Zero-Sum Magic and Null Set of Regular Graphs *†

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Abstract

For every $h \in \mathbb{N}$, a graph G with the vertex set $V(G)$ and the edge set $E(G)$ is said to be *h-magic* if there exists a labeling $l : E(G) \to \mathbb{Z}_h \setminus \{0\}$ such that the induced vertex labeling $s : V(G) \to \mathbb{Z}_h$, defined by $s(v) = \sum_{uv \in E(G)} l(uv)$ is a constant map. When this constant is zero, we say that G admits a *zero-sum h-magic*. The *null set* of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that G admits a zero-sum h-magic. In 2012, the null sets of 3-regular graphs were determined. In this paper we determine the null set of every r-regular graph, for $r \neq 5$. More precisely, we show that if G is an r-regular graph, then for even r $(r > 2)$, $N(G) = \mathbb{N}$ and for odd r $(r \neq 5)$, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$. Moreover, we prove that if G is a 2-edge connected $(2k + 1)$-regular graph and $k \in \mathbb{N} \setminus \{2\}$, then $N(G) = \mathbb{N} \setminus \{2\}$. Also, we show that if G is a 2-edge connected bipartite graph, then $N(G) \subseteq \mathbb{N} \setminus \{2, 3, 4, 5\}$.

1 Introduction

Let G be a finite and undirected graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph in which multiple edges are admissible is called a *multigraph*. An *r-regular* graph is a graph each of whose vertex has degree r. The degree of a vertex u in G is denoted by $d_G(u)$. A *cut edge* is an edge whose removal would increase the number of connected components. A graph G is *n-edge connected* if the minimum number of edges whose removal would disconnect G is at least n. We denote the complete graph and the cycle of order n by K_n and C_n, respectively. A *wheel* is a graph with n vertices, formed by connecting a single vertex to all vertices of C_{n-1} and denoted by W_n. A *pendant edge* is an edge incident with a vertex of degree 1.

A subgraph F of G is a *factor* of G if F is a spanning subgraph of G. If a factor F has

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all of its degrees equal to k, it is called a *k-factor*. Thus a 2-factor is a disjoint union of cycles that cover all vertices of G. A *k-factorization* of G is a partition of the edges of G into disjoint k-factors. For integers a and b with $1 \leq a \leq b$, an *$[a,b]$-multigraph* is defined to be a multigraph G such that for every $v \in V(G)$, $a \leq d_G(v) \leq b$. For a set $\{a_1,\dots,a_r\}$ of non-negative integers an *$\{a_1,\dots,a_r\}$-multigraph* is a multigraph each of whose vertices has degree from the set $\{a_1,\dots,a_r\}$. Analogously, an *$[a,b]$-factor* and an *$\{a_1,\dots,a_r\}$-factor* can be defined.

Let G be a graph. A *zero-sum flow* for G is an assignment of non-zero real numbers to the edges of G such that the sum of values of all edges incident with each vertex is zero. Let k be a natural number. A *zero-sum k-flow* is a zero-sum flow with values from the set $\{\pm 1,\dots,\pm(k-1)\}$.

For an abelian group A, written additively, any mapping $l : E(G) \to A$ is called a *labeling* of a graph G. Given a labeling on the edge set of G, one can introduce a vertex labeling $s : V(G) \to A$, defined by $s(v) = \sum_{uv \in E(G)} l(uv)$, for $v \in V(G)$. A graph G is said to be *A-magic* if there is a labeling $l : E(G) \to A \setminus \{0\}$ such that for each vertex v, the sum of the labels of edges incident with v is all equal to the same constant, that is $s(v) = c$, for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A-magic. A graph G is called an *h-magic graph* if there is a $\mathbb{Z}_h$-magic labeling of G. A graph G is said to be *zero-sum h-magic* if there is an edge labeling of G in $\mathbb{Z}_h \setminus \{0\}$ such that the sum of values of all edges incident with each vertex is zero. If $s(v) = 0$, for a vertex $v \in V(G)$, then we say that *zero-sum h-magic rule* holds in v. The *null set* of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that G admits a zero-sum h-magic. For convenience, the notation 1-*magic* will be used to indicate $\mathbb{Z}$-magic. Clearly, if a graph is h-magic, it is not necessarily k-magic (for $h \neq k$).

