Matching covered graphs with three removable classes

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Abstract

The notion of removable classes arises in connection with ear decompositions of matching covered graphs introduced by Lovász and Plummer. The last (single or double) ear of an ear decomposition is defined as a removable class. Every matching covered graph not induced by a circuit has at least three removable classes. In this paper, we characterize matching covered graphs with precisely three removable classes and show, as a corollary, that every non-planar matching covered graph has at least four removable classes. Let G be a matching covered graph. A matching covered subgraph H of G is conformal if $G - VH$ has a perfect matching. Given $S \subseteq EG$, what is a minimal conformal subgraph of G that contains S? It is known that if $|S| = 2$ then it is induced by a circuit. As an application of the main result, we answer this question for $|S| = 3$.

Keywords: graph theory; perfect matchings; matching covered graphs

1 Matching covered graphs

The graphs considered here are loopless, but they may have multiple edges. The notation and terminology we use is essentially that of Bondy and Murty [1].

A connected graph G is *matching covered* if each of its edges lies in a perfect matching. Some authors refer to matching covered graphs as *1-extendable* graphs. The treatise by Lovász and Plummer [11] and the seminal work by Lovász [10] on the matching lattice contain the basic theory of matching covered graphs. For the convenience of the reader, we shall briefly review here the terminology and results which are pertinent to this article.

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1.1 Conformal subgraphs

Let G be a matching covered graph. A matching covered subgraph H of G is *conformal* if $G - VH$ has a perfect matching.

An *even subdivision* of a graph G is a graph obtained from G by replacing each edge e with an odd path joining the ends of e but having none of its internal vertices in VG. Thus, any graph is an even subdivision of itself. It is easy to see that a graph G is matching covered if and only if every even subdivision of G is matching covered. The following is a classical result due to Lovász.

Theorem 1. *([9]) Every nonbipartite matching covered graph contains a conformal subgraph that is an even subdivision of K_4 or $\overline{C_6}$.*

The following result plays an important role in the proof of our main result.

Theorem 2. *([2]) Any set of $k \geqslant 2$ edges of a matching covered graph G lies in a conformal subgraph formed by the union of at most $k-1$ M-alternating circuits, for some perfect matching M of G.*

It follows from the theorem above that any two edges of a matching covered graph lie in a conformal circuit. (See also [8].) This paper extends this result to three edges. More precisely, given a matching covered graph G and a set S of three edges of G, we characterize minimal conformal subgraphs of G that contain the edges of S.

1.2 Cuts, contractions and splicings

Let G be a connected graph. For any set X of vertices of G, we denote the coboundary of X by $\partial_G X$. Thus, $\partial_G X$ consists precisely of those edges that have one end in X and one end in the complement $\overline{X}$ of X. If G is understood, we write simply ∂X instead of $\partial_G X$. The set ∂X is called a *cut* and the sets X and $\overline{X}$ are its *shores*. A cut is *odd* if both its shores have an odd number of vertices and is *trivial* if one of its shores is a singleton.

Given a cut $C = \partial X$ of G, where X is a nonempty proper subset of VG, the two graphs obtained by contracting X to a single vertex x and $\overline{X}$ to a single vertex $\overline{x}$ are called the C-*contractions* of G. A graph G is the *splicing* of two graphs G_1 and G_2 if it has a cut C such that G_1 and G_2 are isomorphic to the two C-contractions of G. The following assertion may be easily verified.

Proposition 3. *Any splicing of two matching covered graphs is also matching covered.*

1.3 Separating and tight cuts

A cut C in a matching covered graph G is *separating* if both C-contractions are matching covered. Each shore of C then induces a connected subgraph of G. The following result is deduced directly from this definition.

