Colorful subhypergraphs in Kneser hypergraphs

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Submitted: Jul 11, 2013; Accepted: Dec 25, 2013; Published: Jan 12, 2014
Mathematics Subject Classifications: 05C65

Abstract

Using a Z_q-generalization of a theorem of Ky Fan, we extend to Kneser hypergraphs a theorem of Simonyi and Tardos that ensures the existence of multicolored complete bipartite graphs in any proper coloring of a Kneser graph. It allows to derive a lower bound for the local chromatic number of Kneser hypergraphs (using a natural definition of what can be the local chromatic number of a uniform hypergraph).

Keywords: colorful complete p-partite hypergraph; combinatorial topology; Kneser hypergraphs; local chromatic number.

1 Introduction

1.1 Motivations and results

A *hypergraph* is a pair $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite set and $E(\mathcal{H})$ a family of subsets of $V(\mathcal{H})$. The set $V(\mathcal{H})$ is called the *vertex set* and the set $E(\mathcal{H})$ is called the *edge set*. A *graph* is a hypergraph each edge of which is of cardinality two. A *q-uniform hypergraph* is a hypergraph each edge of which is of cardinality q. The notions of graphs and 2-uniform hypergraphs therefore coincide. If a hypergraph has its vertex set partitioned into subsets $V_1, \ldots, V_q$ so that each edge intersects each V_i at exactly one vertex, then it is called a *q-uniform q-partite hypergraph*. The sets $V_1, \ldots, V_q$ are called the *parts* of the hypergraph. When $q = 2$, such a hypergraph is a graph and said to be *bipartite*. A q-uniform q-partite hypergraph is said to be *complete* if all possible edges exist.

A *coloring* of a hypergraph is a map $c : V(\mathcal{H}) \to [t]$ for some positive integer t. A coloring is said to be *proper* if there is no *monochromatic* edge, i.e. no edge e with $|c(e)| = 1$. The *chromatic number* of such a hypergraph, denoted $\chi(\mathcal{H})$, is the minimal value of t for which a proper coloring exists. Given $X \subseteq V(\mathcal{H})$, the hypergraph with vertex set X and with edge set $\{e \in E(\mathcal{H}) : e \subseteq X\}$ is the *subhypergraph of* $\mathcal{H}$ *induced by* X and is denoted $\mathcal{H}[X]$.

Given a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, we define the *Kneser graph* $\mathrm{KG}^2(\mathcal{H})$ by

$$\begin{array}{rcl} V(\mathrm{KG}^2(\mathcal{H})) & = & E(\mathcal{H}) \\ E(\mathrm{KG}^2(\mathcal{H})) & = & \{\{e, f\} : e, f \in E(\mathcal{H}),\, e \cap f = \emptyset\}. \end{array}$$

The "usual" Kneser graphs, which have been extensively studied – see [20, 21] among many references, some of them being given elsewhere in the present paper – are the special cases $\mathcal{H} = ([n], \binom{[n]}{k})$ for some positive integers n and k with $n \geqslant 2k$. We denote them $\mathrm{KG}^2(n, k)$. The main result for "usual" Kneser graphs is Lovász's theorem [11].

Theorem (Lovász theorem)**.** *Given n and k two positive integers with $n \geqslant 2k$, we have $\chi(\mathrm{KG}^2(n, k)) = n - 2k + 2$.*

The 2-colorability defect $\mathrm{cd}^2(\mathcal{H})$ of a hypergraph $\mathcal{H}$ has been introduced by Dol'nikov [3] in 1988 for a generalization of Lovász's theorem. It is defined as the minimum number of vertices that must be removed from $\mathcal{H}$ so that the hypergraph induced by the remaining vertices is of chromatic number at most 2:

$$\mathrm{cd}^2(\mathcal{H}) = \min\{|Y| : Y \subseteq V(\mathcal{H}), \chi(\mathcal{H}[V(\mathcal{H}) \setminus Y]) \leqslant 2\}.$$

Theorem (Dol'nikov theorem)**.** *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then $\chi(\mathrm{KG}^2(\mathcal{H})) \geqslant \mathrm{cd}^2(\mathcal{H})$.*

It is a generalization of Lovász theorem since $\mathrm{cd}^2([n], \binom{[n]}{k}) = n - 2k + 2$ and since the inequality $\chi(\mathrm{KG}^2(n, k)) \leqslant n - 2k + 2$ is the easy one.

The following theorem proposed by Simonyi and Tardos in 2007 [19] generalizes Dol'nikov's theorem. The special case for "usual" Kneser graphs is due to Ky Fan [7].