Recently, Choi, Georges and Mauro in [6] proved that if G is 3-regular graph, then $N(G)$ is $\mathbb{N} \setminus \{2\}$ or $\mathbb{N} \setminus \{2,4\}$. In this article, we extend this result by showing that if G is an r-regular graph, then for even r ($r > 2$), $N(G) = \mathbb{N}$ and for odd r ($r \neq 5$), $\mathbb{N} \setminus \{2,4\} \subseteq N(G)$. Moreover, we prove that if G is a 2-edge connected $(2k+1)$-regular graph and $k \in \mathbb{N} \setminus \{2\}$, then $N(G) = \mathbb{N} \setminus \{2\}$.

The original concept of A-magic graph is due to J. Sedlacek [14, 15], who defined it to be a graph with a real-valued edge labeling such that distinct edges have distinct non-negative labels and the sum of the labels of the edges incident with vertices are the same.
Stanley considered $\mathbb{Z}$-magic graphs and showed that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations, [17, 18]. Recently, there have been considerable research articles in graph labeling, interested readers are referred to [7, 11, 12, 13, 19].

In [11], the null set of some classes of regular graphs are determined.

Theorem 1. *If $n \geq 4$, then* $N(K_n) = \begin{cases} \mathbb{N}, & \text{if } n \text{ is odd;} \\ \mathbb{N} \setminus \{2\}, & \text{if } n \text{ is even.} \end{cases}$

Theorem 2. $N(C_n) = \begin{cases} \mathbb{N}, & \textit{if } n \textit{ is even;} \\ 2\mathbb{N}, & \textit{if } n \textit{ is odd.} \end{cases}$

Recently, the following theorem was proved, in [2] and [3].

Theorem 3. *Let $r \geq 3$, be a positive integer. Then every r-regular graph admits a zero-sum 5-flow.*

This theorem implies that if G is an r-regular graph ($r \geq 3$), then $\mathbb{N} \setminus \{2, 3, 4\} \subseteq N(G)$.

Before establishing our results we need some theorems.

An interesting theorem due to Kano [9] states that:

Theorem 4. *Let $r \geq 3$ be an odd integer and let k be an integer such that $1 \leq k \leq \frac{2r}{3}$. Then every r-regular graph has a $[k-1, k]$-factor each component of which is regular.*

Also, the following theorems were proved.

Theorem 5.[4, p.179] *Let $r \geq 3$ be an odd integer, and G be a 2-edge connected $[r-1, r]$-multigraph having exactly one vertex w of degree $r-1$. Then for every even integer k, $2 \leq k \leq \frac{2r}{3}$, G has a k-factor.*

Theorem 6.[5] *Every 2-edge connected $(2r+1)$-regular multigraph contains a 2-factor.*

Theorem 7.[10] *Every $2r$-regular multigraph admits a 2-factorization.*

2 Regular Graphs

Let G be an r-regular graph. In this section we prove that for every even natural number r ($r > 2$), $N(G) = \mathbb{N}$ and for every odd natural number r ($r \neq 5$), $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.

We start this section with the following theorem.

Theorem 8. *Let r be an odd integer and $r \geq 3$. Then every r-regular multigraph with at most one cut edge admits a zero-sum 4-magic.*

Proof. Obviously, we may suppose that G is connected. First assume that G is a 2-edge connected r-regular multigraph. By Theorem 6, G has a 2-factor, say H. Now, assign 1 and 2 to the edges of H and the edges of $G \backslash E(H)$, respectively. It is not hard to see that G admits a zero-sum 4-magic.