Proposition 4. *Let G be a matching covered graph. A cut C of G is separating if and only if every edge of G lies in a perfect matching that contains precisely one edge in C.*

A separating cut C of a matching covered graph G is *tight* if $|M \cap C| = 1$ for every perfect matching M of G. Thus, every tight cut is also separating. The converse does not hold in general. For example, the graph $\overline{C_6}$ and the Petersen graph have separating cuts that are not tight.

A matching covered graph is *solid* if each separating cut is tight. Every bipartite matching covered graph is solid. Odd wheels and Möbius ladders are also examples of solid matching covered graphs [7]. In fact, these graphs have no separating cut. The Petersen graph and $\overline{C_6}$ are examples of nonsolid matching covered graphs.

A matching covered graph free of nontrivial tight cuts is called a *brace* if it is bipartite, a *brick* otherwise.

1.4 Ear decompositions

An *ear* in a matching covered graph G is a path P of odd length such that both ends of P have degree at least 3 in G but all the internal vertices of P have degree 2 in G. For an ear P, the graph $G - P$ is the graph obtained from G by deleting all edges and internal vertices of P, and P is said to be *removable* if $G - P$ is matching covered. A *double ear* in G is a pair $\{P_1, P_2\}$ of vertex-disjoint ears. A double ear $\{P_1, P_2\}$ is *removable* if neither P_1 nor P_2 is removable, but $G - (P_1 \cup P_2) = (G - P_1) - P_2$ is matching covered. The following theorem is one of the basic results of the theory of matching covered graphs [11, Chapter 5].

Theorem 5 (Ear decomposition)**.** *Let G be a matching covered graph not induced by a circuit, and let H be a conformal matching covered subgraph of G. Then there exists a sequence $(G_1, G_2, \ldots, G_r)$ of subgraphs of G such that $G_1 = G$, $G_r = H$ and, for $2 \leqslant i \leqslant r$, G_i is obtained from G_{i-1} by deleting either a removable ear or a removable double ear of G_{i-1}.*

A single removable ear, or the pair of ears that constitute a removable double ear, will be referred to as a *removable class*. An edge e is also described as *removable* if $G - \{e\}$ is matching covered. The following results present basic properties of removable classes.

Lemma 6. *The removable classes of a matching covered graph not induced by a circuit are pairwise disjoint.*

Proof: Let G be a matching covered graph not induced by a circuit. Let R be a removable class of G and let e be any edge of R. Let us show that R is the only removable class that contains e.

As G is not induced by a circuit, R includes a unique ear P that contains e. If R is a single ear then $R = \{P\}$ and R is the only removable class that contains e. Suppose therefore that R is a double ear. Then there must be another ear Q such that $R = \{P, Q\}$. Moreover, $G - P$ is connected (because G is 2-connected) but not matching covered, and so there exists an edge of $EG - P$ not in any 1-factor of $G - P$. This edge must be in Q since $G - (P \cup Q)$ is matching covered, and so Q is uniquely determined. Hence e belongs to a unique removable class. □

Corollary 7. *Every matching covered graph not induced by a circuit has at least three removable classes.*

Proof: Let G be a matching covered graph not induced by a circuit. Consider an ear decomposition $(G_1, G_2, \ldots, G_r)$ of G. By Theorem 5, G_2 is obtained from G by deleting a removable class, say Q_1, of G.

Now consider an alternating circuit C containing an edge of Q_1. We claim that $Q_1 \subset C$. This is clearly true if Q_1 is a single ear. If Q_1 is a double ear $\{Q_1', Q_1''\}$ and C contains Q_1' but not Q_1'' then Q_1'' would be a removable single ear of G (because $G[EG_2 \cup C]$ is matching covered), in contradiction to the definition of a removable double ear. As asserted, $Q_1 \subset C$. Now, applying the ear decomposition theorem with $H = C$, we get a second removable class Q_2 of G.