Theorem (Simonyi-Tardos theorem)**.** *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Let $r = \mathrm{cd}^2(\mathcal{H})$. Then any proper coloring of $\mathrm{KG}^2(\mathcal{H})$ with colors $1, \ldots, t$ (t arbitrary) must contain a completely multicolored complete bipartite graph $K_{\lceil r/2\rceil, \lfloor r/2\rfloor}$ such that the r different colors occur alternating on the two parts of the bipartite graph with respect to their natural order.*

In 1976, Erdős [4] initiated the study of *Kneser hypergraphs* $\mathrm{KG}^q(\mathcal{H})$ defined for a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ and an integer $q \geqslant 2$ by

$$\begin{array}{rcl} V(\mathrm{KG}^q(\mathcal{H})) & = & E(\mathcal{H}) \\ E(\mathrm{KG}^q(\mathcal{H})) & = & \{\{e_1, \ldots, e_q\} : e_1, \ldots, e_q \in E(\mathcal{H}),\, e_i \cap e_j = \emptyset \text{ for all } i, j \text{ with } i \neq j\}. \end{array}$$

A Kneser hypergraph is thus the generalization of Kneser graphs obtained when the 2-uniformity is replaced by the q-uniformity for an integer $q \geqslant 2$. There are also "usual" Kneser hypergraphs, which are obtained with the same hypergraph $\mathcal{H}$ as for "usual" Kneser graphs, i.e. $\mathcal{H} = ([n], \binom{[n]}{k})$. They are denoted $\mathrm{KG}^q(n,k)$. The main result for them is the following generalization of Lovász's theorem conjectured by Erdős and proved by Alon, Frankl, and Lovász [2].

Theorem (Alon-Frankl-Lovász theorem)**.** *Given n, k, and q three positive integers with $n \geqslant qk$, we have* $\chi(\mathrm{KG}^q(n,k)) = \left\lceil \frac{n-q(k-1)}{q-1} \right\rceil$.

There exists also a q-colorability defect $\mathrm{cd}^q(\mathcal{H})$, introduced by Kříž, defined as the minimum number of vertices that must be removed from $\mathcal{H}$ so that the hypergraph induced by the remaining vertices is of chromatic number at most q:

$$\mathrm{cd}^q(\mathcal{H}) = \min\{|Y| : Y \subseteq V(\mathcal{H}), \chi(\mathcal{H}[V(\mathcal{H}) \setminus Y]) \leqslant q\}.$$

The following theorem, due to Kříž [9, 10], generalizes Dol'nikov's theorem. It also generalizes the Alon-Frankl-Lovász theorem since $\mathrm{cd}^q([n], \binom{[n]}{k}) = n - q(k-1)$ and since again the inequality $\chi(\mathrm{KG}^q(n,k)) \leqslant \left\lceil \frac{n-q(k-1)}{q-1} \right\rceil$ is the easy one.

Theorem (Kříž theorem)**.** *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then*

$$\chi(\mathrm{KG}^q(\mathcal{H})) \geqslant \left\lceil \frac{\mathrm{cd}^q(\mathcal{H})}{q-1} \right\rceil$$

for any integer $q \geqslant 2$.

Our main result is the following extension of Simonyi-Tardos's theorem to Kneser hypergraphs.

Theorem 1. *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Let p be a prime number. Then any proper coloring c of $\mathrm{KG}^p(\mathcal{H})$ with colors $1, \ldots, t$ (t arbitrary) must contain a complete p-uniform p-partite hypergraph with parts $U_1, \ldots, U_p$ satisfying the following properties.*

- *It has $\mathrm{cd}^p(\mathcal{H})$ vertices.*
- *The values of $|U_j|$ for $j = 1, \ldots, p$ differ by at most one.*
- *For any j, the vertices of U_j get distinct colors.*

We get that each U_j is of cardinality $\lfloor \mathrm{cd}^p(\mathcal{H})/p \rfloor$ or $\lceil \mathrm{cd}^p(\mathcal{H})/p \rceil$.

Note that Theorem 1 implies directly Kříž's theorem when q is a prime number p: each color may appear at most $p-1$ times within the vertices and there are $\mathrm{cd}^p(\mathcal{H})$ vertices. There is a standard derivation of Kříž's theorem for any q from the prime case, see [22, 23]. Theorem 1 is a generalization of Simonyi-Tardos's theorem except for a slight loss: when $p = 2$, we do not recover the alternation of the colors between the two parts.

Whether Theorem 1 is true for non-prime p is an open question.

2 Local chromatic number and Kneser hypergraphs

In a graph $G = (V, E)$, the *closed neighborhood* of a vertex u, denoted $N[u]$, is the set $\{u\} \cup \{v : uv \in E\}$. The *local chromatic number* of a graph $G = (V, E)$, denoted $\chi_\ell(G)$, is the maximum number of colors appearing in the closed neighborhood of a vertex minimized over all proper colorings:

$$\chi_\ell(G) = \min_c \max_{v \in V} |c(N[v])|,$$

where the minimum is taken over all proper colorings c of G. This number has been defined in 1986 by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [5]. For Kneser graphs, we have the following theorem, which is a consequence of the Simonyi-Tardos theorem: any vertex of the part with $\lfloor r/2 \rfloor$ vertices in the completely multicolored complete bipartite subgraph has at least $\lceil r/2 \rceil + 1$ colors in its closed neighborhhod (where $r = \mathrm{cd}^2(\mathcal{H})$).

Theorem (Simonyi-Tardos theorem for local chromatic number)**.** *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. If $\mathrm{cd}^2(\mathcal{H}) \geqslant 2$, then*

$$\chi_\ell(\mathrm{KG}^2(\mathcal{H})) \geqslant \left\lceil \frac{\mathrm{cd}^2(\mathcal{H})}{2} \right\rceil + 1.$$

Note that we can also see this theorem as a direct consequence of Theorem 1 in [18] (with the help of Theorem 1 in [13]).