Now, suppose that G has a cut edge, say uv. Let $G' = G \setminus \{uv\}$. Clearly, G' has two components, say G_1 and G_2. Assume that $u \in V(G_1)$ and $v \in V(G_2)$. Now, consider G_1. Note that G_1 is a 2-edge connected $[r-1, r]$-multigraph which u is the only vertex of G_1 of degree $r-1$. By Theorem 5, G_1 has a 2-factor, say H. Now, define a function $l : E(G_1) \to \{1, 2\}$, where $l(e) = 1$ for every $e \in E(H)$ and $l(e) = 2$ for every $e \in E(G_1) \setminus E(H)$. Then $s(u) = 2$ and $s(x) = 0$, for every $x \in V(G_1) \setminus \{u\}$ (mod 4). By a similar argument one can obtain an edge labeling for G_2 with values 1 and 2 such that $s(v) = 2$ and $s(x) = 0$, for every $x \in V(G_2) \setminus \{v\}$ (mod 4). Now, define $l(uv) = 2$. Clearly, we obtain a zero-sum 4-magic and we are done. □

Remark 1. If G is a $2r$-regular multigraph, then by assigning 2 to all edges of G, one can obtain a zero-sum 4-magic.

The following remark shows that there are some regular graphs with no zero-sum 4-magic.

Remark 2. *Let r be an odd integer ($r \geq 3$) and G be an r-regular multigraph. If there is a vertex u such that every edge adjacent to u is a cut edge, then G does not admit a zero-sum 4-magic.*

Proof. For contradiction assume that G admits a zero-sum 4-magic, say l. Since G admits a zero-sum 4-magic it is not hard to see that there exists at least one edge adjacent to u, say uv, with label 1 or 3. Assume that G' is the connected component of $G \setminus \{u\}$ containing v. Clearly, we have $\sum_{x \in V(G')} s(x) = 2\sum_{e \in E(G')} l(e) + l(uv)$. But $\sum_{x \in V(G')} s(x) = 0$ (mod 2). On the other hand, $2\sum_{e \in E(G')} l(e) + l(uv) = 1$ (mod 2), a contradiction. □

Lemma 1. *Let G be a $\{1, 7\}$-multigraph with no component that is isomorphic to K_2. Suppose that the induced subgraph on all vertices of degree 7 has no cut edge. If h is a pendant edge of G with value from $\{1, 2\}$, then there exists a function l on $E(G)$ which agrees on h and has the following properties:*
(i) For every $e \in E(G)$, $l(e) \in \{1, 2\}$;
(ii) For every $v \in V(G)$ of degree 7, the zero-sum 3-magic rule holds in v.

Proof. First assume that G is a multigraph with exactly one pendant edge $h = uv$ with value 1. Assume that $d_G(v) = 1$. Let $G' = G \setminus \{v\}$. Note that G' is a 2-edge connected $[6, 7]$-multigraph in which u is the only vertex of degree 6. By Theorem 5, G' has a 2-factor H. Define $l(e) = 2$, for every $e \in E(H)$ and define $l(e) = 1$, for every $e \in E(G') \setminus E(H)$. Hence we obtain the desired labeling for G.

Now, if the value of h is 2, we multiply all labels of the edges by 2 (mod 3).

Next, suppose that the number of pendant edges of G is at least two and the value of h is 1. Consider two copies of G, say G_1 and G_2. Assume that $u_i v_i$, $1 \leq i \leq k$ ($k \geq 2$) are all edges of G_1, such that $u_i, v_i \in V(G_1)$ and $d_{G_1}(v_i) = 1$. Also, suppose that u'_i and v'_i are the vertices corresponding to u_i and v_i ($i = 1, \ldots, k$) in G_2. Let G^* be the multigraph obtained by removing the vertices $v_1, \ldots, v_k$ and $v'_1, \ldots, v'_k$ and joining u_i and u'_i in $G_1 \cup G_2$, for $i = 1, \ldots, k$. Since none of the connected components of G is K_2, G^* is a 2-edge connected 7-regular multigraph. Thus by Theorem 6, G^* has a 2-factor, say H. If the edge in G^* corresponding to h belong to $E(H)$, then let $l(e) = 2$ for every $e \in E(G^*) \setminus E(H)$ and $l(e) = 1$ for every $e \in E(H)$. Otherwise, define $l(e) = 1$ for every $e \in E(G^*) \setminus E(H)$ and $l(e) = 2$ for every $e \in E(H)$. Hence we obtain the desired labeling.

Now, assume that the value of h is 2. In this case by multiplying all values by 2 (mod 3), we obtain the desired labeling and the proof is complete. □

In the following theorem, we prove that for every r-regular graph G ($r \geq 3$, $r \neq 5$), $3 \in N(G)$.