By Theorem 2, there exists an alternating circuit D containing an edge of Q_1 and an edge of Q_2. By the same reasoning as above, we have $Q_1 \cup Q_2 \subset D$. By applying the ear decomposition theorem again with $H = D$, we get a third removable class of G. □

An attempt to continue the reasoning in this proof to find a fourth removable class fails because the "new" removable class is not necessarily distinct from the three already found.

1.5 A dependence relation

Let G be a matching covered graph, and let e and f be any two edges of G. Then e *depends* on f, or e *implies* f, if every perfect matching that contains e also contains f. We write $e \Rightarrow f$ to indicate that e depends on f. We say that two edges e and f are *mutually dependent* if $e \Rightarrow f$ and $f \Rightarrow e$. In this case we write $e \Leftrightarrow f$. Clearly $\Leftrightarrow$ is an equivalence relation.

A set Q of mutually dependent edges of G such that no edge in $EG - Q$ depends on an edge of Q is called a *minimal class* of G. Each minimal class therefore consists of a set of independent edges, and distinct minimal classes are disjoint. For instance, a removable singleton or doubleton of a matching covered graph is a minimal class. Let e be an edge of G. Any minimal class Q that contains an edge that depends on e is said to be a *minimal class induced by* e. In this case every edge of Q depends on e. Minimal classes in a brick have some attractive properties.

Theorem 8. *([10, Lemma 3.4]) Let G be a brick and Q a minimal class of G. Then $|Q| \leqslant 2$. Moreover, if $|Q| = 2$, then $G - Q$ is bipartite.*

2 Extremal graphs and their properties

A matching covered graph not induced by a circuit is *extremal* if it has precisely three removable classes.

It is easy to see that forming an even subdivision does not change the number of removable classes of a matching covered graph. The following result summarizes this property.

Proposition 9. *A graph G is extremal if and only if every even subdivision of G is extremal.*

The next result provides a more precise idea of an extremal graph.

Lemma 10. *An extremal graph is an even subdivision of a cubic graph.*

Proof: Let G be an extremal graph. Let Q_1, Q_2 and Q_3 be the three removable classes of G. Let e_i, $i = 1, 2, 3$, be an edge of Q_i.

By Theorem 2, G has a 2-connected conformal subgraph containing e_1, e_2 and e_3 formed by the union of two M-alternating circuits C and D, for some perfect matching M of G. Then, $H = C \cup D$ contains Q_1, Q_2 and Q_3. If G is distinct from H then, by considering an ear decomposition of G finishing with H, we find that the first removable class of G is distinct from Q_1, Q_2 and Q_3, which is a contradiction. Thus, $G = H$.

As circuits C and D are alternating with the same perfect matching M, each vertex common to C and D is incident with an edge that is also common to C and D. It follows that G is an even subdivision of a cubic graph. □

In view of Proposition 9 and Lemma 10, to characterize extremal graphs, it suffices to characterize cubic extremal graphs. Thus, we turn our attention to characterizing cubic extremal graphs. Let Θ be the graph with just two vertices and three links joining them.

Proposition 11. *Every cubic extremal graph is simple, unless it is the Θ graph.*

Proof: Let G be a cubic extremal graph, and let e and f be parallel edges of G. Then, $G - \{e\}$ and $G - \{f\}$ are matching covered. Then $\{\{e\}\}$ is a removable class of G. Analogously, $\{\{f\}\}$ is a removable class of G.

If $G - \{e\}$ is not a circuit then, by Corollary 7, $G - \{e\}$ has at least three removable classes. At most one removable class of $G - \{e\}$ contains $\{f\}$. The removable classes of $G - \{e\}$ that do not contain $\{f\}$ are removable classes of G. Then, G has at least four removable classes, a contradiction.

Thus, $G - \{e\}$ is a circuit. Then, G is a subdivision of Θ. As G is cubic, it follows that G is, in fact, the Θ graph. □

3 Extremal graphs free of separating cuts

As every tight cut is also separating, an extremal graph free of separating cuts is also free of tight cuts, that is, it is a brick or a brace. By Lemma 10, it is cubic. Thus, in this section, we characterize cubic extremal bricks and braces free of separating cuts. We shall make use of an important property concerning removable edges in braces.