We use the following natural definition for the local chromatic number $\chi_\ell(\mathcal{H})$ of a uniform hypergraph $\mathcal{H} = (V, E)$. For a subset X of V, we denote by $\mathcal{N}(X)$ the set of vertices v such that v is the sole vertex outside X for some edge in E:

$$\mathcal{N}(X) = \{v : \exists e \in E \text{ s.t. } e \setminus X = \{v\}\}.$$

We define furthermore $\mathcal{N}[X] := X \cup \mathcal{N}(X)$. Note that if the hypergraph is a graph, $\mathcal{N}[\{v\}] = N[v]$ for any vertex v. The definition of the local chromatic number of a hypergraph is then:

$$\chi_\ell(\mathcal{H}) = \min_c \max_{e \in E,\, v \in e} |c(\mathcal{N}[e \setminus \{v\}])|,$$

where the minimum is taken over all proper colorings c of $\mathcal{H}$. When the hypergraph $\mathcal{H}$ is a graph, we get the usual notion of local chromatic number for graphs.

The following theorem is a consequence of Theorem 1 and generalizes the Simonyi-Tardos theorem for local chromatic number to Kneser hypergraphs.

Theorem 2. *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then*

$$\chi_\ell(\mathrm{KG}^p(\mathcal{H})) \geqslant \min\left(\left\lceil \frac{\mathrm{cd}^p(\mathcal{H})}{p} \right\rceil + 1, \left\lceil \frac{\mathrm{cd}^p(\mathcal{H})}{p-1} \right\rceil \right)$$

for any prime number p.

Proof. Denote $\mathrm{cd}^p(\mathcal{H})$ by r. Let c be any proper coloring of $\mathrm{KG}^p(\mathcal{H})$. Consider the complete p-uniform p-partite hypergraph $\mathcal{G}$ in $\mathrm{KG}^p(\mathcal{H})$ whose existence is ensured by Theorem 1. Choose U_j of cardinality $\lceil r/p \rceil$.

If $\lceil r/(p-1) \rceil > \lceil r/p \rceil$, then there is a vertex v of $\mathcal{G}$ not in U_j whose color is distinct of all colors used in U_j. Choose any edge e of $\mathcal{G}$ containing v and let u be the unique vertex of $e \cap U_j$. We have then $|c(\mathcal{N}[e \setminus \{u\}])| \geqslant |U_j| + 1 = \lceil r/p \rceil + 1$.

Otherwise, $\lceil r/(p-1) \rceil = \lceil r/p \rceil$, and for any edge e, we have $|c(\mathcal{N}[e \setminus \{u\}])| \geqslant \lceil r/p \rceil = \lceil r/(p-1) \rceil$, with u being again the unique vertex of $e \cap U_j$. □

As for Theorem 1, we do not know whether this theorem remains true for non-prime p.

3 Combinatorial topology and proof of the main result

3.1 Tools of combinatorial topology

3.1.1 Basic definitions

We use the cyclic and muliplicative group $Z_q = \{\omega^j : j = 1, \ldots, q\}$ of the qth roots of unity. We emphasize that 0 is not considered as an element of Z_q. For a vector $X = (x_1, \ldots, x_n) \in (Z_q \cup \{0\})^n$, we define X^j to be the set $\{i \in [n] : x_i = \omega^j\}$ and $|X|$ to be the quantity $|\{i \in [n] : x_i \neq 0\}|$.

We assume basic knowledges in algebraic topology, see the book by Munkres for instance for an introduction to this topic [17]. A simplicial complex is said to be *pure* if all maximal simplices for inclusion have the same dimension. For K a simplicial complex, we denote by $\mathcal{C}(\mathsf{K})$ its chain complex. We always assume that the coefficients are taken in $\mathbb{Z}$.

3.1.2 Special simplicial complexes

For a simplicial complex K, its first barycentric subdivision is denoted by $\mathrm{sd}(\mathsf{K})$. It is the simplicial complex whose vertices are the nonempty simplices of K and whose simplices are the collections of simplices of K that are pairwise comparable for $\subseteq$ (these collections are usually called *chains* in the poset terminology, with a different meaning as the one used above in "chain complexes").

As a simplicial complex, Z_q is seen as being 0-dimensional and with q vertices. Z_q^{*d} is the join of d copies of Z_q. It is a pure simplicial complex of dimension $d-1$. A vertex v taken in the μth copy of Z_q in Z_q^{*d} is also written (ϵ, μ) where $\epsilon \in Z_q$ and $\mu \in [d]$. Sometimes, ϵ is called the *sign* of the vertex, and μ its *absolute value*. This latter quantity is denoted $|v|$.

The simplicial complex $\mathrm{sd}(Z_q^{*d})$ plays a special role. We have

$$V\left(\mathrm{sd}(Z_q^{*d})\right) \simeq (Z_q \cup \{0\})^d \setminus \{(0, \ldots, 0)\} :$$

a simplex $\sigma \in Z_q^{*d}$ corresponds to the vector $X = (x_1, \dots, x_d) \in (Z_q \cup \{0\})^d$ with $x_\mu = \epsilon$ for all $(\epsilon, \mu) \in \sigma$ and $x_\mu = 0$ otherwise.