Theorem 9. *Let r be an integer such that $r \geq 3$ and $r \neq 5$. Then every r-regular graph admits a zero-sum 3-magic.*

Proof. First assume that r is an even positive integer and $r \neq 2$. The proof is by induction on r. If $r = 4$, then by Theorem 7, G is decomposed into 2-factors G_1 and G_2. Now, assign 1 and 2 to all edges of G_1 and G_2, respectively. Thus G admits a zero-sum 3-magic. If $r = 6$, then assign 1 to the edges of G to obtain a zero-sum 3-magic. Now, suppose that $r \geq 8$. So, by Theorem 7, G is decomposed into 2-factors. Choose two 2-factors G_1 and G_2. Now, by induction hypothesis $G \setminus (E(G_1) \cup E(G_2))$ admits a zero-sum 3-magic. On the other hand, by the case $r = 4$, $G_1 \cup G_2$ admits a zero-sum 3-magic and the proof is complete.

Now, assume that r is an odd positive integer. If r is divisible by 3, then assign 1 to all edges of G to obtain a zero-sum 3-magic.

If r is not divisible by 3, then $r \equiv 1, 5, 7, 11 \pmod{12}$.

First, suppose that $r = 7$. For finding a zero-sum 3-magic we construct a rooted tree T from G, where every maximal 2-edge connected subgraph of G is considered as a vertex of T and every edge of T is corresponding to a cut edge of G. Now, by traversing T, level by level, we find a zero-sum 3-magic for G. We start from the root of T say H (The root can be taken to be any vertex). Let h be an arbitrary cut edge incident with H. Define the value of h, 1. By Lemma 1, one can assign 1 or 2 to each edge of H and cut edges of G which are incident with H such that every cut edge of G incident with H has value 1 or 2 and moreover the zero-sum 3-magic rule holds in every vertex of H. Now, we move to the next vertex level of T. Let H' be a vertex adjacent to H in T. At this stage there exists just one cut edge of G incident with H' which is labeled by 1 or 2. Now, by applying Lemma 1, we can label the edges of H' and apply the same procedure as in H. By continuing this procedure we obtain a zero-sum 3-magic for G, as desired.

Now, assume that $r = 11$. Then by Theorem 4, G has a $[6, 7]$-factor, say H whose components are regular. Let H_1 and H_2 be the union of 6-regular components and 7-regular components of H, respectively. Also, by Theorem 7, H_1 is decomposed into 2-factors G_1, G_2 and G_3. Now, assign 2 to all edges of H_2, G_1 and G_2 and assign 1 to the edges of $G \setminus (E(H_1) \cup E(H_2))$ and G_3. Then it is not hard to see that G admits a zero-sum 3-magic.

Now, suppose that $r = 12k+1$ or $r = 12k+7$, and $k \geq 1$. By Theorem 4, G has a $[6k-2, 6k-1]$-factor, say H, whose components are regular. Let H_1 and H_2 be the union of $(6k-2)$-regular components and $(6k-1)$-regular components of H, respectively. Since $6k-2$ is even, H_1 admits a zero-sum 3-magic. Now, assign 2 to the edges of H_2 and assign 1 to all edges of $G \setminus (E(H_1) \cup E(H_2))$. Then G admits a zero-sum 3-magic.

Now, assume that $r = 12k+5$ or $r = 12k+11$, and $k \geq 1$. By Theorem 4, G has a $[6k+1, 6k+2]$-factor, say H, whose components are regular. Let H_1 and H_2 be the union of $(6k+1)$-regular components and $(6k+2)$-regular components of H, respectively. Since $6k+2$ is even, H_2 admits a zero-sum 3-magic. Now, assign 2 to all edges of H_1 and assign 1 to all edges of $G \setminus (E(H_1) \cup E(H_2))$. Therefore, G admits a zero-sum 3-magic, as desired. □

Now, we are in a position to prove our main theorem for regular graphs.