Lemma 12. *([5, Lemma 3.2]) Let G be a brace on at least six vertices. Then every edge of G is removable.*

The following result characterizes cubic extremal braces.

Theorem 13. *The only cubic extremal brace is the Θ graph.*

Proof: By Lemma 12, every edge of a brace on six or more vertices is removable. A brace on six or more vertices has at least nine edges. Thus, an extremal brace has at most four vertices. But, there is only one connected cubic bipartite graph on four vertices and it has four removable edges. Therefore, a cubic extremal brace has two vertices. The only cubic extremal graph on two vertices is the Θ graph. □

We now proceed to characterize extremal bricks free of separating cuts. Recall that a brick free of separating cuts is a solid brick. We shall show that K_4 is the only extremal solid brick. The following result will be used.

Lemma 14. *([4, Corollary 6.15]) If G is a solid brick of maximum degree 3 or 4, then, for every vertex v of G, at most one edge incident with v does not lie in a removable class of G.*

Theorem 15. *Graph K_4 is the only solid extremal brick.*

Proof: Let G be a solid extremal brick. Let H be the subgraph of G spanned by the union of the removable classes of G. By Lemma 14, H has minimum degree at least 2 and spans G. By Theorem 8, every removable class of G has one or two edges. As G has precisely three removable classes, H has at most six edges. It follows that $|VG| = |VH| \leqslant 6$.

By Theorem 1, G has a conformal subgraph which is an even subdivision of K_4 or $\overline{C_6}$. By Lemma 10 and Proposition 11, G is a cubic simple graph. Thus, G must in fact be K_4 or $\overline{C_6}$. But, $\overline{C_6}$ is not solid. Therefore, G is K_4. □

4 Extremal graphs which have a separating cut

The following theorem, together with Theorem 15, provide an iterative procedure to construct all extremal graphs.

Theorem 16. *Let G be a cubic extremal graph and let C be a separating cut of G. Then both C-contractions are cubic extremal graphs.*

Proof: By induction on $|VG|$. We may assume that C is nontrivial, as the conclusion is clear otherwise. By Proposition 11, G is simple. Let G_1 and G_2 be the two C-contractions. Then G_1 and G_2 are matching covered graphs. Every matching covered graph is 2-connected. So, $|C| \geqslant 2$. As G is cubic, we must in fact have $|C| \geqslant 3$ (otherwise G_1 and G_2 would be graphs with an odd number of vertices of odd degree).

Case 1. *G_1 or G_2 has no nontrivial separating cut.*

Assume, without loss of generality, that G_2 has no nontrivial separating cut. As every tight cut is a separating cut, G_2 has no nontrivial tight cut, that is, G_2 is a brick or brace.

Suppose that G_2 is a brace. If $|VG_2| \geqslant 6$ then it follows from Lemma 12 that G_2 has at least six removable edges not in C. These edges constitute at least six removable classes in G. We may thus assume that $|VG_2| = 4$. But each vertex of G_2 is of degree at least 3, in contradiction to the fact that G_2 is a simple brace. We conclude that G_2 must be a brick.

Let $Q_1, Q_2, \ldots, Q_k$ be the removable classes of G_1. Then, $G_1 - Q_i$ is matching covered, for $i = 1, 2, \ldots, k$. By Corollary 7, $k \geqslant 3$.