We denote by σ_{q-2}^{q-1} the simplicial complex obtained from a $(q-1)$-dimensional simplex and its faces by deleting the maximal face. It is hence a $(q-2)$-dimensional pseudomanifold homeomorphic to the $(q-2)$-sphere. We also identify its vertices with Z_q. A vertex of the simplicial complex $\left(\sigma_{q-2}^{q-1}\right)^{*d}$ is again denoted by (ϵ, μ) where $\epsilon \in Z_q$ and $\mu \in [d]$. For $\epsilon \in Z_q$ and a simplex τ of $\left(\sigma_{p-2}^{p-1}\right)^{*d}$, we denote by τ^ϵ the set of all vertices of τ having ϵ as sign, i.e. $\tau^\epsilon := \{(\omega, \mu) \in \tau : \omega = \epsilon\}$. Note that if q is a prime number, Z_q acts freely on σ_{q-2}^{q-1}.

3.1.3 Barycentric subdivision operator

Let K be a simplicial complex. There is a natural chain map $\mathrm{sd}_\# : \mathcal{C}(\mathsf{K}) \to \mathcal{C}(\mathrm{sd}(\mathsf{K}))$ which, when evaluated on a d-simplex $\sigma \in \mathsf{K}$, returns the sum of all d-simplices in $\mathrm{sd}(\mathsf{K})$ contained in σ, with the induced orientation. "Contained" is understood according to the geometric interpretation of the barycentric subdivision. If K is a free Z_q-simplicial complex, $\mathrm{sd}_\#$ is a Z_q-equivariant map.

3.1.4 The Z_q-Fan lemma

The following lemma plays a central role in the proof of Theorem 1. It is proved (implicitely and in a more general version) in [8, 14].

Lemma 3 (Z_q-Fan lemma)**.** *Let $q \geqslant 2$ be a positive integer. Let $\lambda_\# : \mathcal{C}\left(\mathrm{sd}(Z_q^{*n})\right) \to \mathcal{C}\left(Z_q^{*m}\right)$ be a Z_q-equivariant chain map. Then there is an $(n-1)$-dimensional simplex ρ in the support of $\lambda_\#(\rho')$, for some $\rho' \in \mathrm{sd}(Z_q^{*n})$, of the form $\{(\epsilon_1, \mu_1), (\epsilon_2, \mu_2), \dots, (\epsilon_n, \mu_n)\}$, with $\mu_i < \mu_{i+1}$ and $\epsilon_i \neq \epsilon_{i+1}$ for $i = 1, \dots, n$.*

This ρ' is an *alternating simplex.*

Proof. The proof is exactly the proof of Theorem 5.4 (p.415) of [8]. The complex X in the statement of this Theorem 5.4 is our complex $\mathrm{sd}(Z_q^{*n})$, the dimension r is $n-1$, and the generalized r-sphere (x_i) is any generalized $(n-1)$-sphere of $\mathrm{sd}(Z_q^{*n})$ with x_0 reduced to a single point. The chain map $h_\bullet^\ell$ is induced by our chain map $\lambda_\#$, instead of being induced by the chain map $\ell_\#$ of [8] (itself induced by the labeling ℓ). It does not change the proof since $h_\bullet^\ell$ only uses the fact that $\ell_\#$ is a Z_q-equivariant chain map. In the statement of Theorem 5.4 of [8], α_i is always a lower bound on the number of "alternating patterns" (i.e. simplices ρ' as in the statement of the lemma) in $\ell_\#(x_i)$, even for odd i since the map f_i in Theorem 5.4 of [8] is zero on non-alternating elements. Since $\alpha_0 = 1$, we get that $\alpha_i \neq 0$ for all $0 \leqslant i \leqslant n-1$. □

In particular, for $q = 2$, it gives the Ky Fan theorem [6] used for instance in [7, 15, 18] to derive properties of Kneser graphs.

3.2 Proof of the main result

Proof of Theorem 1. We first sketch some steps in the proof. We assume given a proper coloring c of $\mathrm{KG}^p(\mathcal{H})$. With the help of the coloring c, we build a Z_p-equivariant chain map $\psi_\# : \mathcal{C}(\mathrm{sd}(Z_p^{*n})) \to \mathcal{C}(Z_p^{*m})$, where $m = n - \mathrm{cd}^p(\mathcal{H}) + t(p-1)$. We apply Lemma 3 to get the existence of some alternating simplex ρ' in $\mathrm{sd}(Z_p^{*n})$. Using properties of $\psi_\#$ (especially the fact that it is a composition of maps in which simplicial maps are involved), we show that this alternating simplex provides a complete p-uniform p-partite hypergraph in $\mathcal{H}$ with the required properties.

Let $r = \mathrm{cd}^p(\mathcal{H})$. Following the ideas of [12, 22], we define

$$f : (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\} \to Z_p \times [m]$$

with $m = n - r + t(p-1)$. We choose a total ordering $\preceq$ on the subsets of $[n]$. This ordering is only used to get a clean definition of f.

If $X \in (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}$ ***is such that*** $|X| \leqslant n - r$**,** then $f(X)$ is defined to be $(\epsilon, |X|)$ with ϵ being the first nonzero component in X.

