interpretation for coefficients in the expansion of the n-th derangement polynomial on the basis $\{q^k(1+q)^{n-1-2k}\}, k = 0, 1, \ldots, \lfloor (n-1)/2 \rfloor$. In this note, we define a cyclic analogous of the MFS-action on derangements and give a new proof for the result of Shin and Zeng.

2 Main results

Let $\pi \in \mathfrak{S}_n$. We say that π is a *derangement* of $[n]$ if $\pi_i \neq i$ for all $i \in [n]$. Denote by D_n the set of all derangements of $[n]$. An element $i \in [n]$ is an *excedance* of π if $\pi_i > i$. Denote by $Exc(\pi)$ the set of all excedances in π and let $exc(\pi) = |Exc(\pi)|$. The *n-derangement polynomial* $D_n(q)$ is the generating function of statistic excedance over the set D_n, i.e.,

$$D_n(q) = \sum_{\pi \in D_n} q^{exc(\pi)} = \sum_{j=1}^{n-1} d(n,j) q^j, \tag{2}$$

where $d(n,j) = |\{\pi \in D_n : exc(\pi) = j\}|$.

Recall that a permutation $\pi \in \mathfrak{S}_n$ may be regarded as a disjoint union of its distinct cycles $C_1, C_2, \ldots, C_k$, written $\pi = C_1 C_2 \cdots C_k$. Let $c(\pi)$ denote the number of cycles of π. For a derangement π, each cycle contains at least two elements. The *standard cycle representation* of π is defined by requiring that (i) each cycle is written with its largest element first, and (ii) the cycles are written in increasing order of their largest elements [8]. For example, the standard cycle representation of $\pi = 456321 \in D_6$ is (52)(6143). Throughout the paper all permutations are written in standard cycle representation.

Definition 2 ([7])**.** Let $\pi \in \mathfrak{S}_n$. The entry $x = \pi_i (i \in [n])$ is called a *cyclic valley* if $i = \pi^{-1}(x) > x < \pi(x)$; a *cyclic peak* if $i = \pi^{-1}(x) < x > \pi(x)$; a *cyclic double ascent* if $i = \pi^{-1}(x) < x < \pi(x)$; a *cyclic double descent* if $i = \pi^{-1}(x) > x > \pi(x)$; a *fixed point* if $\pi(x) = x$.

Let $Cval(\pi)$, $Cpeak(\pi)$, $Cdasc(\pi)$, $Cddes(\pi)$ and $Fix(\pi)$ denote the set of all cyclic valley, cyclic peaks, cyclic double ascents, cyclic double descents and fixed points of π, respectively. The corresponding cardinalities are $cval(\pi)$, $cpeak(\pi)$, $cdasc(\pi)$, $cddes(\pi)$ and $fix(\pi)$, respectively. It is easy to see that the union of sets $Cval(\pi)$, $Cpeak(\pi)$, $Cdasc(\pi)$, $Cddes(\pi)$ and $Fix(\pi)$ is $[n]$ for any $\pi \in \mathfrak{S}_n$. For a derangement π, the set $Fix(\pi)$ is empty. The following proposition is immediate by Definition 2.

Proposition 3. *Let $\pi = C_1C_2\cdots C_k$ be a permutation of $[n]$. Then*

$$Exc(\pi) = Cval(\pi) \cup Cdasc(\pi)$$

and

$$exc(\pi) = cval(\pi) + cdasc(\pi).$$

Let $\pi = C_1C_2\cdots C_k$. Following Stanley [8], let $o(\pi)$ be the permutation obtained from π by erasing the parentheses of cycles. For example, if $\pi = (71435)(826)$, then $o(\pi) = 71435862$. The map $o : \mathfrak{S}_n \to \mathfrak{S}_n$ defined above is a bijection. The following result is direct.

Proposition 4. *Let $\pi = C_1C_2\cdots C_k \in D_n$. Suppose that $o(\pi)(0) = 0$ and $o(\pi)(n+1) = n+1$. Then*

$$Cpeak(\pi) = Peak(o(\pi)), \qquad Cval(\pi) = Val(o(\pi)),$$

$$Cdasc(\pi) = Dasc(o(\pi)) \qquad \textit{and} \qquad Cddes(\pi) = Ddes(o(\pi)),$$

where the sets $Peak(o(\pi)), Val(o(\pi)), Dasc(o(\pi))$ and $Ddes(o(\pi))$ are defined similar to Definition 1 with the only difference $o(\pi)(0) = 0$.