If some Q_i does not contain an edge of C then $G - Q_i$ is matching covered, that is, Q_i is a removable class of G. Suppose that a removable class Q_i of G_1 contains an edge e of C. Let R_i be a minimal class of G_2 induced by e, and let f be any edge of $EG_2 - R_i$. Then f does not depend on an edge of the minimal class R_i, and so $G_2 - R_i$ has a 1-factor containing f. Moreover it is connected, by Theorem 8, since its minimum degree is at least 3. Consequently $G_2 - R_i$ is matching covered. If R_i does not contain an edge of C then $G - R_i$ is matching covered, so that R_i includes a removable class of G. In the remaining case, $R_i \cap C = \{e\}$. Then $G - (Q_i \cup R_i)$ is matching covered, by Proposition 3. Therefore $Q_i \cup R_i$ includes a removable class of G. Thus, every removable class Q_i of G_1 yields at least one removable class of G. The resulting removable classes of G are disjoint since minimal classes of G_2 induced by distinct edges of C are disjoint. But G has precisely three removable classes. Thus, G_1 has precisely three removable classes. As asserted, G_1 is extremal. Thus, G_1 is cubic, since $|C| \geqslant 3$.

Let us now show that G_2 is extremal. If G_1 has no nontrivial separating cut then G_2 is an extremal graph by the argument above. We may thus assume that G_1 has a nontrivial separating cut.

Let $D = \partial(Y)$ be a nontrivial separating cut of G_1 such that the shore, say Y, that does not contain the contraction vertex of G_1 is minimal. Note that every separating cut of G_1 is also a separating cut of G. Then, D is a separating cut of G. Let H_1 and H_2 be the two D-contractions of G. Then H_1 and H_2 are matching covered graphs. Fix notation such that H_2 is obtained by contracting Y, so that C is a separating cut of H_2. By the choice of D, graph H_1 has no nontrivial separating cut. By the argument above, H_2 is cubic and extremal. As C is a separating cut of H_2, the inductive hypothesis shows that both C-contractions of H_2 are extremal graphs. One of the C-contractions of H_2 is G_2. Thus, G_2 is extremal.

Case 2. *G_1 and G_2 both have nontrivial separating cuts.*

Let $D = \partial(Y)$ be a nontrivial separating cut of G_1 such that the shore Y that does not contain the contraction vertex of G_1 is minimal. Then D is also a separating cut of G. Let H_1 and H_2 be the two D-contractions of G. Then H_1 and H_2 are matching covered graphs. Fix notation such that H_2 is obtained by contracting Y. By Case 1, H_1 and H_2 are extremal. But C is a separating cut of H_2. By the inductive hypothesis, both C-contractions of H_2 are extremal graphs. One of the C-contractions of H_2 is G_2. Thus, G_2 is extremal. Analogously, G_1 is extremal. By Lemma 10, G_1 and G_2 are cubic. $\square$

5 The list of extremal graphs

Now, we present a list of the nonbipartite cubic extremal graphs: there are nine of them. The following result is useful to guarantee that every extremal graph is a brick.

Lemma 17. *([6, Corollary 2.8]) Any splicing of two cubic bricks is a (cubic) brick.*

By Theorems 15 and 16, every cubic extremal graph distinct from Θ is obtained by iterative splicings of K_4. As K_4 is a cubic brick, it follows from the above lemma that every nonbipartite extremal graph is a brick. As bricks are 3-connected graphs, every minimal class is a removable class, by Theorem 8.

We shall make use of the following arguments to ensure that a cubic graph is not extremal. The next result is a consequence of the definition of minimal classes.

Proposition 18. *Let G be a cubic extremal graph. Then each removable class lies in precisely one perfect matching of G.*

Proof: Let $v \in VG$. Let e_1, e_2 and e_3 be the three edges incident on v. For $i = 1, 2, 3$, let Q_i be a minimal class of G induced by e_i. As G is extremal, Q_1, Q_2 and Q_3 are the three removable classes of G. By the definition of an induced minimal class, every perfect matching that includes Q_i also contains edge e_i. This assertion holds for any vertex of G. Thus, for each i and each vertex w there exists a unique edge that is incident on w and belongs to every perfect matching that includes Q_i. We conclude that each class Q_i lies in a unique perfect matching of G. □

Let H be a cubic graph and let $v \in VH$. We shall denote by $(H \odot K_4)_v$ the graph obtained by splicing H and K_4 at vertex v.