If $X \in (Z_p \cup \{0\})^n \setminus \{(0,\ldots,0)\}$ ***is such that*** $|X| \geqslant n - r + 1$**,** by definition of the colorability defect, at least one of the X^j's with $j \in [p]$ contains an edge of $\mathcal{H}$. Choose $j \in [p]$ such that there is $S \subseteq X^j$ with $S \in E(\mathcal{H})$. In case several S are possible, choose the maximal one according to the total ordering $\preceq$. Its defines $F(X) := S$ and $f(X) := (\omega^j, n - r + c(F(X)))$.

Note that f induces a Z_p-equivariant simplicial map $f : \mathrm{sd}(Z_p^{*n}) \to \mathsf{L} * \mathsf{M}$, where $\mathsf{L} := Z_p^{*(n-r)}$ and $\mathsf{M} := \left(\sigma_{p-2}^{p-1}\right)^{*t}$.

Let W_a be the set of simplices $\tau \in \mathsf{M}$ such that $|\tau^\epsilon| = 0$ or $|\tau^\epsilon| = a$ for all $\epsilon \in Z_p$. Let $W = \bigcup_{a=1}^m W_a$. Choose an arbitrary equivariant map $s : W \to Z_p$. Such a map can be easily built by choosing one representative in each orbit (Z_p acts freely on each W_a). We build also an equivariant map $s_0 : \sigma_{p-2}^{p-1} \to Z_p$, again by choosing one representative in each orbit of the action of Z_p. We define now a simplicial map $g : \mathrm{sd}(\mathsf{L} * \mathsf{M})) \to Z_p^{*m}$ as follows.

Take a vertex in $\mathrm{sd}(\mathsf{L} * \mathsf{M})$. It is of the form $\sigma \cup \tau \neq \emptyset$ where $\sigma \in \mathsf{L}$ and $\tau \in \mathsf{M}$.

If $\tau \neq \emptyset$**.** Let $\alpha := \min_{\epsilon \in Z_p} |\tau^\epsilon|$.

- If $\alpha = 0$, define $\bar{\tau} := \{\epsilon \in Z_p : \tau^\epsilon = \emptyset\}$ and $g(\sigma \cup \tau) = (s_0(\bar{\tau}), n - r + |\tau|)$ (we have indeed $\bar{\tau} \in \sigma_{p-2}^{p-1}$).
- If $\alpha > 0$, define $\bar{\tau} := \bigcup_{\epsilon : |\tau^\epsilon| = \alpha} \tau^\epsilon$ and $g(\sigma \cup \tau) := (s(\bar{\tau}), n - r + |\tau|)$.

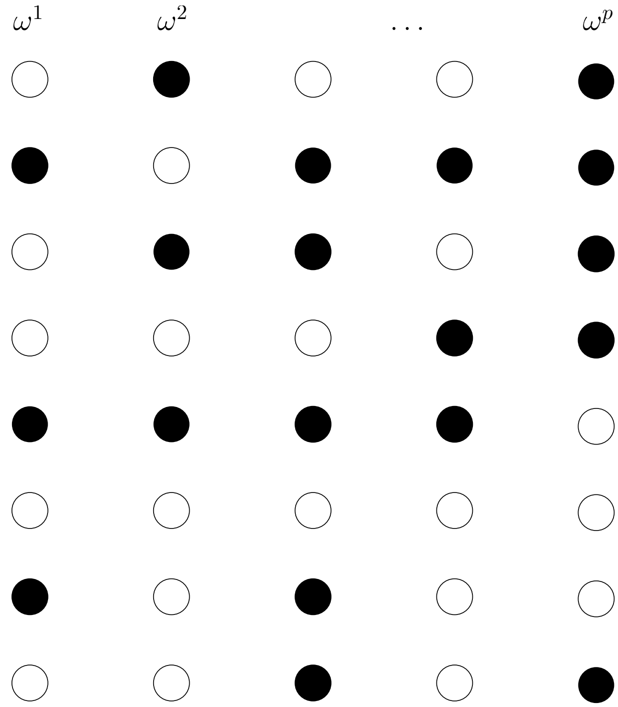

Figure 1: An example of a simplex $\tau \in \mathsf{M}$.

The definition of $\bar{\tau}$ is illustrated on Figures 1 and 2.

If $\tau = \emptyset$***.*** Choose (ϵ, μ) in σ with maximal μ. Define $g(\sigma \cup \tau) := (\epsilon, \mu)$. Note that L is such that there is only one ϵ for which the maximum is attained.

We check now that g is a simplicial map. Assume for a contradiction that there are $\sigma \subseteq \sigma'$, $\tau \subseteq \tau'$ such that $g(\sigma \cup \tau) = (\epsilon, \mu)$ and $g(\sigma' \cup \tau') = (\epsilon', \mu)$ with $\epsilon \neq \epsilon'$. If $\tau = \emptyset$, then $\mu \leqslant n - r$ and $\tau' = \emptyset$. We should then have $\epsilon = \epsilon'$, which is impossible. If $\tau \neq \emptyset$, then $|\tau| = |\tau'|$, and thus $\tau = \tau'$. We should again have $\epsilon = \epsilon'$ which is impossible as well.