Theorem 10. *Let G be an r-regular graph ($r \geq 3$, $r \neq 5$). If r is even, then $N(G) = \mathbb{N}$, otherwise $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.*

Proof. First assume that r is even. Clearly, by assigning 1 to all edges of G, G admits a zero-sum 2-magic. Moreover, by Theorem 3 the existence of zero-sum k-flow implies that the existence of zero-sum k-magic labeling. Thus by Theorems 3, 9 and Remark 1, $N(G) = \mathbb{N}$. Next, assume that r is an odd integer. Then by Theorems 3 and 9 we are done. □

Lemma 2. *If G is a 2-edge connected $(2k+1)$-regular graph and $k \in \mathbb{N} \setminus \{2\}$, then $N(G) = \mathbb{N} \setminus \{2\}$.*

Proof. Since the degree of each vertex is odd, $2 \notin N(G)$. Now, the result follows from Theorems 3, 8 and 9. □

We close this section with the following conjecture.

Conjecture 1. *Every 5-regular graph admits a zero-sum 3-magic.*

It is easily seen that a 5-regular graph G admitting a zero-sum 3-magic is equivalent to G having a factor with the degree sequence 1 or 4.

3 Bipartite Graphs

In this section we show that if G is a 2-edge connected bipartite graph, then $\mathbb{N}\setminus\{2,3,4,5\}\subseteq N(G)$. Before establishing this result we need some definitions and theorems.

Let G be a directed graph. A *k-flow* on G is an assignment of integers with maximum absolute value at most $k-1$ to each edge of G such that for each vertex of G, the sum of the labels on incoming edges is equal to that of the labels on outgoing edges. A *nowhere-zero k-flow* is a k-flow with no zeros.

A *$\mathbb{Z}_k$-flow* on G is an assignment of element of $\mathbb{Z}_k$ to each edge of G such that for any vertex of G, the sum of the labels on incoming edges is equal to that of the labels on outgoing edges (mod k). A *nowhere-zero $\mathbb{Z}_k$-flow* is a $\mathbb{Z}_k$-flow with no zero, for every $k\in\mathbb{N}$.

The following theorem was proved, in [16].

Theorem 11. *Every 2-edge connected graph admits a nowhere-zero 6-flow.*

The following well-known theorem is due to Tutte.

Theorem 12.[8, p.294] *If G is a directed graph and $k\geq 1$ is an integer, then G admits a nowhere-zero k-flow if and only if G admits a nowhere-zero $\mathbb{Z}_k$-flow.*

In [11], the null set of a complete bipartite graph was determined.

Theorem 13. *If $m,n\geq 2$, then* $N(K_{m,n})=\begin{cases}\mathbb{N}, & \text{if } m{+}n \text{ is even;}\\ \mathbb{N}\setminus\{2\}, & \text{if } m{+}n \text{ is odd.}\end{cases}$

In the following theorem we determine a necessary condition for the existence of a zero-sum h-magic in bipartite graphs.

Theorem 14. *Let G be bipartite in which G admits a zero-sum h-magic, for some $h\in\mathbb{N}$. Then G is 2-edge connected.*

Proof. Assume that G admits a zero-sum h-magic labeling, say l. By contrary let $e=uv$ be a cut edge of G. Note that $G\setminus\{e\}$ is bipartite graph. Let H be one of the connected component of $G\setminus\{e\}$ with two parts X and Y such that $Y\cap\{u,v\}\neq\emptyset$. It is not hard to see that in G, $\sum_{x\in X}s(x)=\sum_{y\in Y}s(y)-l(uv)$. On the other hand, by assumption

$$\sum_{x\in X}s(x)=\sum_{y\in Y}s(y)\equiv 0 \pmod h.$$

This implies that $l(uv)\equiv 0 \pmod h$, which is a contradiction. □

Here, we determine the null set of a 2-edge connected bipartite graph which is somehow a generalization of Theorem 13.

Theorem 15. *Let G be a 2-edge connected bipartite graph. Then G admits a zero-sum k-magic, for $k \in \mathbb{N} \setminus \{2, 3, 4, 5\}$.*

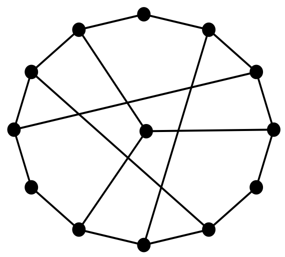
Proof. First, orient all edges from one part of G to the other part and call the resultant directed graph by G'. By Theorem 11, G' admits a nowhere-zero 6-flow. Thus G' admits a nowhere-zero k-flow, for every $k \in \mathbb{N} \setminus \{2, 3, 4, 5\}$ and so by Theorem 12, G' admits a nowhere-zero $\mathbb{Z}_k$-flow, for $k \in \mathbb{N} \setminus \{2, 3, 4, 5\}$. Now, by removing the direction of all edges we conclude that G admits a zero-sum k-magic, for every $k \in \mathbb{N} \setminus \{2, 3, 4, 5\}$ and the proof is complete. □

In the following remark, we show that there are some 2-edge connected bipartite graph with no zero-sum k-magic, for $k = 2, 3, 4$.