Corollary 19. *Let H be a cubic extremal graph and let $v \in VH$. Let Q be a removable class of H which has no edge incident with v. Let*

$$G = (H \odot K_4)_v.$$

If there is a perfect matching M in G containing Q and the three edges incident with v then G is not extremal.

Proof: By hypothesis, $H - Q$ is matching covered, and Q has no edge incident with v. It follows that Q is a removable class of G. But, Q lies in two perfect matchings of G, namely, M and the perfect matching of G whose restriction to EH is the perfect matching of H that includes Q. By Proposition 18, G is not extremal. □

The above corollary suggests the following definition. A vertex v of a cubic graph H is said to satisfy the *extension condition* if the graph obtained from H by deleting v and all the three neighbours of v does not have a perfect matching containing a removable class of H.

The following theorem is an immediate consequence of Theorem 15 and Theorem 16.

Theorem 20. *Let G be a cubic extremal graph different from K_4. Then*

$$G = (H \odot K_4)_v,$$

where H is a cubic extremal graph on $|V(G)| - 2$ vertices and v is a vertex of H that satisfies the extension condition.

Theorem 20 suggests how all cubic extremal graphs distinct from Θ may be generated. We start with K_4, the only cubic extremal graph on four vertices. For $n \geqslant 4$, suppose that the set $\mathcal{G}_n$ of all extremal cubic graphs on n vertices is known. Then each graph in the set $\mathcal{G}_{n+2}$, the set of cubic extremal graphs on $n+2$ vertices, is of the form $(H \odot K_4)_v$, where H is a member of $\mathcal{G}_n$ and v is a vertex of H that satisfies the extension condition. All graphs (up to isomorphism) that can be generated in this way are shown in Figure 1.

The three removable classes of each extremal graph are indicated in the figure with numbers 1, 2 and 3. Vertices that satisfy the extension condition are labelled u and v. Up to automorphisms, no other vertex satisfies the extension condition.

There is only one extremal cubic brick on sixteen vertices, namely G_9 (Figure 1) and no vertex of this graph satisfies the extension condition. Therefore, there are no cubic extremal graphs on eighteen vertices. It is quite interesting that this procedure cannot be carried on forever.

Corollary 21. *Any non-planar matching covered graph has at least four removable classes.*

6 Application

Let $\mathcal{F}$ be the set of the nine graphs in Figure 1. The next result generalizes the theorem [8] that any two edges of a matching covered graph lie in a conformal circuit.

Corollary 22. *Let G be a matching covered graph and let S be a set of three edges of G. Then S is contained in a conformal subgraph of G which is induced by a circuit or is an even subdivision of Θ or of a member of $\mathcal{F}$.*

Proof: We use induction on $|EG|$. The corollary holds if G is induced by a circuit, and so we suppose it is not.

Suppose that G contains a removable class Q such that $Q \cap S = \emptyset$. Then $G - Q$ is a conformal matching covered subgraph of G containing S. If $G - Q$ is induced by a circuit then we are done. Otherwise the inductive hypothesis shows that S is included in a conformal subgraph H of $G - Q$ which is induced by a circuit or is an even subdivision of Θ or of a member of $\mathcal{F}$. The conformal property is transitive. Thus, H is a conformal subgraph of G.