We define the cyclic analogous of the MFS-action on derangements in the following way. Let $\pi = C_1C_2\cdots C_k$. Suppose that $o(\pi)(0) = 0$ and $o(\pi)(n+1) = n+1$. For $x \in [n]$, define the map $\theta_x : D_n \to D_n$ by

$$\theta_x(\pi) = o^{-1}(\varphi'_x(o(\pi))).$$

The map is well-defined. To see this, let $\pi = C_1C_2\cdots C_k \in D_n$. If x is a cyclic valley of π, then x is a valley of $o(\pi)$, $\varphi'_x(o(\pi)) = o(\pi)$ and $\theta_x(\pi) = \pi$. If x is a cyclic peak of π, then x is a peak of $o(\pi)$, $\varphi'_x(o(\pi)) = o(\pi)$ and $\theta_x(\pi) = \pi$. If x is a cyclic double ascent of C_i in π, where $C_i = (w_0w_1xw_2)$ and w_1 denotes the maximal contiguous subword immediately to the left of x whose letters are all smaller than x. Then x is a double ascent of $o(\pi)$, $\varphi'_x(o(\pi)) = o(C_1C_2\cdots C_{i-1}\bar{C}_iC_{i+1}\cdots C_k)$ and $\theta_x(\pi) = C_1C_2\cdots C_{i-1}\bar{C}_iC_{i+1}\cdots C_k \in D_n$, where $\bar{C}_i = (w_0xw_1w_2)$. If x is a cyclic double descent of C_i in π, where $C_i = (w_0xw_1w_2)$ and w_1 denotes the maximal contiguous subword immediately to the right of x whose letters are all smaller than x. Then x is a double descent of $o(\pi)$, $\varphi'_x(o(\pi)) = o(C_1C_2\cdots C_{i-1}\bar{C}_iC_{i+1}\cdots C_k)$ and $\theta_x(\pi) = C_1C_2\cdots C_{i-1}\bar{C}_iC_{i+1}\cdots C_k \in D_n$, where $\bar{C}_i = (w_0w_1xw_2)$. Hence the map θ_x is well-defined for all $x \in [n]$.

Table 1 gives an example of the maps θ_x on $\pi = (623)(87514)$ for all $x \in [8]$, where $o(\pi) = 62387514$.

x	1	2	3	4
$\varphi'_x(o(\pi))$	62387514	62387514	63287514	62387514
$\theta_x(\pi)$	(623)(87514)	(623)(87514)	(632)(87514)	(623)(87514)
x	5	6	7	8
$\varphi'_x(o(\pi))$	62387145	62387514	62385147	62387514
$\theta_x(\pi)$	(623)(87145)	(623)(87514)	(623)(85147)	(623)(87514)

Table 1.

The function θ_x is an involution and $\theta_x\theta_y = \theta_y\theta_x$ for all $x, y \in [n]$. For any subset $S \subseteq [n]$, define the function $\theta_S(\pi) : D_n \to D_n$ by

$$\theta_S(\pi) = \prod_{x \in S} \theta_x(\pi).$$

The group $\mathbb{Z}_2^n$ acts on D_n via the functions $\theta_S, S \in [n]$ and call this action the *CMFS-action.*

For $\pi \in D_n$, let $Orb^c(\pi)$ denote the orbit including π under the CMFS-action. There is a unique derangement in $Orb^c(\pi)$, denoted by $\tilde{\pi}$, such that $\tilde{\pi}$ has no cyclic double ascents. The next is the main results of this note.

Theorem 5. *Let* $\pi \in D_n$. *Then*

$$\sum_{\sigma \in Orb^c(\pi)} q^{exc(\sigma)} = q^{exc(\tilde{\pi})}(1+q)^{n-2exc(\tilde{\pi})} = q^{cpeak(\pi)}(1+q)^{n-2cpeak(\pi)}.$$

Proof. If x is a cyclic double descent of some cycle C_i in π, then x is a cyclic double ascent of cycle C'_i in $\theta_x(\pi)$, where $\pi = C_1C_2\cdots C_k$ and $\theta_x(\pi) = C'_1C'_2\cdots C'_k$. We have $Cdasc(\theta_x(\pi)) = Cdasc(\pi) \cup \{x\}$ and $Cval(\theta_x(\pi)) = Cval(\pi)$. It follows that $Exc(\theta_x(\pi)) = Exc(\pi) \cup \{x\}$ and $exc(\theta_x(\pi)) = exc(\pi) + 1$ from Proposition 3. Then

$$\sum_{\sigma \in Orb^c(\pi)} q^{exc(\sigma)} = q^{exc(\tilde{\pi})}(1+q)^{cddes(\tilde{\pi})}.$$

For any $\pi = C_1C_2\cdots C_k \in D_n$, delete all double descents and double ascents of $o(\pi)$, then we get an alternating permutation

$$0 < x_1 > x_2 < x_3 > \cdots > x_{n-cddes(\pi)-cdasc(\pi)} < n+1,$$

where $o(\pi)(0) = 0$ and $o(\pi)(n+1) = n+1$. Thus

$$cpeak(\pi) = peak(o(\pi)) = val(o(\pi)) = cval(\pi).$$

Note that the union of sets $Cval(\tilde{\pi})$, $Cpeak(\tilde{\pi})$ and $Cddes(\tilde{\pi})$ is the set $[n]$. Hence $exc(\tilde{\pi}) = cpeak(\tilde{\pi}) = cpeak(\pi)$ and $cddes(\tilde{\pi}) = n - 2exc(\tilde{\pi}) = n - 2cpeak(\pi)$. □

The following corollary is an immediate consequence of Theorem 5.

Corollary 6 ([7]). *The derangement polynomials can be expanded as*

$$D_n(q) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_i q^i (1+q)^{n-2i},$$

where $b_i = 2^{-n+2i} |\{\pi \in D_n : cpeak(\pi) = i\}|$ *and* $b_0 = 0$.

Acknowledgements

The authors thank the anonymous referee for his/her careful reading and helpful comments.

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