Remark 3. In a bipartite graph the existence of a zero-sum k-flow is equivalent to the existence of a zero-sum k-magic. To see this first orient all edges from one part to the other part. Therefore, G admits a zero-sum k-flow if and only if G admits a nowhere-zero k-flow. Thus by Theorem 12, G admits a nowhere-zero $\mathbb{Z}_k$-flow. Therefore, G admits a zero-sum k-magic.

Let G be the following graph. By a computer search one can see that G does not admit a zero-sum 4-flow, see [1]. So G does not admit a zero-sum 4-magic.

Since G does not admit a zero-sum 4-flow, G does not admit a zero-sum k-flow, for $k \leq 4$. Hence G does not admit a zero-sum k-magic, for $k = 2, 3, 4$.



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References

[1] S. Akbari, A. Daemi, O.Hatami, A. Javanmard, A. Mehrabian, Zero-sum flows in regular graphs, Graphs and Combinatorics 26 (2010), 603-615.

[2] S. Akbari, N. Gharghani, G.B. Khosrovshahi, S. Zare, A note on zero-sum 5-flow in regular graphs, The Electronic Journal of Combinatorics 19(2), 2012, #P7.

[3] S. Akbari, M. Kano, S. Zare, 0-Sum and 1-Sum Flows in Regular Graphs, submitted.

[4] J. Akiyama and M. Kano, Factors and Factorizations of Graphs, Springer, 2007, Version 1.0.

[5] F. Bäbler, Über die zerlegung regulärer Streckencomplexe ungerader Ordnung, Comment. Math. Helv. 10 (1938), 275–287.

[6] J.-.O, Choi, J.P. Georges, D. Mauro, On zero-sum $\mathbb{Z}_k$-magic labelings of 3-regular graphs, Graphs and Combinatorics, to appear.

[7] J. Gallian, A dynamic survey in graphs labeling (ninth edition), Electronic Journal of Combinatorics (2005).

[8] R.L. Graham, M. Grotschel and L. Lovasz, Handbook of Combinatorics, The MIT Press-North-Holland, Amesterdam (1995).

[9] M. Kano, Factors of regular graph, J. Combin. Theory Ser. B 41 (1986), 27-36.

[10] J. Petersen, Die Theorie der regularen Graphen, Acta Math(15) (1891), 193–220.

[11] E. Salehi, Zero-sum magic graphs and their null sets, Ars Combinatoria 82 (2007), 41-53.

[12] E. Salehi, On Zero-Sum Magic Graphs and Their Null Sets, Bulletin of the Institute of Mathematics, Academia Sinica 3 (2008), 255-264.

[13] E. Salehi, and S. Hansen, Zero-Sum Magic and Null Sets of Planar Graphs, to appear in the Congressus Numerantium 72 (2010), 55-64.

[14] J. Sedlacek, On magic graphs, Math. Slov. 26 (1976), 329-335.

[15] J. Sedlacek, Some properties of magic graphs, in graphs, hypergraph, Bloc Syst. (1976) Proc. Symp. Comb.

[16] P.D. Seymour, Nowhere-zero 6-flows, J. Combin. Theory Ser. B 30 (1981) 130135.l. Zielona Gora (1976), 247-253.

[17] R.P. Stanley, Linear homogeneous diophantine equations and magic labelings of graphs, Duke Mathematics Journal 40 (1973), 607-632.

[18] R.P. Stanley, Magic labeling of graphs, symmetric magic squares, systems of parameters and Cohen- Macaulay rings, Duke Mathematics Journal 40 (1976), 511-531.

[19] W.D. Wallis, Magic Graphs, Birkhauser Boston (2001).