Note that g is increasing: for $\sigma \subseteq \sigma'$ and $\tau \subseteq \tau'$, we have $|g(\sigma \cup \tau)| \leqslant |g(\sigma' \cup \tau')|$.

We get our map $\psi_\#$ by defining: $\psi_\# = g_\# \circ \mathrm{sd}_\# \circ f_\#$. It is a Z_p-equivariant chain map from $\mathcal{C}(\mathrm{sd}(Z_p^{*n}))$ to $\mathcal{C}(Z_p^{*m})$.

This chain map $\psi_\#$ satisfies the condition of Lemma 3. Hence, there exists $\rho \in Z_p^{*m}$ of the form $\rho = \{(\epsilon_1, \mu_1), \ldots, (\epsilon_n, \mu_n)\}$ with $\mu_i < \mu_{i+1}$ and $\epsilon_i \neq \epsilon_{i+1}$ for $i = 1, \ldots, n-1$ such that ρ is in the support of $\psi_\#(\rho')$ for some $\rho' \in \mathrm{sd}(Z_p^{*n})$.

We exhibit now some properties of ρ and ρ'.

Since g is a simplicial map, we know that there is a permutation π and a sequence $\sigma_{\pi(1)} \cup \tau_{\pi(1)} \subseteq \cdots \subseteq \sigma_{\pi(n)} \cup \tau_{\pi(n)}$ of simplices of $\mathsf{L} * \mathsf{M}$ such that $g(\sigma_i \cup \tau_i) = (\epsilon_i, \mu_i)$ with $\mu_i < \mu_{i+1}$ and $\epsilon_i \neq \epsilon_{i+1}$ for $i = 1, \ldots, n-1$. To ease the following discussion, we define $\tau_0 := \emptyset$.

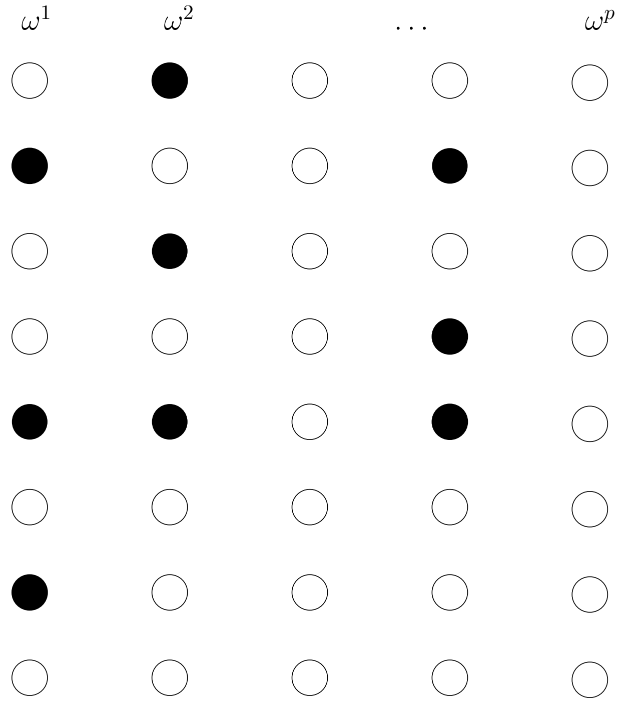

Figure 2: The simplex $\bar{\tau}$ which leads to the definition of g.

Since g is increasing, we get that $\pi(i) = i$ for all i. Using the fact that f is simplicial, we get that $|\sigma_n \cup \tau_n| = n$, and then that $|\sigma_i \cup \tau_i| = i$. Since $|\sigma_n| \leqslant n - r$, we have $\tau_n \neq \emptyset$. Note that $\tau_i = \tau_{i+1}$ implies that $\tau_i = \emptyset$ (otherwise μ_i would be equal to μ_{i+1}). Therefore, defining z to be the largest index such that τ_z is empty, we have $z < n$ and a sequence $\tau_{z+1} \subsetneq \tau_{z+2} \subsetneq \cdots \subsetneq \tau_n$. Finally, noting that $\sigma_{i+1} \cup \tau_{i+1}$ has only one more element than $\sigma_i \cup \tau_i$ for $i = 1, \ldots, n-1$, we get that $|\tau_{z+\ell}| = \ell$ for $\ell = 0, \ldots, n-z$.

Consider now the sequence $(\omega_1, \nu_1), \ldots, (\omega_{n-z}, \nu_{n-z})$, where (ω_ℓ, ν_ℓ) is the unique vertex of $\tau_{z+\ell} \setminus \tau_{z+\ell-1}$ for $\ell = 1, \ldots, n-z$. The sign $\omega_{\ell+1}$ is necessarily such that $\tau_{z+\ell}^{\omega_{\ell+1}}$ has minimum cardinality among the $\tau_{z+\ell}^{\epsilon}$, otherwise the set of ϵ for which $|\tau_{z+\ell+1}^{\epsilon}|$ is minimum would be the same as for $|\tau_{z+\ell}^{\epsilon}|$, and, according to the definition of the maps s and s_0, we would have $\epsilon_{\ell+1} = \epsilon_\ell$.