We may thus assume that every removable class of G contains an edge of S. By Corollary 7, G has at least three removable classes. By Lemma 6, the removable classes of G are disjoint. As $|S| = 3$, it follows that G has precisely three removable classes, that is, G is extremal. Thus, G is an even subdivision of Θ or of a member of $\mathcal{F}$. □

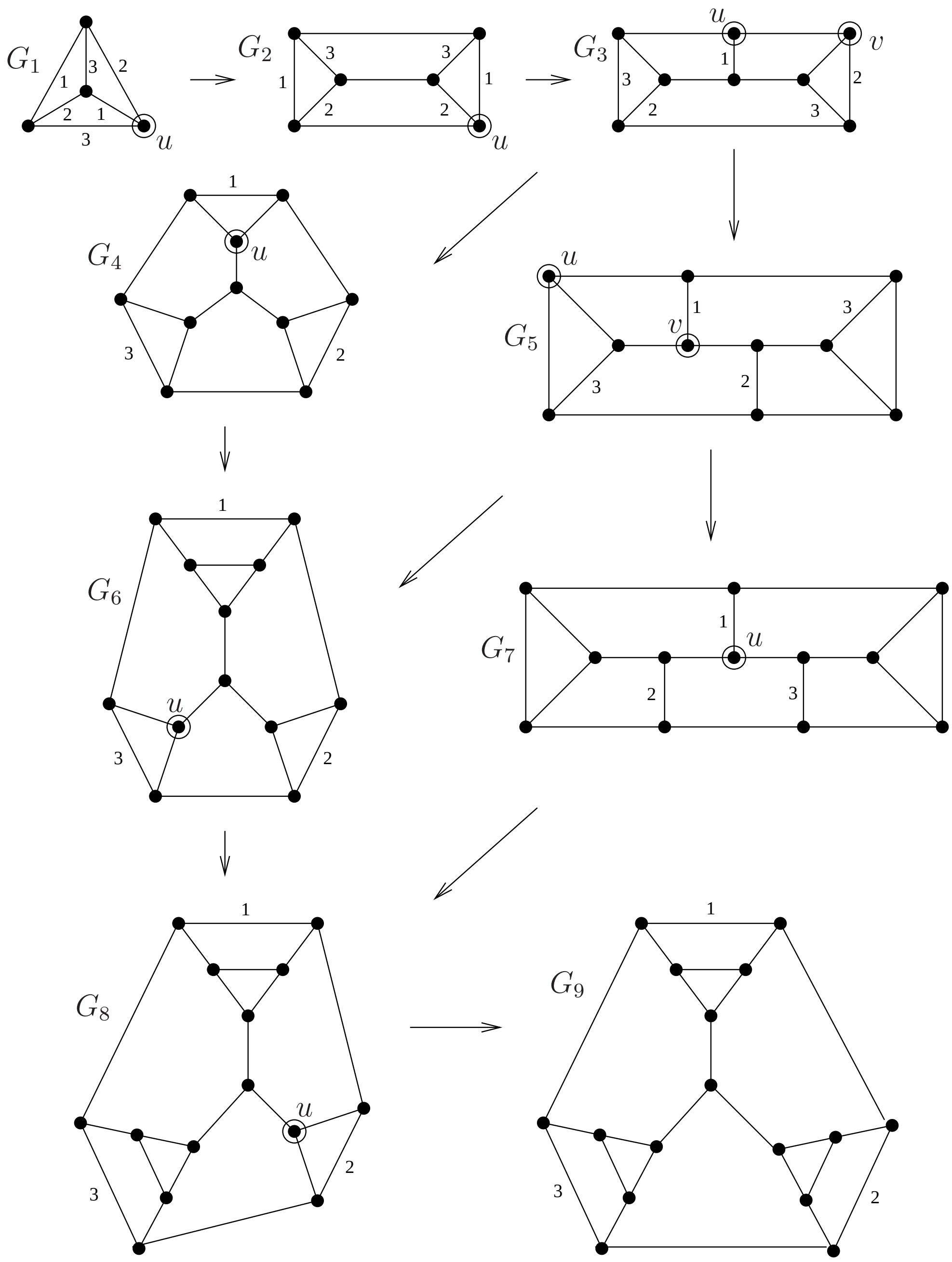

Figure 1: Cubic extremal graphs

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