We clearly have $\big||\tau_{z+1}^{\epsilon}| - |\tau_{z+1}^{\epsilon'}|\big| \leqslant 1$ for all ϵ, ϵ' since $|\tau_{z+1}| = 1$. Now assume that for $k \geqslant z+1$ we have $\big||\tau_k^{\epsilon}| - |\tau_k^{\epsilon'}|\big| \leqslant 1$ for all ϵ, ϵ'. Since the element added to τ_k to get τ_{k+1} is added to a τ_k^{ϵ} with minimum cardinality, we have $\big||\tau_{k+1}^{\epsilon}| - |\tau_{k+1}^{\epsilon'}|\big| \leqslant 1$ for all ϵ, ϵ'. By induction we have in particular

$$\Big||\tau_n^{\epsilon}| - |\tau_n^{\epsilon'}|\Big| \leqslant 1 \quad \text{for all } \epsilon, \epsilon'. \tag{1}$$

We can now conclude. Using the fact that f is simplicial, we get that $\rho' = \{X_1, \ldots, X_n\}$ where the X_i are signed vectors with $|X_i| = i$ and $X_1 \subseteq \cdots \subseteq X_n$. Moreover, we have

$f(\{X_{z+1},\ldots,X_n\})=\tau_n$. Each X_i provides a vertex $F(X_i)$ of $\mathrm{KG}^p(\mathcal{H})$ for $i=z+1,\ldots,n$. For each j, define U_j to be the set of such vertices $F(X_i)$ such that the sign of $f(X_i)$ is ω^j. The U_j are subsets of vertices of $\mathrm{KG}^p(\mathcal{H})$. For two distinct j and j', if $F(X_i)\in U_j$ and $F(X_{i'})\in U'_j$, we have $F(X_i)\cap F(X_{i'})=\emptyset$. Thus, the U_j induce in $\mathrm{KG}^p(\mathcal{H})$ a complete p-partite p-uniform hypergraph with $n-z$ vertices. Equation (1) indicates that the cardinalities of the U_j differ by at most one. Since the $f(X_i)$ are all distinct, each U_j has all its vertices of distinct colors.

It remains to prove that $z=n-r$ (actually, $z\leqslant n-r$ would be enough). First, we have $\mu_i\geqslant i$ for all $i=1,\ldots,n$ and $\mu_{z+1}=n-r+1$, thus $z\leqslant n-r$. Second, $|f(X_{z+1})|\geqslant n-r+1$, which implies $|X_{z+1}|\geqslant n-r+1$, i.e. $z\geqslant n-r$. We get $z=n-r$, as required. □

4 Alternation number

4.1 Definition

Alishahi and Hajiabolhassan [1], going on with ideas introduced in [16], defined the *q-alternation number* $\mathrm{alt}^q(\mathcal{H})$ of a hypergraph $\mathcal{H}$. Using this parameter, we can improve upon some theorems involving the q-colorability defect. The q-alternation number is defined as follows.

Let q and n be positive integers. An *alterning sequence* is a sequence $s_1,s_2,\ldots,s_n$ of elements of Z_q such that $s_i\neq s_{i+1}$ for all $i=1,\ldots,n-1$. For a vector $X=(x_1,\ldots,x_n)\in(Z_q\cup\{0\})^n$ and a permutation $\pi\in\mathcal{S}_n$, we denote $\mathrm{alt}_\pi(X)$ the maximum length of an alternating subsequence of the sequence $x_{\pi(1)},\ldots,x_{\pi(n)}$. Note that by definition this subsequence has no zero element.

Example. Let $n=9$, $q=3$, and $X=(\omega^2,\omega^2,0,0,\omega^1,\omega^3,0,\omega^3,\omega^2)$, we have $\mathrm{alt}_{\mathrm{id}}(X)=4$. If π is a permutation acting only on the first four positions, then $\mathrm{alt}_{\mathrm{id}}(X)=\mathrm{alt}_\pi(X)$. If π exchanges the last two elements of X, we have $\mathrm{alt}_\pi(X)=5$.

Let $\mathcal{H}=(V,E)$ be a hypergraph with n vertices. We identify V and $[n]$. The *q-alternation number* $\mathrm{alt}^q(\mathcal{H})$ of a hypergraph $\mathcal{H}$ with n vertices is defined as:

$$\mathrm{alt}^q(\mathcal{H})=\min_{\pi\in\mathcal{S}_n}\max\{\mathrm{alt}_\pi(X):\ X\in(Z_q\cup\{0\})^n\text{ with }\ E(\mathcal{H}[X^j])=\emptyset\text{ for }j=1,\ldots,q\}. \tag{2}$$

Note that this number does not depend on the way V and $[n]$ have been identified.

4.2 Improving the results with the alternation number

Alishahi and Hajiabolhassan improved the Kříž theorem by the following theorem.

Theorem (Alishahi-Hajiabolhassan theorem)**.** *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then*

$$\chi(\mathrm{KG}^q(\mathcal{H}))\geqslant\left\lceil\frac{|V(\mathcal{H})|-\mathrm{alt}^q(\mathcal{H})}{q-1}\right\rceil$$

for any integer $q \geqslant 2$.

It is an improvement since we have

$$|V(\mathcal{H})| - \mathrm{alt}^q(\mathcal{H}) \geqslant \mathrm{cd}^q(\mathcal{H})$$

as it can be easily checked. This inequality is often strict, see [1].

Theorem 1 and Theorem 2 can be similarly improved with the alternation number. Let π be the permutation on which the minimum is attained in Equation (2). We replace $r = \mathrm{cd}^p(\mathcal{H})$ by $r = |V(\mathcal{H})| - \mathrm{alt}^p(\mathcal{H})$ in both proofs of Theorem 1 and Theorem 2, and we replace $|X|$ in the definition of f by $\mathrm{alt}_\pi(X)$ in the proof of Theorem 1. There are no other changes and we get the following theorems.

Theorem 4. *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Let p be a prime number. Then any proper coloring c of $\mathrm{KG}^p(\mathcal{H})$ with colors $1, \ldots, t$ (t arbitrary) must contain a complete p-uniform p-partite hypergraph with parts $U_1, \ldots, U_p$ satisfying the following properties.*

- *It has $|V(\mathcal{H})| - \mathrm{alt}^p(\mathcal{H})$ vertices.*
- *The values of $|U_j|$ for $j = 1, \ldots, p$ differ by at most one.*
- *For any j, the vertices of U_j get distinct colors.*

Theorem 5. *Let $\mathcal{H}$ be a hypergraph and assume that $\emptyset$ is not an edge of $\mathcal{H}$. Then*

$$\chi_\ell(\mathrm{KG}^p(\mathcal{H})) \geqslant \min\left(\left\lceil \frac{|V(\mathcal{H})| - \mathrm{alt}^p(\mathcal{H})}{p} \right\rceil + 1, \left\lceil \frac{|V(\mathcal{H})| - \mathrm{alt}^p(\mathcal{H})}{p-1} \right\rceil\right)$$

for any prime number p.

The special case of Theorem 4 when $p = 2$ is proved in [1] in a slightly more general form.

4.3 Complexity

It remains unclear whether the alternation number, or a good upper bound of it, can be computed efficiently. However, we can note that given a hypergraph $\mathcal{H}$, computing the alternation number for a fixed permutation is an NP-hard problem.

Proposition 6. *Given a hypergraph $\mathcal{H}$, a permutation π, and a number q, computing*

$$\max\{\mathrm{alt}_\pi(X) : X \in (Z_q \cup \{0\})^n \text{ with } E(\mathcal{H}[X^j]) = \emptyset \text{ for } j = 1, \ldots, q\}$$

is NP-hard.

Proof. The proof consists in proving that the problem of finding a maximum independent set in a graph can be polynomially reduced to our problem for $q = 2$, $\pi = \mathrm{id}$, and $\mathcal{H}$ being some special graph.

Let G be a graph. Define G' to be a copy of G and consider the *join* $\mathcal{H}$ of G and G'. The join of two graphs is the disjoint union of the two graphs plus all edges vw' with v a vertex of G and w' a vertex of G'. We number the vertices of G arbitrarily with a bijection $\rho : V \to [|V|]$. It gives the following numbering for the vertices of $\mathcal{H}$. In $\mathcal{H}$, a vertex v receives number $2\rho(v) - 1$ and its copy v' receives the number $2\rho(v)$. Let $n = 2|V|$. As usual, we denote the maximum cardinality of an independent set of G by $\alpha(G)$.

Let $I \subseteq V$ be a independent set of G. Define $Y = (y_1, \ldots, y_n) \in (Z_2 \cup \{0\})^n$ as follows:

$$y_{2\rho(v)-1} = +1 \text{ and } y_{2\rho(v)} = -1 \text{ for all } v \in I, \text{ and } y_i = 0 \text{ for the other indices } i.$$

By definition of the numbering, we have $\mathrm{alt}_{\mathrm{id}}(Y) = 2|I|$ and thus

$$\max\{\mathrm{alt}_{\mathrm{id}}(X) : X \in (Z_2 \cup \{0\})^n \text{ with } E(\mathcal{H}[X^j]) = \emptyset \text{ for } j = 1, 2\} \geqslant 2\alpha(G)$$

Conversely, any $X = (x_1, \ldots, x_n) \in (Z_2 \cup \{0\})^n$ with $E(\mathcal{H}[X^j]) = \emptyset$ for $j = 1, 2$ gives an independent set I in G and another I' in G': take a longest alternating subsequence in X and define the set I as the set of vertices v such that $x_{2\rho(v)-1} \neq 0$ and the set I' as the set of vertices v such that $x_{2\rho(v)} \neq 0$. We have $\mathrm{alt}_{\mathrm{id}}(X) = |I| + |I'|$ because two components of X with distinct index parities cannot be of same sign: each vertex of G is the neighbor of each vertex of G'. Thus

$$\max\{\mathrm{alt}_{\mathrm{id}}(X) : X \in (Z_2 \cup \{0\})^n \text{ with } E(\mathcal{H}[X^j]) = \emptyset \text{ for } j = 1, 2\} \leqslant 2\alpha(G).$$

□

The same proof gives also that computing the two-colorability defect $\mathrm{cd}^2(\mathcal{H})$ of any hypergraph $\mathcal{H}$ is an NP-hard problem.

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